Two-fluid flowing equilibria of compact plasmas

Loren C. Steinhauer
Redmond Plasma Physics Laboratory, University of Washington, Redmond, Washington 98052

Hideaki Yamada
Graduate School of Science and Technology, Niigata University, Ikarashi, Niigata 950-2181, Japan

Akio Ishida
Department of Environmental Science, Faculty of Science, Niigata University, Ikarashi, Niigata 950-2181, Japan

(Received 14 March 2001; accepted 5 June 2001)

The properties of two-fluid flowing equilibria are explored. This is facilitated by limiting attention to compact toroids in a ''stationary-energy'' state with uniform density. Flowing equilibria are found to fall into two classes, *force-free* and *non-force-free*, referring to the absence or presence of a **j**$\times$**B** force. The force-free class may have significant flows. Spheromaks are in this class. The non-force-free class is diamagnetic and has Alfvénic poloidal flows. Field reversed configurations (FRCs) are in this class. Both classes admit arbitrarily large equilibria. Both classes occupy certain ''allowed'' regions in ''helicity space,'' a two-dimensional parameter map with the electron and ion helicities as coordinates. Allowed regions for the two classes overlap; in the overlap region the non-force-free class is energetically favorable. This sheds light on the FRC-spheromak bifurcation observed in experiments. Two-dimensional analytic equilibria are also found that span both classes. These may play a role similar to the familiar Hill's vortex and Bessel function models in static, magnetohydrodynamic equilibria.
[DOI: 10.1063/1.1388034]

I. INTRODUCTION

In magnetic fusion the need for plasmas that are stable to global and local magnetohydrodynamic (MHD) modes is foundational. Stability issues underlie research on beta limits, disruptions, and equilibrium control in tokamaks. Further, the equilibrium should be controlled so as to minimize the energy transport. In this context, an even more impressive goal would be to find *natural states* of a plasma that are stable to both MHD and microinstabilities and thus have a minimal transport rate. Such states, if they exist, would have obvious interest to fusion. Moreover, they might arise spontaneously in space. Theoretical *preferred* states have been found using reduced plasma models: in the ideal MHD model the lowest-energy *force-free* state (Taylor state) is absolutely stable.[1,2] In fusion, however, force-free states have limited interest because significant plasma pressure is needed in a practical system. Besides, force-free states appear experimentally only in certain arrangements (reversed-field pinch, spheromak) and then only in the central ''core'' of the plasma.

The possibility of natural plasma states with finite pressure has re-emerged in studies of the two-fluid model, which is more general than the MHD formulation.[3–5] A key feature of these states is significant flow. As such they constitute a new class of plasma. They contrast markedly with standard magnetohydrodynamic models of fusion plasmas in which the flow energy is negligible compared with the magnetic energy. Some stationary-energy states may be states of minimum energy.[6] Such states in a two-fluid have been found in idealized one-dimensional (1D) geometries,[3–5] but not yet in a realistic geometry, although the general formulation has been developed.[7] By ''realistic'' geometry is meant a plasma that is (1) large size, (2) axisymmetric two-dimensional (2D), and (3) *isolated* from its surroundings.

We present the first detailed analysis of 2D stationary-energy states of a flowing two-fluid. This includes, importantly, their classification into two distinct types of equilibria. Simple analytic solutions are found for a compact toroid geometry. A compact plasma object with comparable fields and flows might be described as a magnetofluid vortex. The results presented here are limited in two respects. (1) Uniform density is assumed, which simplifies the analysis. (2) Attention is limited to the region inside the separatrix of a compact toroid. Since the self-consistent region outside the separatrix is not considered, the issue of isolation from the surroundings is not addressed.

Analytic solutions are helpful for illuminating the basic properties of equilibria and for preliminary stability analysis. The Hill's vortex or Solov'ev[8] equilibria played such a role for static-MHD equilibria of field reversed configurations (FRC). The flowing equilibria found here can play a similar role. They are more general than the Hill's vortex because they span both FRC and spheromak-like equilibria. Like the Hill's vortex, they are limited to the region inside the separatrix. Analytic solutions of flowing equilibria were found previously but are quite restrictive. Some used 1D analogies to FRCs,[4] tokamaks,[4] and reversed field pinches.[5] Others that found 2D flowing equilibria were limited to the MHD formalism and had only a rigid rotation.[9,10]

The outline of the paper is as follows. Section II reviews the basic elements of two-fluid equilibria, including the natural length scale and the invariants. Section III develops the general properties of two-fluid equilibria in a compact toroid. The uniform density, and ''stationary state'' assumptions are clarified in Sec. III A. The flowing-equilibrium equations are presented in Sec. III B. In Sec. III C the ''characteristic'' equation is found and equilibria are shown to separate into force-free and non-force-free classes with quite different properties. As points of reference, two familiar reduced cases are reviewed in Sec. III D. Section IV examines global integrals and finds the functional form of the organized energy of the system. The properties of force-free and non-force-free classes are explored in Secs. IV B and IV C, respectively. Section IV D addresses stability implications, and Sec. IV E shows the selection of preferred states and the tendency toward bifurcation. Section V presents a simple analytic equilibrium and examines its structural properties. Section VI concludes with a discussion and summary of these results.

II. BASIC TWO-FLUID EQUILIBRIUM PHYSICS

The two-fluid model is a relatively general formulation that includes as limits two familiar cases, simple fluids and MHD. The relationship between these reduced models and the more general formulation is illustrated in Table I of Ref. 4. A two fluid includes both the flow effects found in a simple fluid and the magnetic field effects found in MHD. Flow and field effects are linked through the canonical momentum for each fluid species $\mathbf{P}_\alpha = m_\alpha \mathbf{u}_\alpha + q_\alpha \mathbf{A}/c$, where $\mathbf{A}$ is the vector potential, and $q_\alpha m_\alpha \mathbf{u}_\alpha$ are the charge, mass, and flow velocity of species α (i=ions, e=electrons), respectively. A two-fluid plasma with massless electrons has a natural length scale that appears in the equilibrium formulation,[7,11] as well as in stability analyses,[12] namely the ion skin depth

$$l_i = (m_i c^2/4\pi e^2 n)^{1/2},$$

where n is the density. The ion skin depth is quite small in relation to the size of fusion and space plasmas; l_i is a few cm in fusion plasmas and a few km in the magnetosphere. Thus an important question challenging the relevance of ''natural'' plasma states is, can they have a scale much larger than the natural scale l_i. This question was answered in the affirmative in Ref. 11. Note that the reduced cases of a simple fluid and MHD are scaleless.

Natural equilibrium states may be regulated by the robust global integrals of a two-fluid. A multi-fluid has an ideal magnetofluid invariant for each charge species, the generalized helicities

$$K_\alpha = (c^2/8\pi q_\alpha^2) \int d\tau\, \mathbf{P}_\alpha \cdot \nabla \times \mathbf{P}_\alpha, \tag{1}$$

where the integral is over the system volume, with increment $d\tau$. The ''species'' helicities are composites, linking fluid and field behavior through the canonical momentum. The constant $c^2/8\pi q_\alpha^2$ gives the helicities the units of energy length. In addition, the global angular momentum is invariant in axisymmetric systems with suitable boundary conditions. For massless electrons,

$$L = \int d\tau\, m_i n r u_{i\theta}. \tag{2}$$

Relaxation theory postulates that weakly dissipative systems approach the state of maximum *entropy* of the ideal (nondissipative) system subject to its constraints.[13,14] This is the organized (or *coherent*) energy form as opposed to the disorganized or thermal energy form. For massless electrons and quasineutrality the organized energy form is the *magnetofluid* energy

$$W_{\mathrm{MF}} = \int d\tau (m_i n u_i^2/2 + B^2/8\pi), \tag{3}$$

where $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic field. Stationary values of W_{MF} with constraints on K_e, K_i, L lead to Euler equations for the species flow [Eq. (49) of Ref. 4; Eq. (32) of Ref. 6]:

$$\mathbf{u}_\alpha = \Omega r \hat{\boldsymbol{\theta}} + \lambda_\alpha (c^2/4\pi e^2 n) \nabla \times \mathbf{P}_\alpha. \tag{4}$$

Appearing here are the Lagrange multipliers associated with the electron and ion helicities (λ_e,λ_i) and the angular momentum (Ω). The same Euler equations for the stationary state arise if the organized energy form is minimized subject to the same constraints.[4] An important question concerns the ruggedness of the helicities. While these are not strictly constant in a dissipative plasma, they are *less* susceptible to dissipation than W_{MF} and thus should act as regulators of the equilibrium. Several arguments for the ruggedness of the helicities were advanced in Ref. 4.

The reduced cases of the foregoing are recovered as follows. The single helicity invariant in a simple fluid is the fluid helicity, found by setting the charge $e=0$ in K_i. Beltrami states are stationary-energy states of a simple fluid. The single helicity invariant in MHD is the magnetic helicity found by setting $m_e=0$ in K_e. Taylor states are stationary-energy states of MHD. The organized energy form in a simple fluid is the first term (flow energy) in Eq. (3), and in MHD it is the second term (magnetic energy).

An alternative and more intuitive approach to natural states of a two-fluid is the double-Beltrami formulation [Ref. 5; and Eqs. (27) and (28) of Ref. 11]. These states are the two-fluid magnetoplasma extension of Beltrami states in a simple fluid. In fact, double-Beltrami states are identical to the two-fluid stationary states [Eq. (18) of Ref. 3; Eq. (49) of Ref. 4]. The only difference is that the latter account for a global angular momentum invariant [Eq. (49) of Ref. 4]. The Beltrami coefficients [Eq. (12) of Ref. 11] are in one-to-one correspondence with the Lagrange multipliers associated with the constraints in the minimum energy procedure (λ_α in Refs. 3, 4, and 7).

III. EQUILIBRIUM ANALYSIS

A. Assumptions

Static MHD equilibria are governed by a single equation (Grad–Shafranov) with two surface functions. By contrast, the system for two-fluid flowing equilibria is bewilderingly complex. With massless electrons and quasineutrality, it is composed of a coupled system of three nonlinear equations: two second-order partial differential equations for the magnetic and flow stream functions, and a Bernoulli equation for the density [Eqs. (38)–(40) of Ref. 7]. Moreover, there are no less than *six* arbitrary surface functions. The goal here is to find meaningful 2D solutions to this complicated system. Here 2D means equilibria that depend on both r and z (cylindrical coordinates), and have limited dimensions in both r and z directions. To simplify the problem three assumptions are made: a stationary-energy state is assumed; the conservation of angular momentum is ignored; and uniform density is assumed.

(1) The assumption of a stationary-energy state greatly simplifies the equilibrium equations. In this case the six surface functions take very simple forms.[7] The only arbitrary quantities remaining are the three Lagrange parameters associated with the invariants. The stationary-energy condition is related to stability: the lowest energy state (subject to constraints that are stable to visco-resistive modes) satisfies a *sufficient* condition for stability to all two-fluid modes. (2) If the angular momentum constraint is ignored (by setting $\Omega=0$), only two parameters remain λ_e, λ_i. (3) Finally, suppose that uniform density is assumed (as in Refs. 5 and 11), which is equivalent to an ideal fluid with infinite adiabatic index.[7] This causes the differential equations to become linear, and decouples the Bernoulli equation. Although a realistic plasma has varying density, uniform density has been a common artifice in analyzing the equilibrium and stability of magnetically confined plasmas.

B. Equilibrium equations

The variational principle that extremized the magnetofluid energy [Eq. (3)] subject to constraints on the helicities [Eq. (1)] and angular momentum [Eq. (2)] leads to Euler equations for the flow of each species [Eq. (4)]. For massless electrons these reduce to a coupled pair of differential equations for the magnetic field and the ion flow velocity. In matrix operator form these are

$$\begin{bmatrix} \nabla\times-\lambda_e & -1 \\ \lambda_i & (l_i^2\lambda_i\nabla\times-1) \end{bmatrix}\begin{Bmatrix} c\mathbf{B}/4\pi en \\ \mathbf{u}_i \end{Bmatrix} = -\Omega r\hat{\boldsymbol{\theta}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \tag{5}$$

where $\nabla\times$ is the curl operator, and λ_e, λ_i, Ω are the Lagrange multipliers associated with the ion helicity, electron helicity, and angular momentum, and $\hat{\boldsymbol{\theta}}$ is the unit vector in the azimuthal direction. The first line of Eq. (5) arises from the Euler equation for electron flow using Ampere's law to eliminate $\mathbf{u}_e$ in favor of $\mathbf{B}$; the second line is the Euler equation for the ion flow. For uniform density and no angular momentum constraint, Eq. (5) is equivalent to Eq. (10) of Ref. 5.

In axisymmetric equilibria the poloidal (r,z) field and flow can be expressed using stream functions: $\mathbf{B}_p=(\hat{\boldsymbol{\theta}}/r)\times\nabla\psi$ and $\mathbf{u}_{ip}=(\hat{\boldsymbol{\theta}}/nr)\times\nabla\psi$, where ψ is the magnetic stream function, ψ_i is the flow stream function. Then Eq. (5), becomes

$$\overleftrightarrow{\mathbf{D}}(\Delta^*)\begin{Bmatrix} c\psi/4\pi e \\ \psi_i/\lambda_i \end{Bmatrix} = -\Omega n r^2\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{6}$$

using the matrix operator

$$\overleftrightarrow{\mathbf{D}}(\Delta^*)=\begin{bmatrix} \Delta^*+\lambda_e^2-1/l_i^2 & \lambda_i\lambda_e+1/l_i^2 \\ -(\lambda_i\lambda_e+1/l_i^2) & \lambda_i^2(l_i^2\Delta^*-1)+1/l_i^2 \end{bmatrix} \tag{7}$$

and $\Delta^*=r^2\nabla\cdot[(1/r^2)\nabla]$ is the familiar Grad–Shafranov operator. With constant density, the ion skin depth l_i is constant. Auxiliary equations give the toroidal field and flow components [Eq. (31) in Ref. 7]:

$$B_\theta=-\frac{1}{r}\left(\lambda_e\psi+\frac{4\pi e}{c}\psi_i\right), \tag{8}$$

$$u_{i\theta}=\frac{1}{r}\frac{e}{m_i c}\left(\psi-\frac{4\pi e}{c}\frac{\psi_i}{\lambda_i}\right). \tag{9}$$

The uniform density form of Bernoulli's equation gives the pressure

$$p+m_i n(u_i^2/2-r\Omega u_{i\theta})=\text{const.} \tag{10}$$

C. Classes of equilibria

The angular momentum constraint is ignored here $(\Omega=0)$ as discussed in Sec. III A so that the right-hand sides of Eqs. (5) and (6) are zero. Then Eq. (6) is a coupled pair of linear partial differential equations that satisfy the modified Helmholtz equation

$$(\Delta^*+\Lambda^2)\{\psi,\psi_i\}=0 \tag{11}$$

with the boundary condition $\psi=\psi_i=0$ at the separatrix. The eigenvalue Λ is *real* and is a *geometric* property determined by the boundary conditions.[15] It should be kept in mind that Λ cannot be freely chosen once the boundary geometry is fixed. With $\Delta^*\to-\Lambda^2$ the differential operator in Eq. (7) becomes algebraic, $\overleftrightarrow{\mathbf{D}}\to\overleftrightarrow{\mathbf{D}}(-\Lambda^2)$. A nontrivial solution requires $\|\overleftrightarrow{\mathbf{D}}(-\Lambda^2)\|=0$ (the characteristic equation) where $\|\cdots\|$ denotes the determinant. Since the square of Λ appears in $\overleftrightarrow{\mathbf{D}}(-\Lambda^2)$, either sign of Λ is allowed, corresponding to left- or right-handedness of the fields or flows, as will be seen. Note further that $1/|\Lambda|$ is the length scale; thus large-scale equilibria have $|\Lambda|l_i\ll1$. The characteristic equation is

$$[(1+\Lambda^2 l_i^2)^2-\lambda_e^2\Lambda^2 l_i^4]\lambda_i^2+2\lambda_e\lambda_i+\lambda_e^2-\Lambda^2=0 \tag{12}$$

which is a relation between the two parameters λ_e, λ_i. Recall that Λ is fixed by the boundary geometry.

Two classes of solutions arise. If the off-diagonal terms in $\overleftrightarrow{\mathbf{D}}$ [Eq. (7)] are *nonzero*, then the field and flow are *coupled*, and the flow and magnetic stream functions have a particular proportionality, $\psi_i=-(c/4\pi e)(\lambda_e+\Lambda)\psi$. If the off-diagonal terms in $\overleftrightarrow{\mathbf{D}}$ are *zero*, then the field and flow are somewhat decoupled, and the ratio ψ_i/ψ is arbitrary. This possibility was mentioned in Ref. 11. Both classes can be expressed using the following relationships. The two stream functions are related by

$$\psi_i=-\frac{c}{4\pi e}(\lambda_e+\alpha\Lambda)\psi, \tag{13}$$

where the α parameter reflects the arbitrariness of ψ_i/ψ in the non-force-free case only. The field and flow vectors are

$$\mathbf{B}=\alpha\Lambda\psi\frac{\hat{\boldsymbol{\theta}}}{r}+\frac{\hat{\boldsymbol{\theta}}}{r}\times\nabla\psi, \tag{14}$$

$$\mathbf{u}_i=-\frac{c}{4\pi en}\left[(\alpha\lambda_e+\Lambda)\Lambda\psi\frac{\hat{\boldsymbol{\theta}}}{r}+(\lambda_e+\alpha\Lambda)\frac{\hat{\boldsymbol{\theta}}}{r}\times\nabla\psi\right]. \tag{15}$$

Note that the nominal poloidal field (subscript p) is $B_p\sim\Lambda|\nabla\psi|$ and the nominal toroidal field is $\alpha\Lambda\psi$; thus α itself is the nominal ratio of the toroidal to poloidal fields. The parameters λ_e, λ_i, α for the two classes are as follows. In the *force-free* class the ratio ψ_i/ψ is not arbitrary so that $\alpha=1$, and Eq. (12) determines λ_i:

$$\lambda_i=-\frac{\lambda_e+\Lambda}{1+\Lambda(\lambda_e+\Lambda)l_i^2}. \tag{16}$$

In this class λ_e remains a free parameter. In the *non-force-free* class the vanishing of the off-diagonal terms in $\overleftrightarrow{\mathbf{D}}(\Lambda^2)$ combined with Eq. (12) determines *both* λ_e and λ_i:

$$\lambda_e^2=\Lambda^2+1/l_i^2, \tag{17a}$$

$$\lambda_i=-1/\lambda_e l_i^2. \tag{17b}$$

In this class α (the parameter reflecting the arbitrary ratio ψ_i/ψ) remains free. Note that in Eqs. (14) and (15), the toroidal field component obeys a left- (right-) hand rule with respect to the poloidal field for Λ positive (negative).

The key properties of the *force-free* class are the following. (a) The current density is aligned with the magnetic field, $\mathbf{j}=(c/4\pi)\Lambda\mathbf{B}$, as is easily verified using Ampere's law; hence the designation *force-free*. (b) Although the magnetic field is force free, the pressure [Eq. (10)] may be nonuniform because of inertial effects. (c) The ion flow is also aligned with the magnetic field. (d) The ion flow velocity for large plasmas ($|\Lambda|\to0$) is $\mathbf{u}_i=-\lambda_e l_i V_A\mathbf{B}/B$ where $V_A=B/(4\pi m_i n)^{1/2}$ is the local Alfven speed. Thus, depending on the size of the product $\lambda_e l_i$, the flow may be Alfvénic. It is *antiparallel* (*parallel*) to the field if the sign of λ_e is positive (negative).

The key properties of the *non-force-free* case are the following. (a) The current is *not* aligned with the magnetic field, thus the field is *diamagnetic*. The $\mathbf{j}\times\mathbf{B}/c$ force density is $\mathbf{j}\times\mathbf{B}/c=(1-\alpha^2)\Lambda^2\psi\nabla\psi/4\pi r^2$, which is nonzero in this class ($\alpha\neq1$). A *pure* FRC (zero toroidal field) has $\alpha=0$, in which case the $\mathbf{j}\times\mathbf{B}/c$ force is maximal. (b) *Near* FRCs (small but nonzero toroidal field) occur whenever α is small but not zero. (c) Only the poloidal parts of the flow and field vectors are aligned. (d) The poloidal flow speed for large plasmas ($|\Lambda|\to0$) is $u_{ip}=\pm V_{Ap}$, i.e., the poloidal flow is *Alfvénic*. (e) The toroidal flow, using $\Lambda\psi\sim|\nabla\psi|$ is $u_{i\theta}\sim\Lambda l_i V_{Ap}$. Thus for large plasmas ($\Lambda l_i\ll1$) the toroidal flow is much smaller than the poloidal. (f) In this class the fields and flows *partially* decouple, i.e., the strict alignment and proportionality of magnitudes found in the force-free case is broken. Even so, if the two separatrices (flow and field) coincide, as assumed here, then the $\mathbf{u}_i$ and $\mathbf{B}$ are roughly coupled, i.e., nearly aligned with magnitudes nearly in a fixed proportionality. It is unclear whether this relationship is retained if the two separatrices are not required to coincide.

D. Reduced cases

Before continuing it is useful as a point of reference to show how two important reduced cases are recovered. In a *simple fluid* $e\to0$ so that $e\psi/m_i c\to0$. The first line of Eq. (6) is discarded since it reflects Ampere's law. In the second line only the "lower-right" element of $\overleftrightarrow{\mathbf{D}}$ remains and in it the $-\lambda_i^2$ term is discarded since it is a manifestation of the toroidal magnetic field component. A *simple fluid* is a reduced case of a two-fluid in the limit $\lambda_e\to\infty$. Then the characteristic equation, Eq. (12), reduces to

$$\lambda_i^2=1/(\Lambda l_i^2)^2. \tag{18}$$

In the limit $\lambda_e\to\infty$ Eq. (13) shows that $\psi/\psi_i\to0$, implying the vanishing of the poloidal fields relative to the flows, i.e., a simple fluid. The vanishing of B_θ is less obvious because of the product $\lambda_e\psi$ in Eq. (8). It can be shown by eliminating ψ using Eq. (13), which (for $\lambda_e l_i\gg1$) leads to $B_\theta\propto\psi_i/\lambda_e\to0$. Note that although l_i appears in Eq. (18), it has actually disappeared as a length scale because of the grouping $\lambda_i l_i^2$. This is seen by recognition that the dimensional factor in the ion helicity $c^2/8\pi e^2$ is equal to $nl_i^2/2m_i$. Thus in the minimization principle where the product $\lambda_i K_i$ appears, the factor $\lambda_i c^2/8\pi e^2=(\lambda_i l_i^2)n/2m_i$ appears, i.e., only the combination $\lambda_i l_i^2$ matters. Thus a simple fluid is scaleless.

The second reduced case is an MHD *fluid*, which is recovered by discarding the ion helicity constraint by setting its Lagrange multiplier to zero, $\lambda_i=0$. Then the characteristic equation, Eq. (12), reduces to

$$\lambda_e^2=\Lambda^2. \tag{19}$$

Note the disappearance of the length scale l_i, i.e., the MHD case is also scaleless. Further the second line in Eq. (6) reduces to $\psi-(4\pi e/c)\psi_i/\lambda_i=\Omega(m_i c/e)r^2$. Since $\lambda_i\to0$ the poloidal flow vanishes, $\psi_i\to0$, although the quotient ψ_i/λ_i may be finite. Using this in the first line of Eq. (6) recovers the same equation for the magnetic stream function, Eq. (11). The toroidal field, flow, and pressure are $B_\theta=-\lambda_e\psi/r$, $u_{i\theta}=0$, $p=\text{const}$, which is the familiar force-free result.

IV. GLOBAL INTEGRALS

A. Functional form of magnetofluid energy

Given the foregoing solutions, the global integrals, Eqs. (1) and (3), can be found. The objective is to eliminate the free parameter (λ_e or α) in favor of the helicities in order to find the functional form $W_{\rm MF}=W_{\rm MF}(\Lambda,K_e,K_i)$; in this form the last two arguments are the global constraints and the first is the eigenvalue. The two helicities can be thought of as the coordinates in a two-dimensional map of helicity space; on this map the domains of the equilibrium types and their properties can be classified. The function $W_{\rm MF}(\cdots)$ will be different for the force-free and non-force-free classes. This functional format will lead to conclusions about stability and the tendency to select preferred states.

The helicities, Eq. (1), contain the canonical momentum, which requires the vector potential. A form of the vector potential that serves both classes of equilibria is

$$\mathbf{A}=-\frac{\hat{\boldsymbol{\theta}}}{r}\psi-\frac{\alpha}{\Lambda}\frac{\hat{\boldsymbol{\theta}}}{r}\times\nabla\psi.$$

This leads to magnetic field, Eq. (14). Note that volume integrals will appear in two forms, which in fact are identical. This is the reference energy

$$W_R\equiv\frac{1}{8\pi}\int\frac{|\nabla\psi|^2}{r^2}=\frac{\Lambda^2}{8\pi}\int\frac{\psi^2}{r^2}.$$

That the second integral is identical to the first follows from an integration by parts, accounting for $\psi=0$ on the boundary, and using Eq. (11). The reference energy is positive except in the uninteresting case of vanishing magnetic field. Using Eqs. (14) and (15) the two helicities and the magnetofluid energy are

$$K_e=-\alpha\frac{2W_R}{\Lambda}, \tag{20}$$

$$K_i=-[1+\Lambda(\alpha\lambda_e+\Lambda)l_i^2][\alpha+\Lambda(\lambda_e+\alpha\Lambda)l_i^2]\frac{2W_R}{\Lambda}, \tag{21}$$

$$W_{\mathrm{MF}}=\{(1+\alpha^2)+[(\alpha\lambda_e+\Lambda)^2+(\lambda_e+\alpha\Lambda)^2]l_i^2\}\mathbf{W}_R. \tag{22}$$

Observe the two groups of terms in W_{MF}: the $(1+\alpha^2)$ part represents the magnetic energy and the rest is the flow energy.

B. Force-free class of equilibria

As is customary in constrained minimization procedures, the remaining step is to eliminate the free parameters in favor of the constrained quantities. In the force-free case $(\alpha=1)$ the quantities to eliminate in Eqs. (20)–(22) are λ_e and W_R. Note from Eqs. (20) and (21) with $\alpha=1$, the two helicities must have the same sign. This restricts the "allowed" domain on the helicity map. Then the functional form of the organized energy $W_{\mathrm{MF}}(\Lambda,K_e,K_i)$ follows:

$$W_{\mathrm{MF}}=\left[|\Lambda|l_i+\frac{1}{|\Lambda|l_i}(\sqrt{K_i/K_e}-1)^2\right]\frac{|K_e|}{l_i}. \tag{23}$$

In fact, *two* signs of the square root $(K_i/K_e)^{1/2}$ are allowed here; the positive root is shown since it is energetically favorable (lower W_{MF}). The negative square root has a much larger value of W_{MF}. In Eq. (23) the first and second terms in the square brackets represent the magnetic and flow energies, respectively. At the critical value

$$\Lambda_C=(\sqrt{K_i/K_e}-1)/l_i \tag{24}$$

the two energies are equal, i.e., energy equipartition. For $|\Lambda|>\Lambda_C$ the magnetic energy is larger and for $|\Lambda|<\Lambda_C$ the flow energy is larger. Note that $\Lambda_C l_i$ is small only for K_i quite close to K_e. The flow velocity is

$$\mathbf{u}_i=-V_A\frac{\sqrt{K_i/K_e}-1}{\Lambda l_i}\frac{\mathbf{B}}{B},$$

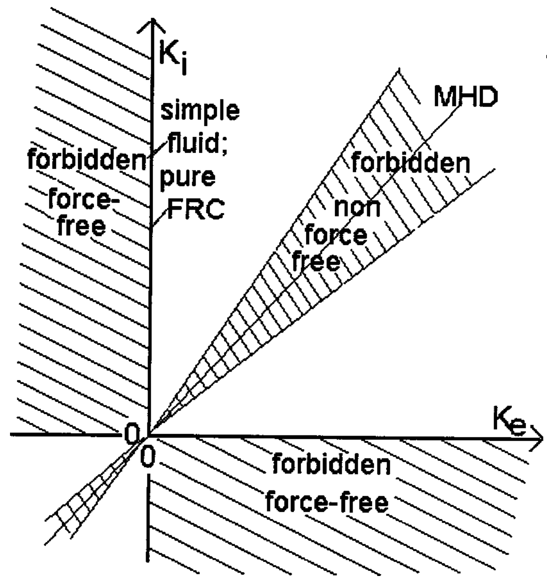

FIG. 1. Domains in helicity space.

where $V_A=B/\sqrt{4\pi m_i n}$ is the local Alfven speed.

Observe then the following properties of *force-free* equilibria. (1) *Helicities*. Both helicities must have the same sign. Thus the allowed combinations of K_e–K_i fill two quadrants (I and III) of the K_e–K_i plane. (2) *Flow*. The flow can be parallel or antiparallel to the field. The flow is quite large unless the two helicities are nearly equal. (3) *Energy partition*. The proportions of magnetic field and flow energies depends on $|\Lambda|$. Equipartition occurs at a critical value $\Lambda=\Lambda_C$ that depends on the ratio of helicities.

The two reduced cases discussed in Sec. III D are limits of the force-free class. The *simple fluid* limit is recovered for $\lambda_e\to\infty$. Then the characteristic equation reduces to Eq. (18). In this limit the ratio $K_e/K_i=0$ (no electron helicity). The functional form of the organized energy, using Eq. (23), is

$$W_{\mathrm{MF}}=|K_i|/|\Lambda|l_i^2. \tag{25}$$

The MHD limit is recovered for $\lambda_i\to0$. Then the characteristic equation reduces to Eq. (19). Here there is no flow and the two helicities are identical. Then the functional form of the organized energy, using Eq. (23), is

$$W_{\mathrm{MF}}=|\Lambda||K_e|. \tag{26}$$

The domain of the force-free class on the helicity map (K_i vs K_e) is portrayed in Fig. 1. Mainly the first and fourth quadrants are shown, since the second and third $(K_e<0)$ are identical, although inverted. Force-free equilibria are forbidden in the shaded regions (II and IV quadrants). Certain lines of constant K_i/K_e (rays from the origin) have special meaning. MHD fluids (Taylor states) lie on $K_i=K_e$ line, and simple fluids (Beltrami states) lie on the K_i axis.

C. Non-force-free class of equilibria

Again, the remaining step is to eliminate the free parameters in favor of the constrained quantities. In the non-force-free case the quantities to eliminate in Eqs. (20)–(22) are α and W_R. Of course λ_e is given by Eq. (17a). From Eqs. (20) and (21),

$$|\alpha| = f/2 \pm \sqrt{(f/2)^2 - 1},$$
$$f \equiv \frac{1}{|\Lambda| l_i (1+\Lambda^2 l_i^2)^{1/2}} \left| \frac{K_i/K_e}{1+\Lambda^2 l_i^2} - 1 - 2\Lambda^2 l_i^2 \right|. \tag{27}$$

Real solutions require $f \geqslant 2$. Regions of K_i–K_e space in which $f<2$ are *forbidden* to the non-force-free class. In large plasmas ($|\Lambda| l_i \ll 1$) the forbidden region on the helicity map is bounded by the lines $K_i \approx K_e(1 \pm 2|\Lambda| l_i)$; this is a narrow wedge-like shaded region near the $K_i = K_e$ line. Then the functional form of the organized energy $W_{\mathrm{MF}}(\Lambda, K_e, K_i)$ follows:

$$W_{\mathrm{MF}} = \frac{1}{l_i} \left| \frac{K_i}{\sqrt{1+\Lambda^2 l_i^2}} - K_e \sqrt{1+\Lambda^2 l_i^2} \right|. \tag{28}$$

Over most of the allowed domain of the non-force-free class there is an approximate equipartition between flow and field energies. In the non-force-free class *only* the poloidal parts of the flow and field vectors are aligned:

$$\mathbf{u}_{ip} = -V_{Ap}[\sqrt{1+\Lambda^2 l_i^2} + O(\Lambda^2 l_i^2)] \frac{\mathbf{B}_p}{B_p}, \tag{29}$$

where $V_{Ap} = B_p/(4\pi m_i n)^{1/2}$ is the Alfvén speed based on the *poloidal* field. For large plasmas the square root term is close to unity; thus the poloidal flow speed is roughly the Alfvén speed based on the local poloidal field. The square-root radical in Eq. (29), which is $\lambda_e l_i$ in view of Eq. (17a), may have either sign so that the poloidal flow is either parallel or antiparallel to the poloidal field.

Observe then the following properties of *non-force-free* equilibria. (1) *Helicities*. The helicities need not have the same sign. Moreover, the electron helicity may vanish without forcing the uninteresting case of no magnetic fields; indeed, pure FRCs (zero toroidal field) have $K_e = 0$. (2) *Flow*. Only the poloidal parts of the ion flow and magnetic field are aligned. (3) *Energy partition*. There is an approximate equipartition of energy between the magnetic and flow energies for most non-force-free equilibria.

The domain of the non-force-free class is also located on the helicity map (K_i vs K_e) in Fig. 1. Again the domains are bounded by rays from the origin, i.e., lines of constant K_i/K_e. Non-force-free equilibria are forbidden only in the narrow wedge (hashed region in Fig. 1). Evidently, the allowed domain for non-force-free equilibria is much broader than that for force-free equilibria. Note that pure FRCs lie on the K_i axis, the same line occupied by Beltrami states; of course, pure FRCs are in the non-force-free class while Beltrami states are in the force-free class.

D. Stability implications

In the preceding two sections the functional form of the organized energy form $W_{\mathrm{MF}}(\Lambda, K_e, K_i)$ has been found. Because this is the organized energy, its dependence on the eigenvalue Λ has important stability implications. Given the boundary geometry, Λ must be one of a discrete set of eigenvalues, each representing a unique equilibrium. The lowest-$|\Lambda|$ eigenvalue has the largest structure, comparable to the system size. In interesting fusion and space plasmas, the lowest-$|\Lambda|$ is much smaller than $1/l_i$. Equilibria with finer structure have larger $|\Lambda|$. Suppose the two helicities are stable to tearing modes, as was argued in Refs. 16 and 17. If W_{MF} is monotonically increasing for $|\Lambda| >$ the lowest eigenvalue, then the largest equilibrium structure has the lowest possible energy given the specified values of K_e, K_i. If so this equilibrium is absolutely stable to all modes of a two-fluid plasma, including visco-resistive modes. This is a sufficient condition for stability and is stronger than the criterion for ideal modes.

In the force-free class the W_{MF} vs Λ dependence [Eq. (23)] has a *minimum* at $|\Lambda| = \Lambda_C$, Eq. (24) and is monotonically increasing for $|\Lambda| > \Lambda_C$. If Λ_C is less than the lowest $|\Lambda|$, then that equilibrium is absolutely stable. Since $\Lambda_C l_i$ is small only for K_i quite close to K_e, it follows that large force-free equilibria are only stable near the $K_i = K_e$ line on the helicity map.

In a *simple fluid* (K_i axis in Fig. 1) the W_{MF} vs Λ test applied to Eq. (25) shows that the lowest energy arises for $|\Lambda| \to \infty$ (because $\Lambda_C \to \infty$); thus all Beltrami states *fail* the stability criterion. An MHD *fluid* lies exactly on the $K_i = K_e$ line (see Fig. 1); thus Taylor states are stable.

In the *non-force-free* class the W_{MF} vs Λ test is applied to Eq. (28). It is easily shown that W_{MF} is a monotonically *increasing* function of $|\Lambda|$ if the helicity ratio satisfies

$$-1 \leqslant K_i/K_e \leqslant 1.$$

These are two broad (quarter plane) sectors in the I–IV and II–III quadrants of the helicity map. In these sectors the lowest-$|\Lambda|$ equilibrium is *the* equilibrium of minimum energy and therefore absolutely stable. These stable equilibria are *near* FRCs, i.e., they have a small but nonzero toroidal field. *Pure* FRCs ($K_i/K_e \to \infty$) lie *outside* the stable region, and in fact have minimum energy for $|\Lambda| \to \infty$. Thus *pure* FRCs are not absolutely stable. Nothing can be said at this point about the ideal stability of pure FRCs (or other non-force-free equilibria outside the stable sectors), since ideal modes are a more restricted class of instability.

E. Preferred states and bifurcation

In the unshaded regions in Fig. 1 *either* equilibrium class is allowed. Relaxation processes are likely to select the class with lower energy. Using Eqs. (23) and (28), the expressions for W_{MF} in the two classes, it is easily shown that throughout its allowed region, the non-force-free class has lower W_{MF}. This represents a *bifurcation* between the two classes. The force-free equilibria is required in the region labeled "forbidden (non-force-free)" in Fig. 1, and the non-force-free equilibria is energetically favorable everywhere else. This may explain the bifurcation between FRC and spheromak-like equilibria observed in the Tokyo Spheromak (TS-3) experiments.[18] The former are high-β equilibria of the non-force-free class, and the latter are force-free equilibria of the force-free class.

The boundaries of the non-force-free forbidden region and the bifurcation lines in Fig. 1 are at their approximate location for the example $|\Lambda|l_i \approx 0.2$. As will be seen in the next section $|\Lambda| \approx 3.8/a$. The radial size parameter is, $S_* \equiv a/l_i \approx 3.8/|\Lambda|l_i \sim 20$ for this example, which is in the typical range of FRC experiments.[19] In equilibria relevant to fusion or space plasmas ($|\Lambda|l_i \ll 1$), the forbidden "wedge" for non-force-free equilibria becomes narrow, though still centered on the $K_i = K_e$ line. Thus for large plasmas, $|\Lambda|l_i \ll 1$, the allowed range of non-force-free equilibria spans nearly all the helicity map, and these are energetically preferred in nearly all cases.

V. ANALYTIC FLOWING EQUILIBRIA

Given the boundary geometry, Λ must be one of a set of discrete eigenvalues, each representing a separate equilibrium. It is useful, however, to treat Λ as a free parameter and allow the separatrix boundary to adjust accordingly. By this method analytic solutions for the stream function structure $\psi(r,z)$, $\psi_i(r,z)$ are easily found. Both stream functions are governed by Eq. (11). Its solutions therefore apply to both force-free and non-force-free equilibria. These classes have the same spatial structure, although they differ in the ratio ψ_i/ψ.

A separable solution to Eq. (11) is $\psi(r,z) = rJ_1(\Lambda_\perp r)\cos(\Lambda_\parallel z)$, where

$$\Lambda_\perp^2 + \Lambda_\parallel^2 = \Lambda^2. \tag{30}$$

There are an infinite number of these solutions since any combination of $\Lambda_\perp$, $\Lambda_\parallel$ satisfying Eq. (30) is a solution. Indeed, since Eq. (11) is linear, these solutions can be added in an arbitrary way. Thus the superposed form

$$\psi(r,z) = r\sum_k A_k J_1(\Lambda_{\perp k} r)\cos(\Lambda_{\parallel k} z)$$

is also a solution so long as each $\Lambda_{\perp k}$, $\Lambda_{\parallel k}$ combination satisfies Eq. (30). Suppose now that the separatrix crosses the midplane ($z=0$) at $r=a$, $\psi(a,0)=0$, and suppose the x points are at $z=\pm b$, $\psi/r^2(r=0,z=\pm b)=0$, then the coefficients A_k satisfy the conditions

$$\sum_k A_k J_1(\Lambda_{\perp k} a) = 0, \tag{31a}$$

$$\sum_k \Lambda_{\perp k} A_k \cos(\Lambda_\parallel b) = 0. \tag{31b}$$

Consider the simple case with only two such elements, one of which has $\Lambda_\parallel = 0$,

$$\psi = B_0 r \frac{\Lambda J_1(\Lambda_\perp r)\cos(\Lambda_\parallel z) - \Lambda_\perp J_1(\Lambda r)\cos(\Lambda_\parallel b)}{\Lambda\Lambda_\perp[1-\cos(\Lambda_\parallel b)]},$$

where $B_0 \equiv |B|(r=0,z=0)$. Here Eq. (31a) is already satisfied and Eq. (31b) is satisfied if $\Lambda_\perp$, $\Lambda_\parallel$ are related by

$$\Lambda J_1(\Lambda_\perp a) - \Lambda_\perp J_1(\Lambda \alpha)\cos(\Lambda_\parallel b) = 0.$$

These equilibria have a closed separatrix for $\pi/2 \leq \Lambda_\parallel b \leq \pi$. The parameter $\Lambda_\perp \approx 3.8/a$ where the proportionality constant is near first zero of the J_I Bessel function.

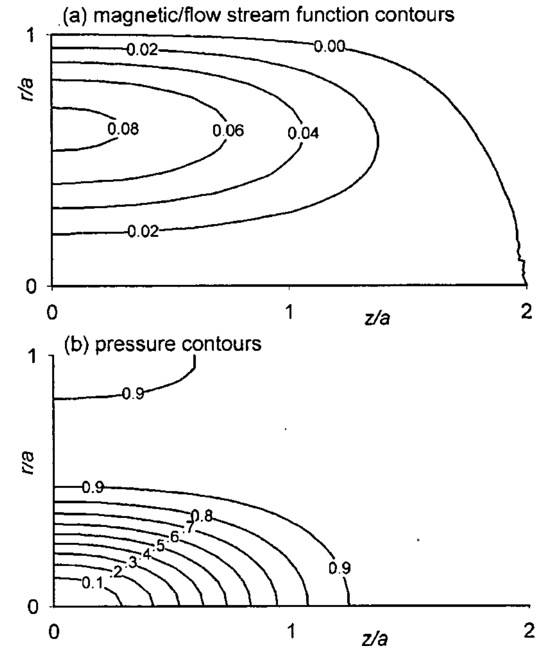

FIG. 2. Contours of (a) ψ or ψ_t and (b) pressure for a racetrack-like equilibrium. This example has $E=2$, $\Lambda_\perp a = 3.703$, $\Lambda_\parallel b = 2.913$.

Internal structures of a racetrack-like separatrix example are shown in Figs. 2 and 3. Figure 2(a) shows the stream function ψ or ψ_i; it applies to either because one is a multiple of the other. This holds for both force-free and non-force-free equilibria. Visually this example resembles static MHD equilibria found elsewhere. Figure 2(b) shows the pressure contours for the same case. Note that the ψ and pressure contours are somewhat different. By contrast, in static MHD equilibria the pressure and ψ contours coincide.

The *separatrix shapes* for two example equilibria cases are shown in Fig. 3. Case 1 ($\Lambda_\parallel b = 2.913$) is the same as Fig. 2. Case 2 has the maximum value of $\Lambda_\parallel b = \pi$, for which there is an x-point on the axis. Also shown are curves for a true ellipse and a true racetrack (constant radius sides joined to a spherical endcap). Case 1 is close to a true racetrack. Case 2, however, does not resemble an ellipse even though its separatrix is slightly pointed at the end, i.e., the separatrix radius (r vs z) approaches the x-point at an angle less than

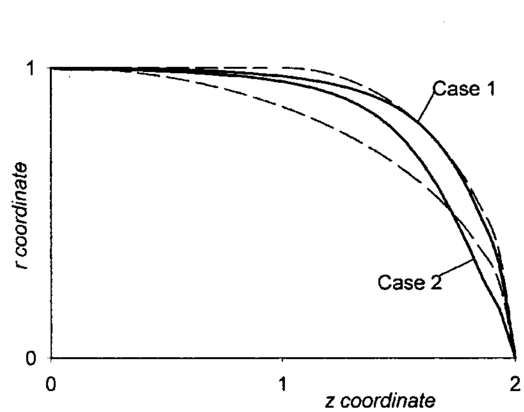

FIG. 3. Separatrix shape for the same example as in Fig. 2.

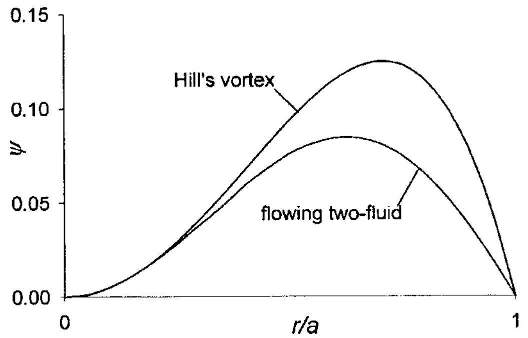

FIG. 4. Radial profile of ψ at midplane; two-fluid and Hill's vortex with $E=2$ and same B_0.

90°. A more elliptical separatrix could be achieved if more than two terms were used in the solution. For lower values of $\Lambda_{\|}b$ the separatrix becomes more rectangular, and a rectangle is reached at $\Lambda_{\|}b=\pi/2$.

The profiles of important quantities at the midplane are shown in Figs. 4 and 5. The ψ profiles for a two-fluid equilibrium and the Hill's vortex with the same elongation and magnetic field at the origin are compared in Fig. 4. The maximum ψ (poloidal flux) is lower by about 30% in the two-fluid equilibrium. A more striking difference appears in the current profiles, Fig. 5. Static MHD equilibria (Grad–Shafranov) have $j_\theta/r=f(\psi)$. In particular, a Hill's vortex has a *flat* profile $j_\theta/r=$const. The two-fluid equilibrium is quite different; indeed j_θ/r peaks on the geometric axis and falls monotonically toward the separatrix. A result is that the radius of the maximum ψ (Fig. 4) shifts slightly inward from $a/\sqrt{2}$.

VI. DISCUSSION

Compact equilibria of a flowing two-fluid have been investigated analytically. This was facilitated by limiting attention to stationary-energy states, and assuming uniform density. Stationary-energy states are of considerable interest because they have a chance of being stable. Two distinct classes of equilibria were found: force-free and non-force-free. The *force-free* class is force-free and may have significant flow. Beltrami states (simple fluid) and Taylor states (MHD) are limiting cases of force-free equilibria. The non-force-free class is diamagnetic and has Alfvénic flows. The range of equilibria in each class lie in particular *allowed* regimes in the helicity map, K_i vs K_e. Over most of the allowed range of the non-force-free class, its equilibria are near FRCs, i.e., the toroidal field is small compared with the poloidal. In many cases there is a near equipartition of energy between magnetic fields and flows. The force-free and non-force-free classes overlap in a certain region of helicity space, and in most of this the non-force-free class has lower energy, i.e., a relaxation process would favor it over the other.

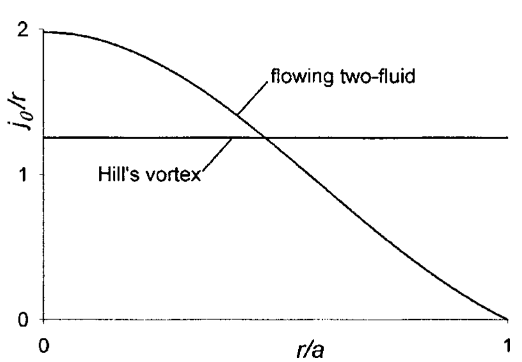

FIG. 5. Current profiles j_θ/r at midplane for a two-fluid and Hill's vortex.

In force-free equilibrium, the lowest-eigenvalue (largest) state has the lowest organized energy for certain conditions; in these instances the equilibrium is stable to all two-fluid modes. In non-force-free equilibria the lowest-eigenvalue (largest) state has the minimum energy in two large sectors of the helicity map and thus are stable to two-fluid modes therein. The allowed domains of the two classes of equilibria overlap in part of the helicity map. In much of this overlap the non-force-free equilibrium is energetically favorable (lower organized energy). This may explain the bifurcation between spheromaks (force-free) and FRCs (non-force-free) observed in experiments.

Several matters call for future investigation. First of all, experimentally, there is a great need to measure the flows. Evidently the flows play a important role in stationary-energy equilibria, contributing to the stability of configurations that are quite unstable in a static state. Theoretically several tasks are at hand. (1) Although the analysis here focussed on compact configurations, it has broader applicability and might be applied to configurations with a toroidal boundary as well. In such cases an additional constraint can be applied, the toroidal magnetic flux. (2) For simplicity sake, the analysis here ignored angular momentum considerations. If the boundary is axisymmetric, this can be included as an additional constraint. Each of these additional constraints will influence the allowed and forbidden domains and the stability. (3) While the uniform density assumption is a widely used artifice in plasma physics, realistical equilibria have varying density, and in the case of isolated equilibria, small density outside the separatrix. Future work should allow nonuniform density. (4) The analysis here regarded the separatrix as the system boundary. This completely neglects the effect of an edge layer and the external magnetic field region. Future work should include the region outside the separatrix. (5) Poloidal flows may experience strong *neoclassical* damping.[20] This needs to be investigated in the context of ''stationary energy'' equilibria, and particularly for FRCs which do not have neoclassical effects in the usual sense.

ACKNOWLEDGMENTS

This work was supported by U.S. Department of Energy Grant No. DE-FG03-98ER54480 (L.C.S.) and by a grant-in-aid from the Japanese Ministry of Education, and by a Collaborating Research Program of the National Institute for Fusion Science, Japan (A.I. and H.Y.). The authors want to thank S. Mahajan for useful discussions and the suggestion that these equilibria with strong flows present a ‘‘new kind of plasma state;’’ B. Dasgupta for useful discussions; M. Schaffer for calling our attention to the neoclassical damping effect on the poloidal flows.

[1]J. B. Taylor, Rev. Mod. Phys. **58**, 741 (1986).
[2]D. Pfirsch and R. N. Sudan, Phys. Plasmas **3**, 29 (1996).
[3]L. C. Steinhauer and A. Ishida, Phys. Rev. Lett. **79**, 3423 (1997).
[4]L. C. Steinhauer and A. Ishida, Phys. Plasmas **5**, 2609 (1998).
[5]S. M. Mahajan and Z. Yoshida, Phys. Rev. Lett. **81**, 4863 (1998).
[6]‘‘Minimum energy state’’ has been widely used where the correct term is *stationary* energy state. A stationary state may or may not have the minimum energy subject to the constraints. Except in degenerate cases, only one of the stationary states can have minimum energy.
[7]L. C. Steinhauer, Phys. Plasmas **6**, 2734 (1999). Note a correction: the *n* inside the square brackets in Eq. (31) should be deleted.
[8]L. S. Solov'ev, Rev. Plasma Phys. **5**, 239 (1976).
[9]E. K. Maschke and H. Perrin, Phys. Plasmas **22**, 579 (1980).
[10]R. A. Clemente and R. Farengo, Phys. Fluids **27**, 776 (1984).
[11]Z. Yoshida and S. M. Mahajan, J. Math. Phys. **40**, 5080 (1999).
[12]A. Ishida, H. Momota, and L. C. Steinhauer, Phys. Fluids **31**, 3024 (1988).
[13]D. Montgomery, L. Turner, and G. Vahala, Phys. Fluids **21**, 757 (1978).
[14]E. Hameiri and J. H. Hammer, Phys. Fluids **25**, 1855 (1982).
[15]Strictly speaking an eigenvalue problem is not rigorously set up here, although most likely it could be. Thus the term ‘‘eigenvalue’’ is used loosely.
[16]W. Park, D. A. Monticello, and R. B. White, Phys. Fluids **27**, 137 (1984).
[17]See discussion in Sec. III D of Ref. 4.
[18]Y. Ono, A. Morita, T. Itagaki, and M. Katsurai, *Plasma Physics and Controlled Nuclear Fusion Research* (International Atomic Energy Agency, Vienna, 1992), Vol. 2, p. 619; Y. Ono, Fusion Technol. **27**, 369 (1995).
[19]N. Iwasawa, A. Ishida, and L. C. Steinhauer, Phys. Plasmas **7**, 931 (2000).
[20]F. L. Hinton and S. K. Wong, Phys. Fluids **28**, 3082 (1985).