Double mode condensates of a flowing plasma as possible relaxed states

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Stationary energy (SE) states are found for a flowing two-fluid plasma. These are of interest because quiescent relaxed plasma equilibria, if they exist, should be a subset of the SE states. The platform for the analysis is a two-fluid flowing plasma, which is more realistic than the single-fluid model used in most treatments of relaxed plasmas. The two-fluid model is characterized by *two* helicities, one for each species. Including flow allows for the possibility that flow may be an important ingredient in relaxed states. The analysis expands the flow and field vectors in a complete basis set of divergence-free vectors. This reduces the problem to algebra. It leads at once to the prediction of SE states that are a two-point spectrum of the basis set, i.e., *double-mode condensates*. The properties of these SE states are shown to depend on their location in helicity space, a two-dimensional parameter space of the ion and electron helicities. The migration of a SE state as a result of resistive and viscous dissipation is also shown.
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I. VARIATIONAL PRINCIPLE FOR A TWO-FLUID PLASMA

Introduction

Basic magnetofluid stability is the foundational, ''zero-order'' issue on which practical magnetic fusion is built. Even when it can be assured on the global-scale, stability remains the foundational issue in beta limits, with strong implications on the economics of fusion energy. Into this arena has entered a growing recognition of the importance of flow. Plasma flow, especially sheared flow, adds a whole new dimension to the quest. Flow is particularly intriguing because, on one hand it is observed more and more in experiments as improved diagnostics are brought to bear. On the other hand, classical stability theory is particularly ill-suited to deal with flow. Even the theory of flowing equilibria raises substantial barriers: the system of equations is exceedingly complex compared with the simple, single equation for static equilibria. Added to the intrigue is the emerging possibility that magnetoplasmas with large flows (comparable to the Alfvén speed) may be *natural states* of a plasma.[1–3] This represents a clear paradigm shift. In the standard scenario, the major actors are curvature (good or bad) and pressure gradients, and flow is only a bit player. In the alternate scenario, flow becomes a lead actor.

The objective of this paper is to find natural or *preferred* states of a plasma without a precondition of little or no flow. Since common stability methods for flowing plasmas remain murky at best, an alternate approach is adopted, a kind of short-cut, namely the search for stationary-energy (SE) states. While a particular SE state may not be stable, it is certain that a stable plasma is a SE state. Stated another way, the set of stable states lies wholly within the set of SE states. At the very least, focusing on SE states greatly simplifies the difficult task of finding flowing equilibria.

The search for *preferred states* has broader interest than simply the possibility of self-organization, i.e., the plasma spontaneously finding the preferred state it wants. Preferred states are important even if the plasma does not relax spontaneously. They are desirable because they are quiescent or ''contented'' states with low turbulence and hence good confinement.

Stationary-energy theory is not the end-all in the quest for stability. There are several questions it does not answer. The first set of questions concerns self-organization, if it occurs. *Which* of the many SE states will be chosen by the plasma? What are the details of the dynamical process by which the SE state is reached? The second set of questions concerns control. How much do the *initial conditions* influence the SE state chosen by the plasma? Can *external controls* be employed to encourage the plasma toward a more attractive SE state? The third set concerns stability *per se*. Can an SE state be satisfactorily metastable even if not absolutely stable? Is the energy-minimization principle too conservative, thereby excluding interesting SE states that might appear in a more generous principle? Even though it does not answer these questions, finding SE states is an important step on the road toward understanding the preferred states of a flowing plasma and their stability.

The specific method for finding SE states pursued here is as follows. The platform for the analysis is the two-fluid plasma model including the two ''two-fluid'' helicities as invariants. To simplify the analysis, an incompressible (constant density) plasma is assumed; this linearizes the problem. The flows and fields are expanded in a complete basis set (eigenfunctions of the curl); this reduces the differential equations to algebra. It also conceals the entire geometric content in the eigenvalues of the basis set. In general the resulting SE states are two-eigenvalue states, i.e., the eigenvalue spectrum collapses to two points, hence the designation *double-mode condensate*. Thus SE states are the set of all two-eigenvector states with a particular flow-field relationship.

The properties of double-mode states, and the simpler reduced case of single-mode states can be expressed as a function of their position in helicity space, the two-dimensional map whose coordinates are the electron and ion helicities. Moreover the evolution of the SE states can be found; this produces a migration in helicity space.

The outline of the paper is as follows. Section II further discusses the foundations for the SE principle employed here, offering a justification for the stationary-energy principle (Sec. II A) and presenting the two-fluid platform and its ideal invariants (Sec. II B). Section III solves the variational problem for SE states, finding expressions for the flow-field relationships and the energy as a function of the invariants. Section IV finds the evolution of the SE states caused by dissipative processes. Section V concludes the paper with a discussion.

II. FOUNDATIONS OF STATIONARY-ENERGY PRINCIPLE

A. Rationale for stationary-energy states

The search for preferred states of a plasma is not new. The oldest and most venerated approach is the minimum-energy principle of Woltjer and Taylor.[4,5] Unfortunately this approach predicts states that are force-free and thus have zero beta (plasma pressure/magnetic pressure); consequently these states have limited interest in fusion applications. This unsatisfactory result may be the result of adopting an overly conservative minimization principle. Several attempts have been made to broaden the Taylor–Woltjer energy-based approach to allow for plasmas with finite beta[4–10] and significant flows.[1–3,6,7] In addition, others have investigated approaches based on statistical principles,[11] minimum dissipation rate,[12,13] and minimum autocorrelation function.[14]

The energy to be minimized is the *ordered* energy of the system, the sum of the mechanical flow (f) energy of fluid motion and the magnetic (m) energy:[6,15]

$$W_{mf}=\int d\tau\left(\frac{1}{2}m_i n u^2+\frac{B^2}{8\pi}\right). \tag{1}$$

Here $d\tau$ is the volume element and the integral is over the system domain, n is the plasma density, m_i and $\mathbf{u}$ are the ion mass and flow velocity, and $\mathbf{B}$ is the magnetic field. (For massless electrons the bulk flow velocity and ion fluid velocity are indistinguishable.) The ordered energy W_{mf} is the *complement* of the *disordered* energy, namely the thermal energy $W_{\rm th}=\int d\tau p/(\gamma-1)$. It is the complement in the sense that the sum of the two $W_{mf}+W_{\rm th}$ is invariant in a closed system. In general, W_{mf} itself is *not* an invariant.

There are at least four reasons, *a priori*, for adopting the stationary-energy approach. First, W_{mf} is non-negative and thus bounded below. Thus it is meaningful to find a state with *minimum* W_{mf} compared with neighboring states. By contrast the ideal invariants (to be introduced shortly) may be positive or negative. Second, W_{mf} is *not* an ideal invariant in a compressible plasma. As such, minimizing it implies a dynamical relaxation process in which it can ''fall down,'' while the ideal invariants change relatively little. Third, SE states are *genuine two-fluid equilibria*. It was shown in Ref. 16 that SE states are a *subclass* of two-fluid equilibria. Fourth, W_{mf} is a measure of the *ordered* energy and as such has a thermodynamic connotation. Minimizing the ordered energy is intuitively equivalent to maximum *disordered* energy, of which entropy is the thermodynamic measure. An equivalence between minimizing W_{mf} and maximizing the entropy was established in Sec. IV A of Ref. 2.

Before proceeding one qualification must be made. The analysis here adopts the incompressible plasma paradigm. This is done on the strictly pragmatic grounds that it simplifies the analysis. In particular, the equations become linear, to which powerful mathematical tools can be applied. It must be kept in mind though that incompressibility is only an artifice. An unfortunate result of this artifice is that W_{mf} becomes an ideal invariant. The only ideal transfer mechanism between the ordered and disordered energy forms is through the work term, which is excluded by incompressibility. Thus apart from dissipative processes, W_{mf} cannot change, i.e., it is an ideal invariant, but only artificially so. It should never be regarded as having equal standing with the helicity invariants, which are genuine invariants of compressible plasma species.

B. Two-fluid platform

The two-fluid plasma model is adopted because it is more general than the single-fluid platform assumed in many previous analyses. This broader base is important because the limited range of SE states found by the Woltjer–Taylor principle[4,5] sprang in part from too narrow a platform, namely the *single*-fluid magnetohydrodynamic (MHD) plasma model.

The magnetofluid invariants in an ideal two-fluid plasma are the electron and ion helicities

$$K_e=\frac{1}{8\pi}\int d\tau\mathbf{A}\cdot\nabla\times\mathbf{A}, \tag{2}$$

$$K_i=\frac{1}{8\pi}\int d\tau\left(\mathbf{A}+\frac{m_i c}{e}\mathbf{u}\right)\cdot\nabla\times\left(\mathbf{A}+\frac{m_i c}{e}\mathbf{u}\right). \tag{3}$$

Here K_e assumes massless electrons. The ideal invariance of these was proved elsewhere.[2,10] The angular momentum may also be invariant if the system has an axisymmetric boundary with suitable boundary conditions,[2,4] but this case is left for future investigation. Then the variational problem to be solved is

$$\delta(W_{MF}-\lambda_e K_e-\lambda_i K_i)=0. \tag{4}$$

It is important to clarify the ''standing'' of K_e and K_i *vis a vis* other invariants that have been used, particularly in single-fluid MHD analyses. The more conservative Taylor theory accepts only a single invariant, the magnetic helicity, $K_m=\int d\tau\mathbf{A}\cdot\nabla\times\mathbf{A}$, which is equivalent to the electron helicity here. More generous theories have also included the cross helicity $K_x=\int d\tau\mathbf{u}\cdot\mathbf{B}$. However, ideal MHD (single-fluid) actually has a *third* invariant, the fluid helicity $K_f=\int d\tau\mathbf{u}\cdot\nabla\times\mathbf{u}$, the same as appears in a simple Euler fluid. That all three, K_m, K_x, K_f are ideal invariants in single-fluid MHD was also proved in Sec. II A of Ref. 2. The relationship between the invariants for single-fluid MHD and the more general two-fluid model is shown in the Appendix. It is demonstrated there that the (K_m, K_x, K_f) set *overstates* the invariance, while the (K_m) set *understates* it.

C. Durability of the helicities

A realistic plasma has dissipative processes that cause both energy and helicities to change. However the relaxation theory that assumes invariant helicities is still valid, at least approximately, if the helicities are more durable than the energy that is minimized. However, it is not sufficient for this ''durability'' to hold in an equilibrium with large length scale. In turbulent processes, very fine length scales can arise spontaneously, as in reconnections. On a fine scale, a helicity may suffer rapid dissipation. The durability of K_e (the magnetic helicity assuming massless electrons) is reasonably well established. However, the ion helicity has been challenged in Ref. 17 on the grounds that viscosity makes K_i even *more vulnerable* to dissipation than the ordered energy.

The question of the durability of the helicities was taken up in Ref. 2 using three tests of ''ruggedness.'' These tests, which been introduced elsewhere, are the following: (1) *inverse cascade*—the ''direction'' of the relaxation must be toward large-scale structures; (2) *selective decay*—the helicity invariant must decay more slowly than the magnetofluid energy; and (3) *stability to resistive modes*—the helicities must be stable to resistive modes, i.e., the time scale for change in the field topology must be faster than that for change in the helicity. The electron helicity was previously shown to pass all three tests.

Applying these tests to K_i is more difficult since the flow-field coupling in a turbulent process must be taken into account. In Ref. 2 the well-known paradigm of quasilinear theory was employed, namely that the frequencies and responses of linear theory are retained even though the fluctuations may grow to nonlinear amplitudes. By this the electromechanical coupling operator was found, which links the flow and field for each kind of fluctuation (Alfvén, ion-cyclotron, whistler, etc.). The lowest-energy fluctuation was considered dominant on energetic grounds. Using the flow-field coupling in a two-fluid, K_i was found to pass all three tests, just as did K_e. Note that in *single-fluid* MHD the magnetic helicity passes the ruggedness tests but the cross helicity and the fluid helicity, *fail*.

III. DOUBLE-MODE CONDENSATES

A. Introduction

The approach taken by Taylor[5] finds the minimum magnetic energy states in single-fluid MHD assuming invariant magnetic helicity. The Euler–Lagrange system is a single linear equation. Vector analysis leads to a characteristic equation that is *linear* in the eigenvalue Λ, i.e., it has a single value for a given solution; thus the stationary-equilibrium state is composed of a single ''mode,'' i.e., *single-mode condensates*. In the two-fluid model the problem is more complicated. Here the Euler–Lagrange system is a coupled *pair* of equations (linear if uniform density is assumed). Vector analysis leads to a characteristic equation that is *quadratic* in the eigenvalue with *two* roots Λ_1, Λ_2. Then stationary-energy states are the superposition of two eigenvectors, or *double-mode condensates*. These are identical to the double-Beltrami states investigated in Ref. 3.

In the single-fluid MHD theory only the magnetic vector potential $\mathbf{A}$ appears. Assuming the Coulomb gauge $\nabla\cdot\mathbf{A}=0$ the variational problem can be analyzed by expanding in a divergence-free vector basis set. In the two-fluid theory, two vectors appear: $\mathbf{A}$ and the flow velocity $\mathbf{u}$. If the plasma is assumed to be incompressible $(n=const)$ then both vectors are divergence-free and the vector expansion can again be used. The entire geometric content of the problem is contained in the *two* eigenvalues.

The objective here is to find functional relationship between the ordered energy and the two helicity invariants, $W_{mf}=F(K_i,K_e)$. The standard procedure for a constrained minimization is followed. (1) The variational principle, Eq. (4) leads to Euler–Lagrange equations for $\mathbf{u}_i(\mathbf{r},\lambda_i,\lambda_e)$ and $\mathbf{A}(\mathbf{r},\lambda_i,\lambda_e)$; (2) From these the global integrals are constructed, $W_{mf}=f_W(\lambda_i,\lambda_e), K_i=f_i(\lambda_i,\lambda_e), K_e=f_e(\lambda_i,\lambda_e)$. (3) The second and third functions are inverted to find the Lagrange multipliers $\lambda_i=g_i(K_i,K_e), \lambda_e=g_e(K_i,K_e)$. (4) Finally these are substituted into f_W to give $W_{mf}=F(K_i,K_e)$. The same approach was followed less rigorously in Ref. 18.

B. Analysis

The independent variations $\delta\mathbf{A},\delta\mathbf{u}$ lead to a pair of Euler–Lagrange equations

$$\begin{bmatrix} (\nabla\times)-(\lambda_i+\lambda_e) & -\lambda_i(\nabla\times) \\ \lambda_i & \lambda_i(\nabla\times)-1/\ell_i^2 \end{bmatrix} \begin{Bmatrix} (\nabla\times)\mathbf{A} \\ m_i c\mathbf{u}/e \end{Bmatrix}=0 \quad (5)$$

where $(\nabla\times)$ is the curl operator, and $\ell_i=(m_i c^2/4\pi e^2 n)^{1/2}$ is the ion skin depth, which is the fundamental length scale of a two-fluid. In the constant-density model, $\ell_i=const$ so that these equations are linear.

Adopt the Coulomb gauge, $\nabla\cdot\mathbf{A}=0$. With $n=const$, the continuity equation is $\nabla\cdot\mathbf{u}=0$. Thus both $\mathbf{A}$ and $\mathbf{u}$ are divergence free. Expand them in eigenfunctions of the curl $\mathbf{Y}_k$ (also called Beltrami functions) which satisfy $\nabla\times\mathbf{Y}_k=\Lambda_k\mathbf{Y}_k$, with eigenvalues Λ_k. In three-dimensional geometry, the index k is three-dimensional $k=(l,m,n)$. The eigenvectors $\mathbf{Y}_k$ are orthogonal so that $\int d\tau\mathbf{Y}_k\cdot\mathbf{Y}_{k'}=\delta_{kk'}$ where $\delta_{kk'}=\delta_{ll'}\delta_{mm'}\delta_{nn'}$ is a generalized Kronecker delta. These eigenvectors form a complete basis for divergence-free vectors.[19] Then

$$\mathbf{A}=\sum_k A_k\mathbf{Y}_k; \quad \mathbf{u}=\sum_k u_k\mathbf{Y}_k, \quad (6)$$

where A_k and u_k are the expansion coefficients. Substitute the expanded forms into Eq. (5) and take the inner product $\int d\tau\mathbf{Y}_K(\cdots)$ where K is arbitrary. Then for every K

$$\begin{bmatrix} \Lambda_K-\lambda_i-\lambda_e & -\lambda_i\Lambda_K \\ \lambda_i & \lambda_i\Lambda_K-1/\ell_i^2 \end{bmatrix} \begin{Bmatrix} \Lambda_K A_K \\ m_i c u_K/e \end{Bmatrix}=0. \quad (7)$$

Observe two major simplifications. (1) The differential-operator matrix in Eq. (5) reduces to an algebraic matrix, and the analysis becomes purely algebraic. (2) The *geometric* content of the problem is contained entirely in the eigenvalues Λ_k. One need not know the structure of the $\mathbf{Y}_k$ except to view the spatial structure of solutions.

Equation (7) has nontrivial solutions only if the determinant of the operator matrix is zero. This defines the *characteristic equation*

$$\lambda_i\Lambda_K^2-(1/\ell_i^2+\lambda_i\lambda_e)\Lambda_K+(\lambda_i+\lambda_e)/\ell_i^2=0. \tag{8}$$

It is quadratic in Λ and thus has *two* roots, hereafter labeled Λ_1,Λ_2. the subscripts being shorthand for the two indices K_1 and K_2 for which Eq. (8) is satisfied.

In a fixed "compact" (singly connected) system domain the eigenvalues $\{\Lambda_k\}$ form a real and discrete set.[19] In a toroidal (doubly connected) geometry, the eigenvalues are less restricted. This "discretization" is handled here by viewing Eq. (8) as a *permissibility* condition on the Lagrange multipliers λ_e,λ_i. Another form of the permissability condition gives the permitted value of λ_i in terms of λ_e and the eigenvalue

$$\lambda_i=\frac{\Lambda_K-\lambda_e}{1+\Lambda_K(\Lambda_K-\lambda_e)\ell_i^2}. \tag{9}$$

Note that the same λ_e, λ_i must satisfy this equation for *both* eigenvalues Λ_1,Λ_2.

Since only two eigenvalues at a time can satisfy the characteristic equation (8), all other the coefficients in the infinite sums Eq. (6) must vanish, i.e., $A_k=0$ for all $k\neq K_1,K_2$. Then the general stationary-energy state is a *double-mode condensate*:

$$\mathbf{A}=A_1\mathbf{Y}_1+A_2\mathbf{Y}_2, \tag{10}$$

$$\mathbf{u}=u_1\mathbf{Y}_1+u_2\mathbf{Y}_2. \tag{11}$$

The existence of such two-part states was explored in Ref. 3. The diamagnetic class of solutions found in Ref. 18 are a subset of these two-part solutions for the case $\Lambda_2=-\Lambda_1$.

Equation (7) connects each coefficient pair (A_K,u_K), $K=1,2$

$$u_K=\Lambda_K(\Lambda_K-\lambda_e)\frac{cA_K}{4\pi en}, \tag{12}$$

i.e., in stationary-energy states the flow and field are coupled in a particular way. Requiring λ_e,λ_i to be the same for both eigenvalues Λ_1,Λ_2, it follows from Eq. (9)

$$\lambda_e=\frac{\Lambda_1+\Lambda_2}{2}\pm\sqrt{\frac{1}{\ell_i^2}+\left(\frac{\Lambda_1-\Lambda_2}{2}\right)^2}. \tag{13}$$

Since Eq. (13) has two roots, there will be two stationary-energy states for each $\Lambda_1-\Lambda_2$ combination.

There is also a *third family* of stationary-energy states if one limits attention to solutions with a *single* eigenvalue Λ_1: $\mathbf{A}=A_1\mathbf{Y}_1$ and $\mathbf{u}=u_1\mathbf{Y}_1$. These are *single-mode condensates*. The force-free class of solutions found in Ref. 18 are equivalent to these. Again the coefficient pair (A_1,u_1) is governed by Eq. (11) with $K\to1$. The multiplier λ_i is again given by Eq. (9) with $K\to1$, however, λ_e need not satisfy Eq. (13) since these states use only a single eigenvalue.

Apply the double-mode solution to the expressions for the three global integrals to find the functional forms $K_e,K_i,W_{mf}=f(\Lambda_1,\Lambda_2,\lambda_e,\lambda_i)$. Exploit here the orthogonality of the basis functions and the normalization $\int|\mathbf{Y}_k|^2d\tau=1$. Then

$$K_e=\Lambda_1\frac{A_1^2}{8\pi}+\Lambda_2\frac{A_2^2}{8\pi}, \tag{14}$$

$$K_i=\Lambda_1[1+\Lambda_1(\Lambda_1-\lambda_e)\ell_i^2]^2\frac{A_1^2}{8\pi}+\Lambda_2[1+\Lambda_2(\Lambda_2-\lambda_e)\ell_i^2]^2\frac{A_2^2}{8\pi}, \tag{15}$$

$$W_{mf}=\Lambda_1^2[1+(\Lambda_1-\lambda_e)^2\ell_i^2]\frac{A_1^2}{8\pi}+\Lambda_2^2[1+(\Lambda_2-\lambda_e)^2\ell_i^2]\frac{A_2^2}{8\pi}. \tag{16}$$

Recall that λ_e must be one of the two roots of Eq. (13). Since K_e and K_i are the specified invariants, Eqs. (14) and (15) form a coupled algebraic system for the coefficients A_1 and A_2. Solving these gives

$$\frac{A_1^2}{8\pi}=\frac{f(\Lambda_2)K_e-K_i}{\Lambda_1[f(\Lambda_2)-f(\Lambda_1)]}, \tag{17}$$

$$\frac{A_2^2}{8\pi}=\frac{K_i-f(\Lambda_1)K_e}{\Lambda_2[f(\Lambda_2)-f(\Lambda_1)]}, \tag{18}$$

where $f(\Lambda)\equiv[1+\Lambda(\Lambda-\lambda_e)\ell_i^2]^2$. Observe that since A_1^2,A_2^2 are non-negative, some combinations of Λ_1,Λ_2, K_e,K_i, will be forbidden. These results are substituted into Eq. (16) to find the functional form $W_{mf}=f(\Lambda_1,\Lambda_2,K_e,K_i)$ for the *double-mode* stationary-energy states

$$W_{mf}=\Lambda_1^2g(\Lambda_1)\frac{A_1^2}{8\pi}+\Lambda_2^2g(\Lambda_2)\frac{A_2^2}{8\pi}, \tag{19}$$

where $g(\Lambda)\equiv1+(\Lambda-\lambda_e)^2\ell_i^2$.

For *single-mode condensates* the functional form was found previously [Eq. (23) of Ref. 18], for which, again, there are two branches:

$$W_{MF}=\left[|\Lambda_1|\ell_i+\frac{1}{|\Lambda_1|\ell_i}(\pm\sqrt{K_i/K_e}-1)^2\right]\frac{|K_e|}{\ell_i}. \tag{20}$$

C. Ordered energy vs location in helicity space

By definition, all "successful" magnetically confined plasmas in experiments have predominantly large-scale structure, whether they are quiescent or not. Thus the focus here is on the states with the largest-scale structure allowed by the system domain. These are either (1) *single-mode states* with $|\Lambda_1|=\Lambda_{\min}$ or (2) *double-mode states* with $\Lambda_1=-\Lambda_2=\Lambda_{\min}$, where $\Lambda_{\min}\sim1/L$ is the magnitude of the lowest eigenvalues and L is the smallest dimension of the system domain.

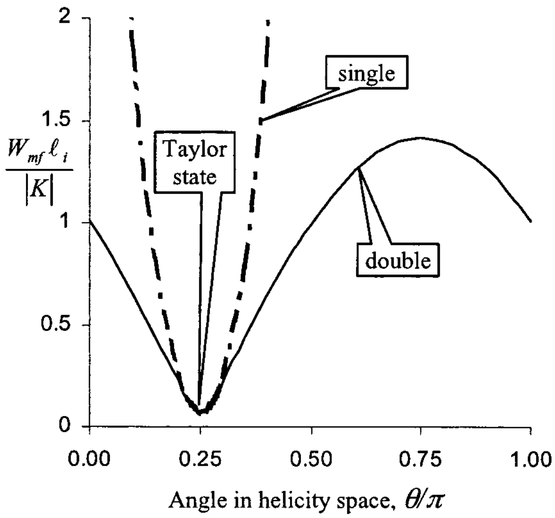

FIG. 1. Ordered energy as a function of position in helicity space for single- and double-mode states: $\Lambda_1 = 0.1/\ell_i$; for double-mode solutions $\Lambda_2 = -\Lambda_1$ is assumed.

In order to portray the energy vs helicity dependence, define the helicity *modulus*

$$|K| \equiv \sqrt{K_e^2 + K_i^2}. \tag{21}$$

The normalized energy $W_{mf}/|K|$ can be expressed as a function of the helicity ratio K_i/K_e and the eigenvalues. A particular helicity ratio corresponds to a polar "ray" in *helicity space* K_i vs K_e. *Helicity space*[18] is the two-dimensional domain of the helicities. The polar angle in helicity space designates a particular ray

$$\theta \equiv \tan^{-1}(K_i/K_e). \tag{22}$$

Figure 1 shows how the energy varies with polar angle in helicity space. The range of θ shown spans the first and second quadrants of helicity space; the pattern repeats in the third and fourth quadrants. The example $|\Lambda_1| = |\Lambda_2| = 0.1/\ell_i$ is shown: This represents a domain size comparable to 10 skin depths, which is relevant to many field-reversed-configuration experiments. Observe three points. (1) Both single- and double-mode states have allowed and forbidden regions.[18] In the single-modes states the forbidden regions for are the second and fourth quadrants of helicity space. This is obscured in Fig. 1 because the single-mode energy goes off scale well before approaching the forbidden regimes. In the double-mode states the forbidden region in the first two quadrants is the range $0.219 < \theta/\pi < 0.284$. This is obscured in the figure because the single-mode curve neatly fills in the gap. (2) The double-mode states have lower energy than the single-mode states throughout the region where they are allowed. (3) The point $\theta/\pi = 1/4$ is the Taylor state, a particular single-mode state.

IV. EVOLUTION OF STATIONARY-ENERGY STATES

A. Validity of slowly changing stationary-energy states

This section investigates the evolution of SE states caused by dissipation. It is presumed that any rapid self-organization phase has long since ended. The action of dissipation causes the global integrals (W_{mf}, K_e, K_i) to evolve. The continuing action of self-organization only serves to maintain the plasma in an approximately SE state, albeit an evolving one. The evolution will cause the plasma to a *migrate* in helicity space (K_e, K_i). The evolution that is considered here is the slow change in *all* the ideal invariants (W_{mf}, K_e, K_i) that results from dissipation. It is not the same as the rapid "catastrophic" relaxation that may take place in a newly formed plasma.

Since W_{mf} is positive definite its evolution is a *decay*. Thus the exponential decay rate $\dot{W}_{mf}/W_{mf}$ is the appropriate measure of its evolution. The helicities, however, are not positive definite. The exponential decay time of, e.g., the ion helicity $K_i/|\dot{K}_i|$ is meaningless in the interesting case which has $K_i = 0$. Better measures of evolution are provided by $\dot{K}_e/|K|$, and $\dot{K}_i/|K|$, where $|K| = (K_e^2 + K_i^2)^{1/2}$ is the *helicity modulus*. In helicity space, $|K|$ is the radius from the origin.

B. Dissipation coefficients

The diffusivities for resistive and viscous effects, respectively, are

$$D_{\rm res} = \eta c^2/4\pi, \quad D_{\rm vis} = \mu/m_i n, \tag{23}$$

where η is the electrical resistivity and μ is the viscosity coefficient. The *magnetic Prandtl number* measures the ratio of viscous to resistive effects

$$\Pr \equiv \frac{4\pi\mu}{\eta c^2 m_i n} = \frac{3}{20\sqrt{2}} \left(\frac{m_i}{m_e}\right)^{1/2} \beta_i \left(\frac{T_e}{T_i}\right)^{3/2}. \tag{24}$$

The latter form assumes a hydrogen plasma, the Braginskii shear viscosity $\mu_\perp \to \eta_1^i$ and the perpendicular resistivity $\eta \to \eta_\perp$. Here $\beta_i \equiv 8\pi p_i/B^2$ is the ion "beta" and T_e, T_i are the electron and ion fluid temperatures. For example, a deuterium plasma with equal temperatures $T_e = T_i$ and $\beta = 1$ (so that $\beta_i = 1/2$) has a magnetic Prandtl number of $\Pr \approx 3.2$. In this case the viscous dissipation exceeds but does not dominate the resistive dissipation. For simplicity hereafter we assume a scalar $\mu = \eta_1^i$, and that both η and μ are spatially uniform; the latter makes Pr uniform. The small electron viscosity is ignored.

Braginskii or Navier–Stokes viscous stress laws are only valid if the gradient length scale does not exceed the mean-free-path. The interest here, however, is in large-scale plasma structures in which these laws remain valid.

C. Rate of change of global integrals for SE states

Selective decay is applied to the single- and double-mode states. The rates of change of the energy and helicities, adapted from Eqs. (39) and (40) of Ref. 2 are

$$\frac{dW_{MF}}{dt} = -D_{\rm res} \int d\tau \times \left\{ \frac{|\nabla \times \mathbf{B}|^2}{4\pi} + \Pr \cdot m_i n \mathbf{u}_i \cdot [\nabla \times (\nabla \times \mathbf{u}_i)] \right\}, \tag{25}$$

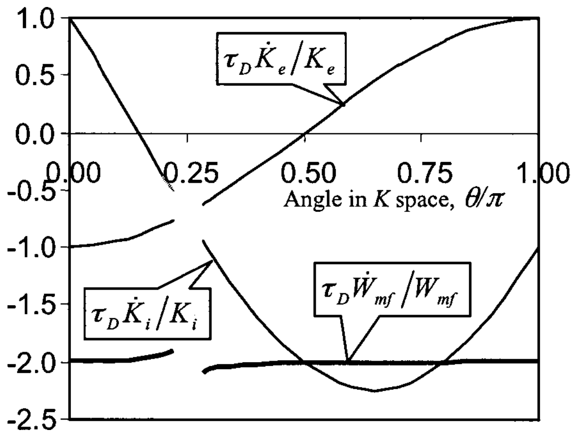

FIG. 2. Rates of change of global integrals vs position in helicity space: Double-mode states with $\Lambda_1 = -\Lambda_2 = 0.1/\ell_i$ and Prandtl number$=3$.

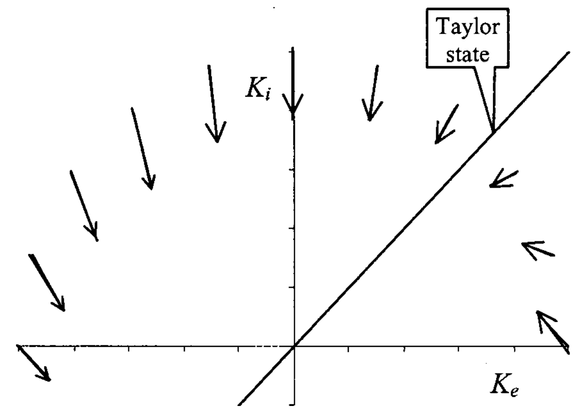

FIG. 3. Evolution of double-mode states: $\Lambda_1 = -\Lambda_2 = 0.1/\ell_i$, Pr$=3$.

$$\frac{dK_e}{dt} = -D_{\rm res}\int d\tau \frac{\mathbf{B}\cdot\nabla\times\mathbf{B}}{4\pi}, \quad (26)$$

$$\frac{dK_i}{dt} = -D_{\rm res}\int d\tau\left[\frac{\nabla\times\mathbf{B}}{4\pi} + \mathrm{Pr}\frac{m_i c}{4\pi e}\nabla\times(\nabla\times\mathbf{u}_i)\right] \cdot\left(\frac{m_i c}{e}\nabla\times\mathbf{u}_i + \mathbf{B}\right). \quad (27)$$

Expand the flows and fields found in Sec. III, exploiting the orthogonality of the eigenvectors. Then

$$\frac{dW_{MF}}{dt} = -D_{\rm res}\sum_{K=1,2}[1+\mathrm{Pr}(\Lambda_K-\lambda_e)^2\ell_i^2]\Lambda_K^4\frac{A_K^2}{4\pi}, \quad (28)$$

$$\frac{dK_i}{dt} = -D_{\rm res}\sum_{K=1,2}[1+\mathrm{Pr}\cdot\Lambda_K(\Lambda_K-\lambda_e)\ell_i^2] \cdot\{1+\Lambda_K(\Lambda_K-\lambda_e)\ell_i^2\}\Lambda_K^3\frac{A_K^2}{4\pi}, \quad (29)$$

$$\frac{dK_e}{dt} = -D_{\rm res}\sum_{K=1,2}\Lambda_K^3\frac{A_K^2}{4\pi}, \quad (30)$$

where the sum is over the two elements of the double-mode state. The yardstick for the three rates of change $\dot{W}_{mf}/W_{mf}$, $\dot{K}_e/|K|$, $\dot{K}_i/|K|$ is the resistive diffusion time scale

$$\tau_D \equiv (D_{\rm res}\Lambda_1^2)^{-1}. \quad (31)$$

The three rates of change are shown in Fig. 2 as a function of position in helicity space for double-mode states. The forbidden gap in the range $0.219<\theta/\pi<0.284$ is clearly visible. The ordered energy, of course, always decays (negative). Both the electron helicity $K_e = |K|\cos\theta$ and its rate $\tau_D\dot{K}_e/|K|$ change sign at $\theta=\pi/2$; thus the electron helicity always decays as well. This is not the case for the ion helicity. Since $K_i=|K|\sin\theta$ it changes sign at $\theta=0,\pi/2$. However, $\tau_D\dot{K}_i/|K|$ changes sign at $\theta/\pi=0.147$ and -0.853. Thus dissipation actually causes the ion helicity to *grow* in magnitude in the ranges $0<\theta/\pi<0.147$ and $-1<\theta/\pi<-0.853$; these correspond to two sectors in helicity space.

Observe that all three rates of change are generally comparable in magnitude. This merely means that no one of the global quantities changes precipitously compared with the others. Moreover, the rates are comparable to the simple resistive timescale, Eq. (31). This is partly the result of the Prandtl number being not far from unity. If Pr were much larger than unity, then W_{mf} and K_i would change more rapidly as a result of viscous dissipation.

The rates of change of the two helicities reflects a migration in helicity space as a result of dissipation. Figure 3 shows arrows representing the changes $\Delta K_e = \dot{K}_e\Delta t$, $\Delta K_i = \dot{K}_i\Delta t$. For the most part the evolution is a simple decay, i.e., toward the center of the map. There is a tendency of states far from the Taylor-state line $\theta/\pi=1/4$ to drift toward it. However, states closer to a Taylor state migrate mostly parallel to the $\theta/\pi=1/4$ line. In any case double-mode states retain their identity.

V. DISCUSSION

A variational principle is used to find stationary-energy (SE) states. The ordered energy W_{mf} is minimized subject to constraints on the species helicities K_e and K_i, which are ideal invariants of a two-fluid. This leads to the functional form $W_{mf}=F(K_i,K_e,\Lambda_1,\Lambda_2)$ where Λ_1,Λ_2 are eigenvalues associated with a particular state. Since these states have *two* eigenvalues, they are called *double-mode condensates*. Single-mode condensates have a single eigenvalue and a different functional form $W_{mf}=G(K_i,K_e,\Lambda_1)$.

Both single- and double-mode states have *allowed* and *forbidden* regions in helicity space. The allowed region for double-mode states is much larger. Where their allowed regions overlap, double-mode states always have lower energy than the single-mode states.

Once the SE states are found, their slow evolution from one SE state to another can be found. This evolution results from the changes in K_e and K_i produced by resistivity and viscosity. It is found that the ordered energy and electron helicity always decay as a result of dissipative processes. However, the ion helicity actually grows in certain domains of helicity space. Dissipation also produces a migration in helicity space. For most part this is like a decay, although in some regions of the helicity map there is a slight tendency toward a Taylor-type state (the line $K_i = K_e$)

This theory does not directly answer the question: Which SE state is the preferred state or ''natural'' state of the plasma. While the preferred state is certainly an SE state, it is unclear which. The rationale pursued here is that the preferred state is that with the lowest-magnitude eigenvalues. This is reasonable because in all successful plasma experiments, more or less by definition, the plasma structure is the largest scale structure that fits within the system domain, at least visually. The question of which SE state is the preferred state requires a more sophisticated treatment. The most well-known selection principle is that the state of absolute minimum energy is the preferred state. This approach has been challenged lately by Yoshida and Mahajan, who propose instead a plasma structure-based criterion, minimizing a ''coercive functional'' that measures the structural complexity of the plasma.[20]

Two other limitations of the present theory are worthy of note. (1) The SE states here ignored the influence of an angular momentum invariant. Many practical system domains are axisymmetric, for which, with suitable boundary conditions, the angular momentum is invariant. This is a subject for future work. (2) The model here assumes uniform density while realistically the density is nonuniform. The constant density artifice was useful because it allowed the problem to be treated using analytical methods. Unfortunately, the constant density artifice makes the ordered energy an invariant, but only artificially. The variable density problem needs to be investigated, although it is much more difficult.

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APPENDIX: COMPARISON OF SINGLE-FLUID AND TWO-FLUID INVARIANTS

Differing *helicity* invariants have been employed in variational models of self-organized fluid and plasma states. Helicity is a quadratic invariant of form $\int d\tau \mathbf{F}\cdot\nabla\times\mathbf{F}$ where $\mathbf{F}$ is some physical vector. It is useful to compare the standing of the invariants in conventional single-fluid MHD and in the more general two-fluid model. It should be kept in mind that single-fluid MHD is a reduced version of a two-fluid, i.e., it contains less physics. Consequently, at some points a single fluid will fail in ways that the more general model does not.

A single-fluid has *three* helicity invariants,[2] K_m, K_x, K_f although previous single-fluid plasma studies, to the author's knowledge, have ignored K_f. The one- and two-fluid helicities [Eqs. (2) and (3)] are related as follows:

$$K_e = \frac{1}{8\pi}K_m; \quad K_i = \frac{1}{8\pi}K_m + \frac{m_i c}{4\pi e}K_x + \frac{m_i^2 c^2}{8\pi e^2}K_f. \quad \text{(A1)}$$

Euler–Lagrange (EL) equations are found by minimizing W_{mf} subject to the invariants, taking independent variations $\delta\mathbf{A}$ and $\delta\mathbf{u}$. Their derivation uses suitable integrations by parts and assumes zero normal components $u_n = 0$, $B_n = 0$ on the domain boundary. For a single fluid (invariant K_m, K_x, K_f), the EL equations are

$$\frac{1}{4\pi}\nabla\times\mathbf{B} - \lambda_m\mathbf{B} - \lambda_x\nabla\times\mathbf{u} = 0, \quad \text{(A2)}$$

$$m_i n\mathbf{u} - \lambda_x\mathbf{B} - \lambda_f\nabla\times\mathbf{u} = 0, \quad \text{(A3)}$$

where the $\lambda_m, \lambda_x, \lambda_f$ are the Lagrange multipliers associated with the three constraints.

Similarly, for a two-fluid (invariant K_e, K_i) the EL equations are

$$\nabla\times\mathbf{B} - (\lambda_e + \lambda_i)\mathbf{B} - \lambda_i\frac{m_i c}{e}\nabla\times\mathbf{u} = 0, \quad \text{(A4)}$$

$$m_i n\mathbf{u} - \lambda_i\frac{m_i c}{4\pi e}\mathbf{B} - \lambda_i\frac{m_i^2 c^2}{4\pi e^2}\nabla\times\mathbf{u} = 0. \quad \text{(A5)}$$

While there are obvious resemblances between the EL systems for the two models, there is a fundamental difference. In the specialized case where the two sets of Lagrange multipliers are related as follows:

$$\lambda_m = \frac{1}{4\pi}(\lambda_e + \lambda_i), \quad \text{(A6)}$$

$$\lambda_x = \frac{m_i c}{4\pi e}\lambda_i, \quad \text{(A7)}$$

$$\lambda_f = \frac{m_i^2 c^2}{4\pi e^2}\lambda_i, \quad \text{(A8)}$$

then the one- and two-fluid models have identical EL systems. In general though the reduced one-fluid system has *too much* freedom, i.e., three independent parameters while only two are warranted by the more general model. Thus the single-fluid model *overstates* the invariance.

This difficulty is not removed by discarding one of the single-fluid invariants, e.g., K_f. Dropping it as an invariant is done formally by setting its Lagrange-multiplier λ_f to zero. Comparing Eqs. (A7) and (A8), this implies that $\lambda_x = 0$, i.e., K_x must be discarded as well. The same result also shows up in another way. If K_f is discarded (by setting $\lambda_f = 0$), Eq. (A3) implies that $\mathbf{u}\|\mathbf{B}$. Solve it for $\mathbf{u}$ and substitute into Eq. (A2); for uniform density it follows that $\nabla\times\mathbf{B} = const\ \mathbf{B}$, which is a force-free state, just as found if only K_m is invariant. Of course, the K_m-only theory *understates* the invariants because there are in fact *two* helicity invariants.

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