Formalism for multi-fluid equilibria with flow

Loren C. Steinhauer[a)]
University of Washington, Redmond Plasma Physics Laboratory, Redmond, Washington 98052

(Received 9 February 1999; accepted 14 April 1999)

A formalism is developed for flowing multifluid equilibria. In the standard reduced case (massless electrons, quasineutrality) this system simplifies to a pair of second-order partial differential equations for the magnetic and ion flow stream functions plus a Bernoulli equation giving the density. Each species has its own characteristic surfaces, which are the drift surfaces, and three arbitrary surface functions associated with each species. In the case of minimum energy equilibria, the surface functions are no longer arbitrary. The flowing equilibrium system is a generalization of the familiar Grad–Shafranov system for magnetostatic equilibria. [S1070-664X(99)04007-0]

I. INTRODUCTION

A. Motivation

In the many experimental studies of magnetically confined plasmas over several decades, one of the less appreciated phenomena has been flow, that is, until recent years. Now it appears that significant flows are a common feature of magnetic plasmas, and not only in instances where it is driven externally, e.g., by neutral beams. Further, various favorable effects have been attributed to flows, including the stabilization of ballooning modes by sheared flow, and reduced transport rate (H-mode or high confinement mode). The latter has been associated with sheared radial electric field, which is the concomitant of sheared flow. Theoretical studies of equilibria of flowing plasmas date to the 1950s, but most have been in the last two decades (see citations in Ref. 1).

In view of the growing interest in flowing equilibria, it is appropriate to develop a suitable formalism for describing them. This will facilitate understanding of the principal features of flowing equilibria, e.g., characteristic surfaces and surface functions; it will foster intuition about such states, and it will produce a well-defined set of equations for numerical computation of two-dimension states. Once those states are found, then the purported stability and transport advantages of flows can be investigated in detail. In short, it is needed to generalize the well-known system governing axisymmetric magnetostatic equilibria, namely the Grad–Shafranov (GS) equation. In brief, the GS system governs axisymmetric equilibria of an ideal magnetohydrodynamic (MHD) fluid with no flow. It is composed of a single second-order partial differential equation for the magnetic stream function, ψ. The characteristic surfaces are the magnetic surfaces with $\psi=\text{const}$. There are two arbitrary surface functions (of ψ), one related to the toroidal field, and the other for the pressure.

Here we develop a formalism for flowing equilibria based on the two-fluid model. In addition to being more general than MHD, which is a single-fluid treatment, the two-fluid model has other notable attributes. A two fluid has a broader range of stability than MHD, which can be analyzed by a symmetric treatment of electrons and ions.[2] Further, the two-fluid model led recently to a generalization of the familiar MHD relaxation theory and the prediction of finite-β minimum energy states.[3,4]

B. Basic properties of a two fluid

It is useful then to review the important properties of a two fluid[2–4] from which clues can be drawn for developing two-fluid equilibria. The elementary building blocks for a multifluid are the canonical momentum $\mathbf{P}_\alpha=m_\alpha\mathbf{u}_\alpha+q_\alpha\mathbf{A}/c$, and the generalized vorticity $\mathbf{\Omega}_\alpha=\nabla\times\mathbf{P}_\alpha$ (or α-vorticity) where m_α, $\mathbf{u}_\alpha$, q_α are the species mass, flow velocity and charge, and $\alpha=i,e$ denotes the species, and $\mathbf{A}$ is the vector potential. The quadratic invariants of a multifluid are the self helicities, or ''α-helicities,'' the ''density'' of which is $\mathbf{P}_\alpha\cdot\mathbf{\Omega}_\alpha$. The α-helicities are generalizations of helicities that appear in a simple fluid and in MHD. For vanishing electron mass the electron helicity reduces to the familiar magnetic helicity, which is an invariant in ideal MHD. The evolution of the α-helicities is governed by the helicity transport equations, derived from Maxwell's equations and the equations of motion. This requires no reducing assumptions, so that finite electron mass, non-neutrality, and distinct species flows are allowed.

The form of the helicity transport equations is, $n_\alpha D_\alpha(\mathbf{P}_\alpha\cdot\mathbf{\Omega}_\alpha/n_\alpha)/Dt=\nabla\cdot[(\cdots)\mathbf{\Omega}_\alpha]+\mathit{friction}$, where n_α is the species density. The generalized vorticity in the divergence term implies the existence of a ''local'' α-helicity associated with these lines, $K_\alpha=(c^2/8\pi q_\alpha^2)\int_C d\boldsymbol{\tau}\mathbf{P}_\alpha\cdot\mathbf{\Omega}_\alpha$, where C is the volume occupied by a bundle of α-vortex lines. The constant factor gives K_α the convenient units of energy length. The total derivative D_α/Dt implies that the local α-helicity convects with its own species. If an α-vortex line does not intersect the system boundary, then in the strictly ideal (frictionless) case, the associated α-helicity is constant. There is an associated circulation theorem, $\Gamma_\alpha=\int_C\mathbf{P}_\alpha\cdot d\mathbf{x}$ where C is the generalized vortex line, and $d\mathbf{x}$ is

[a)]Electronic mail: steinhauer@aa.washington.edu

a differential length vector along that line. Note that each species has its own set of generalized vortex lines, its own local α-helicities, and its own circulation theorem.

In the realistic case with frictional dissipation, viscoresistive instabilities drive reconnections that break individual α-vortex lines and destroy their identity. This is an instance of nonuniform convergence, where even a minute friction is sufficient to rule out the ideal state of preserved local α-helicities. The only quantities immune to these topology altering events are the global α-helicities, $K_\alpha = (c^2/8\pi q_\alpha^2)\int_V d\tau \mathbf{P}_\alpha \cdot \mathbf{\Omega}_\alpha$, where V is the system volume. Even global invariants may not be *rugged* in the sense that they are *more* ''invariant'' than the organized energy form, which is the magnetofluid energy $W_{mf} = \int_V d\tau(\Sigma m_\alpha n_\alpha u_\alpha^2 + B^2/8\pi)$, composed of the flow energy and the magnetic energy (the sum is over species). The ruggedness of the global α-helicities has been supported by three arguments. (1) *Selective decay:* W_{mf} decays more rapidly than K_α in thin reconnection layers. Properly applied, this argument must account for limits on viscous friction coefficients for sharp gradients. (2) *Inverse cascade:* The fluctuation spectrum of $\tilde{W}_{mf}(k)$ and $\tilde{K}_\alpha(k)$ satisfy the necessary conditions for a cascade toward larger scale objects (k is the wave number of the disturbance). (3) *Stability to resistive modes:* K_α is less affected than W_{mf} by resistive modes. Each of these is the generalization of arguments previously applied to verify the ruggedness of the magnetic helicity in weakly dissipative MHD.

A minimum energy state is found formally by minimizing W_{mf} subject to invariant α-helicities, and (given axisymmetric system boundary) the global angular momentum, $L_\theta = \int d\tau r \Sigma m_\alpha n_\alpha u_{\alpha\theta}$. The variation with respect to $\delta\mathbf{u}_\alpha$ leads to the flow equations: $n_\alpha(\mathbf{u}_\alpha - \Omega r\hat{\theta}) = (\lambda_\alpha/m_i l_c^2)\mathbf{\Omega}_\alpha$ where $\lambda_\alpha, \mathbf{\Omega}$ are the Lagrange multipliers associated with invariant α-helicities and angular momentum, and $l_c = c/\omega_{pi} = (m_i c^2/4\pi n e^2)^{1/2}$ is the length scale. An entropy maximization procedure subject to invariant K_α, L_θ, and total energy (W_{mf}+thermal) leads to the same equation. In addition, a global Bernoulli equation links the pressure to the flow by a relation that applies throughout the system volume. Note that an important feature of a two-fluid minimum energy state is the length scale l_c. A two fluid may or may not relax to the minimum energy state depending on whether the fast mechanisms have been stabilized.

The outline of the remainder of this paper is as follows. Section II develops the equations for multifluid equilibria. First (Sec. II A) the various continuity equations describing the fields and flows are replaced in favor of scalar variables (stream functions, etc.). The remaining equations for the fields (electromagnetic, gravitational) are expressed in terms of the scalar variables (Sec. II B). The equations of motion are simplified (Sec. II C) by two actions: introducing the classical thermodynamic enthalpy; and expressing the Lorentz force in terms of the generalized vorticity. Finally (Sec. II D) the three principal components of the equation of motion are examined, leading to the identification of the characteristic surfaces (and surface variables) for each species. This also leads to the identification of three arbitrary surface functions for each species. The result of Sec. II is a closed system of equations describing axisymmetric, multifluid equilibria. Section III presents three reduced cases. The first (Sec. III A) is the standard reduced case of a two fluid with massless electrons and quasineutrality. This leads to a system of two second-order equations for the magnetic and ion stream functions plus an auxiliary ''Bernoulli'' equation for the density. The second reduced case (Sec. III B) is the minimum energy state of a two fluid; its main feature is that the arbitrary surface functions take on specific forms. The third reduced case (Sec. III C) is single-fluid magnetostatics; this familiar case is considered to verify that the general system reduces to the GS equation in the proper limit. The paper concludes with a discussion of the properties of two-fluid equilibria (Sec. IV) and a summary (Sec. V).

II. ANALYSIS OF MULTIFLUID FLOWING EQUILIBRIA

A. Continuity equations: Scalar functions

The steady electromagnetic fields satisfy Maxwell's equations, two of which are Gauss' law of magnetism and the steady Faraday's law, $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} = 0$, where $\mathbf{B}$ and $\mathbf{E}$ are the magnetic and electric fields. These, which act as continuity equations for the fields, imply the existence of certain scalar functions. In an axisymmetric system they can be expressed as

$$\mathbf{E} = -\nabla V_E, \tag{1}$$

$$\mathbf{B} = \hat{\theta}\phi/r + (\hat{\theta} \times \nabla\psi)/r, \tag{2}$$

where V_E is the electrostatic potential, ψ is the magnetic stream function, and ϕ is related to the azimuthal magnetic field; all are functions of r,z only (cylindrical coordinates). The continuity equations of the fluid species are $\nabla \cdot (n_\alpha \mathbf{u}_\alpha) = 0$, where n_α is density, $\mathbf{u}_\alpha$ is the fluid velocity, and the subscript α denotes the species. These also imply the existence of scalar functions such that

$$n_\alpha \mathbf{u}_\alpha = \hat{\theta}\phi_\alpha/r + (\hat{\theta} \times \nabla\psi_\alpha)/r, \tag{3}$$

where ψ_α is the stream function, and ϕ_α is related to the azimuthal flow.

B. Field equations

The remaining Maxwell's equations are Gauss' law and the steady Ampere's law. In view of Eq. (1), $\nabla \cdot \mathbf{E} = 4\pi \Sigma q_\alpha n_\alpha$ (sum over species), becomes the Poisson equation

$$\nabla^2 V_E = -4\pi \Sigma q_\alpha n_\alpha. \tag{4}$$

With Eq. (2), the latter, $\nabla \times \mathbf{B} = (4\pi/c)\Sigma q_\alpha n_\alpha \mathbf{u}_\alpha$, becomes two scalar equations

$$\Delta^* \psi = \frac{4\pi}{c} \sum_\alpha q_\alpha \phi_\alpha, \tag{5}$$

$$\phi = -\frac{4\pi}{c} \sum_\alpha q_\alpha \psi_\alpha, \tag{6}$$

representing the toroidal and poloidal components, respectively. Here $\Delta^* = r^2\nabla\cdot[(1/r^2)\nabla]$ is the familiar GS operator, and $\Delta^*\psi=(4\pi/c)rj_\theta$, where j_θ is the azimuthal current density. The gravitational potential is governed by the gravitational field equations

$$\nabla^2 V_G = 4\pi G \Sigma m_\alpha n_\alpha, \tag{7}$$

where $-m_\alpha \nabla V_G$ is the gravitational force density, and G is the universal gravitational constant. In each of the field equations, Eqs. (4)–(7), the right sides are the influence of species properties on the fields.

C. Equations of motion: Equation of state

The steady equation of motion for a fluid species is

$$m_\alpha \mathbf{u}_\alpha\cdot\nabla\mathbf{u}_\alpha + \frac{\nabla p_\alpha}{n_\alpha} = -m_\alpha \nabla V_G - q_\alpha \nabla V_E + \frac{q_\alpha}{c}\mathbf{u}_\alpha\times\mathbf{B}. \tag{8}$$

The terms on the left side are fluid effects, while the terms on the right side are the gravitational force (first term) and the electromagnetic force (second and third terms), i.e., the Lorentz force. Closure of the fluid species equations requires a further relationship, an equation of state that relates thermodynamic quantities such as the pressure p, the density n, the temperature T, and perhaps other quantities.

Previous treatments of flowing equilibria have adopted various equations of state: *incompressible*, $n=\text{const}$ (e.g., Refs. 1, 5, and 6); *isothermal magnetic surfaces*[6,7] $T=T(\psi)$, where ψ is the magnetic stream function; *barotropic*[8] $p=p(n)$; and isentropic magnetic surfaces[9] $p/n^\gamma = f(\psi)$, where γ is the adiabatic index. The most flexible equation of state proposed to date[10] is $p=p(n;\psi)$, which might be called the *magnetic surface barotropic* equation of state. Here, in the context of a multifluid model, we propose a further generalization in which each species is subject to a barotropic relationship associated with the characteristic surfaces for each species, $\Psi_\alpha=\text{const}$, here called *α surfaces*. (The surface variable Ψ_α remains to be defined). Then the thermal equation of state, here called *α-surface barotropic*, is $p_\alpha = p_\alpha(n_\alpha,\Psi_\alpha)$.

The *α-surface barotropic* equation of state arises from classic thermodynamic arguments. Suppose there exists, for each species, a specific enthalpy (enthalpy per particle) with the canonical form

$$h_\alpha = h_\alpha(p_\alpha, S_\alpha), \tag{9}$$

with conjugate variables p_α and S_α. Here S_α plays the role of an entropy and is constant on a drift surface of the species in the ideal fluid case. The partial derivatives are

$$1/n_\alpha = \partial h_\alpha/\partial p_\alpha, \tag{10}$$

$$\Theta_\alpha = \partial h_\alpha/\partial S_\alpha, \tag{11}$$

where Θ_α is a thermal variable that plays the role of a temperature. From Eqs. (9) and (10) it follows that $1/n_\alpha = \partial h_\alpha/\partial p_\alpha = J(p_\alpha,S_\alpha)$, where J is some function, or $p_\alpha = K(n_\alpha)L(S_\alpha)$, where K,L are functions. The latter is simply the thermal equation of state for a *surface-barotropic* fluid, i.e., one in which the pressure is a function of density on the surfaces $S_\alpha=\text{const}$. This is a generalization of the equation of state suggested by Hameiri [1983] in that it is species specific and the surfaces $S_\alpha=\text{const}$ may not be magnetic surfaces.

An important example of an α-surface-barotropic fluid is an ideal fluid, in which case the canonical form of the enthalpy is

$$h_\alpha = \frac{\gamma}{\gamma-1} p_\alpha^{(\gamma-1)/\gamma}(C_{st}S_\alpha)^{1/\gamma}, \tag{12}$$

where γ is the adiabatic constant and C_{st} is a dimensional constant. From this equation of state the pressure and thermal variables follow:

$$p_\alpha = C_{st} n_\alpha^\gamma S_\alpha, \tag{13}$$

$$\Theta_\alpha = \frac{C_{st}}{\gamma-1} n_\alpha^{\gamma-1}. \tag{14}$$

It will be convenient also to use the expression for the enthalpy in terms of density and the entropy variable

$$h_\alpha(n_\alpha,S_\alpha) = \frac{\gamma C_{st}}{\gamma-1} n_\alpha^{\gamma-1} S_\alpha. \tag{15}$$

The thermodynamic entropy is $s_\alpha = [k/(\gamma-1)]\ln S_\alpha + \text{const}$, where k is the Boltzmann constant. Familiar limits of an ideal gas include the following. (1) *Isothermal* $(\gamma\to 1)$, $h_\alpha = S_\alpha \ln(p_\alpha/S_\alpha)$, $p_\alpha/n_\alpha = S_\alpha$, $\Theta_\alpha = \ln n_\alpha - 1$. Here the conventional temperature $kT_\alpha = p_\alpha/n_\alpha = S_\alpha$ is uniform on the surfaces, $S_\alpha=\text{const}$. (2) *Incompressible* $(\gamma\to\infty)$, $h_\alpha = p_\alpha$, $n_\alpha=\text{const}$, $\Theta_\alpha=0$. In these two limits, only the product $(\gamma-1)s_\alpha$ is defined. In a hot fusion plasma, the parallel thermal conductivity is large, leading to nearly isothermal surfaces, i.e., $\gamma\approx 1$. In astrophysical plasmas, the size scale is so large that parallel thermal conduction does not dominate. Then $\gamma>1$ is determined by other factors such as radiative transfer.

D. Surface functions

Important simplifications of the equations of motion [Eq. (8)] follow from considering their components in certain directions. This procedure was fruitful in the case of magnetostatic equilibria. There the principal directions are the azimuthal $\hat\theta$, "parallel" $\mathbf{B}$, and "perpendicular" $\nabla\psi \sim \hat{\boldsymbol\theta}\times\mathbf{B}$. Note that second and third directions are not orthogonal in general. These choices led to the surface functions of magnetostatic equilibria, $p=p(\psi)$, and $\phi=\phi(\psi)$. In multispecies flowing equilibria, the principle directions are the azimuthal $\hat\theta$, "parallel" Ω_α (generalized vorticity), and "perpendicular" $\hat\theta\times\Omega_\alpha$. The significance of Ω_α in the two-fluid model has been noted elsewhere where it was found that Ω_α defined "tubes" of local self helicity.[3,4] Herein lies a notable difference between magnetostatics and a flowing multifluid: In the latter case each species has its own set of principal directions.

Since Ω_α is the curl of a vector, it has its own continuity relation, $\nabla\cdot\Omega_\alpha=0$, and therefore, can always be expressed in forms analogous to Eqs. (2) and (3). Using these

$$\mathbf{\Omega}_\alpha = \frac{q_\alpha}{c}\left[\frac{\hat{\boldsymbol{\theta}}}{r}\left(\phi + \frac{m_\alpha c}{q_\alpha}\Delta_n^* \psi_\alpha\right) + \frac{\hat{\boldsymbol{\theta}}}{r}\times \nabla \Psi_\alpha\right], \tag{16}$$

where the α-surface variable (units of magnetic flux) is

$$\Psi_\alpha = \psi - \frac{m_\alpha c}{q_\alpha n_\alpha}\phi_\alpha . \tag{17}$$

Here the density-weighted Grad–Shafranov operator is

$$\Delta_n^* = r^2 \nabla \cdot [(1/nr^2)\nabla].$$

Note that $\Delta_n^* \psi_\alpha = r\omega_{\alpha\theta}$, i.e., the azimuthal component of the kinetic vorticity ($\nabla \times \mathbf{u}_\alpha$).

The α-surface function Ψ_α is physically significant because it is identical to $rP_{\alpha\theta}$, the canonical angular momentum. In an axisymmetric geometry, the particle motion preserves $rP_{\alpha\theta}$ so that $\Psi_\alpha = \text{const}$ are the drift surfaces. Therefore, the entropy-like variable introduced earlier is a function (arbitrary) of the drift surface function

$$S_\alpha = S_\alpha(\Psi_\alpha). \tag{18}$$

Another important property concerns the *parallel* thermal conduction (coefficient $K_{\parallel\alpha}$), which is the conduction *along* the direction of free particle motion. In hot plasmas $K_{\parallel\alpha}$ is large so that the surfaces $\Psi_\alpha = \text{const}$ are isothermal (corresponding to $\gamma \to 1$).

Return now to the equation of motion, Eq. (8). From the equations of state Eqs. (9)–(11), the pressure gradient can be expressed as

$$\nabla p_\alpha / n_\alpha = \nabla h_\alpha - \Theta_\alpha \nabla S_\alpha .$$

Further simplifications arise by introducing the α-vorticity, $\mathbf{\Omega}_\alpha = m_\alpha \nabla \times \mathbf{u}_\alpha + q_\alpha \mathbf{B}/c$. Using the identity $\mathbf{u}\cdot\nabla\mathbf{u} = \nabla u^2/2 - \mathbf{u}\times\nabla\times\mathbf{u}$, the equation of motion becomes

$$\nabla H_\alpha - \Theta_\alpha S_\alpha' \nabla \Psi_\alpha = \mathbf{u}_\alpha \times \mathbf{\Omega}_\alpha , \tag{19}$$

where the species *total enthalpy* is

$$H_\alpha = h_\alpha + m_\alpha u_\alpha^2/2 + q_\alpha V_E + m_\alpha V_G . \tag{20}$$

This is a generalization of the total enthalpy of a simple fluid in that it is species specific and it also accounts for electrostatic and gravitational energy forms.

Consider the components of the equation of motion, Eq. (19). The *azimuthal* component, using Eqs. (3) and (16) leads to $\mathbf{\Omega}_\alpha \cdot \nabla \psi_\alpha = 0$. This implies that ψ_α is constant along $\mathbf{\Omega}_\alpha$ and, therefore, on the surfaces $\Psi_\alpha = \text{const}$. Thus ψ_α is a surface quantity

$$\psi_\alpha = (c/4\pi q_\alpha) G_\alpha(\Psi_\alpha), \tag{21}$$

where G_α is an arbitrary function. The convenience factor $c/4\pi q_\alpha$ gives G_α the units of magnetic flux/length. Since ψ_α is the stream function, this equation is consistent with the earlier observation that α-surfaces are drift surfaces. Continuing, the ''parallel'' ($\mathbf{\Omega}_\alpha$) component, using $\mathbf{\Omega}_\alpha \cdot \nabla \mathbf{\Psi}_\alpha = 0$ [from Eq. (16)] leads at once to the equation $\mathbf{\Omega}_\alpha \cdot \nabla H_\alpha = 0$; thus the total enthalpy [Eq. (20)] is also a surface function

$$H_\alpha = H_\alpha(\Psi_\alpha). \tag{22}$$

This is a Bernoulli equation in that it regulates the energy and flow *along* a stream line of the flow. It generalizes the Bernoulli relation for a simple fluid in that it is species specific and accounts for electrostatic and gravitational energies. Finally, consider the *perpendicular* component ($\hat{\boldsymbol{\theta}} \times \mathbf{\Omega}_\alpha \sim \nabla \mathbf{\Psi}_\alpha$). Simplify using Eqs. (21) and (22), and replace $\mathbf{u}_\alpha, \mathbf{\Omega}_\alpha$ using Eqs. (3) and (16). Then the Eq. (19) reduces to a scalar times $\nabla \mathbf{\Psi}_\alpha$ equals zero. The resulting equation is

$$\frac{m_\alpha c^2}{4\pi q_\alpha^2} G_\alpha' r^2 \nabla \cdot \left(\frac{G_\alpha'}{n_\alpha r^2}\nabla \Psi_\alpha\right) = -G_\alpha' \phi + \frac{4\pi q_\alpha}{c}\phi_\alpha + 4\pi n_\alpha r^2 (H_\alpha' - \Theta_\alpha S_\alpha'). \tag{23}$$

The variables and equations are summarized as follows. There are four scalar functions for the fields, ϕ, ψ, V_{es}, V_g; and seven scalar variables for each species, ϕ_α, ψ_α, Ψ_α, h_α, n_α, p_α, and Θ_α. This system is closed by the following equations. The various continuity equations are automatically satisfied in view of the existence of the scalar functions. The remaining equations then are the two components of Ampere's law [Eqs. (5) and (6)], Poisson's [Eq. (4)], and gravitational potential [Eq. (7)]. Equations for each species include the definition of the α surface variable [Eq. (17)]; three equations of state [Eqs. (12)–(14)]; and three components of the equation of motion [Eqs. (21)–(23)]. Some of these equations contain second-order differential operators, and others are simply functional relationships. The system contains three arbitrary surface functions for each species $G_\alpha, H_\alpha, S_\alpha$ [Eqs. (18), (21), and (22)] which are functions of the surface variables Ψ_α. Considerable elimination can be done to simplify this system. The next section considers the simplifications for three classes of reducing assumptions.

III. REDUCED MODELS

A. Standard reduced model

The standard reduced model is a two fluid of hydrogen ions and massless electrons (each of which is an ideal fluid), with quasineutrality ($n_e \approx n_i = n$), and negligible gravity. Then Poisson's and the gravitational equations are unnecessary. For massless electrons $\Psi_e \to \psi$. Also, $\Theta_\alpha = \Theta(n)$. The electrostatic potential is eliminated at once by summing the two Bernoulli equations [Eq. (22)]; the enthalpies are expressed in terms of n and Ψ_i or ψ [Eq. (15)]. Then the system of equations simplifies to

$$\Delta_\psi^* = \frac{4\pi e}{c}(\phi_i - \phi_e), \tag{24}$$

$$\phi = -G_i - G_e , \tag{25}$$

$$\frac{\gamma C_{st}}{\gamma - 1} n^{\gamma - 1}(S_i + S_e) + \frac{1}{2} m_i u_i^2 = H_i + H_e , \tag{26}$$

$$l_c^2 G_i' n r^2 \nabla \cdot \left(\frac{G_i'}{nr^2}\nabla \Psi_i\right) = -G_i' \phi + \frac{4\pi e}{c}\phi_i + 4\pi n r^2 [H_i' - \Theta(n) S_i'], \tag{27}$$

$$0=-G_e'\phi-\frac{4\pi e}{c}\phi_e+4\pi nr^2[H_e'-\Theta(n)S_e'], \tag{28}$$

where the density-dependent collisionless skin depth has been introduced,

$$l_c^2=m_ic^2/4\pi e^2n.$$

The arbitrary functions are $\psi_i=G_i(\Psi_i)$, $\psi_e=G_e(\psi)$, $H_i(\Psi_i)$, $H_e(\psi)$, and $S_i(\Psi_i)$, $S_e(\psi)$. In the isothermal limit $\Theta\rightarrow\ln n-1$ in Eqs. (27) and (28).

By eliminating variables these reduce to a system of two second-order equations for Ψ_i and $\psi(=\Psi_e)$ and an algebraic Bernoulli equation for n

$$\frac{m_ic^2}{4\pi e^2}G_i'r^2\nabla\cdot\left(\frac{G_i'}{nr^2}\nabla\Psi_i\right)$$

$$=\frac{\psi-\Psi_i}{l_c^2}+(G_i+G_e)G_i'+4\pi nr^2[H_i'-\Theta(n)S_i'], \tag{29}$$

$$\Delta_\psi^*=\frac{\psi-\Psi_i}{l_c^2}-(G_i+G_e)G_e'-4\pi nr^2[H_e'-\Theta(n)S_e'], \tag{30}$$

$$h_i(n,S_i)+h_e(n,S_e)+\frac{1}{8\pi nl_c^2r^2}[(\psi-\Psi_i)^2+(nl_c^2)^2G_i'^2|\nabla\Psi_i|^2]$$

$$=H_i+H_e, \tag{31}$$

where the species enthalpies are known functions as in, e.g., the case of an ideal gas [Eq. (15)]. Note the repeated appearance of the term $(\psi-\Psi_i)/l_c^2$; this will prove significant later. The other variables are given by algebraic relations in terms of Ψ_i, ψ, and n

$$\psi_i=\frac{c}{4\pi e}G_i(\Psi_i),\quad \psi_e=-\frac{c}{4\pi e}G_e(\psi),$$

$$\phi=-(G_i+G_e),\quad \phi_i=\frac{en}{m_ic}(\psi-\Psi_i).$$

This is the generalization of the Grad–Shafranov system for two-fluid flowing equilibria, and of the one-fluid flowing system [Eqs. (6) and (7) of Ref. 10].

B. Minimum energy equilibria

A particularly important subset of flowing equilibria are those for a minimum energy state. In the case of magnetostatic plasma the minimized quantity is the magnetic energy and the constraint is on the magnetic helicity.[11] Then the surface functions are no longer arbitrary but must have the form $\phi(\psi)=-\lambda\psi$, and $p(\psi)=\text{const}$ where λ is the Lagrange multiplier associated with the invariant magnetic helicity. Thus the constrained minimization procedure requires special forms of the surface functions that depend on the Lagrange multiplier. In the more general case of a flowing multifluid,[3,4] the minimized quantity is the magnetofluid energy, which includes the magnetic and flow energies. The constraints are the self helicities for each species and (supposing an axisymmetric system boundary) the mechanical angular momentum. This leads to the relaxed flow equations for each species [from Eq. (49) of Ref. 4]

$$n_\alpha\left(\mathbf{u}_\alpha-\mathbf{\Omega}r^2\frac{\hat{\boldsymbol{\theta}}}{r}\right)=\lambda_\alpha\frac{c^2}{4\pi q_\alpha^2}\mathbf{\Omega}_\alpha, \tag{32}$$

where λ_α, $\mathbf{\Omega}$ are the Lagrange multipliers associated with the constraints on the self helicities and the angular momentum.

Physically, a relaxing system probably approaches the state of maximal entropy. In this case the invariants include (in a closed system) the total energy. The total energy is the sum of the magnetofluid energy (the organized energy form) and the thermal energy (the disorganized energy form). The variational principle for maximal entropy with constrained total energy leads to exactly the same relaxed flow equation [Eq. (32)] as was discussed in Ref. 4. Evidently, relaxation corresponds to the conversion of *some* of the organized energy to disorganized energy.

The relaxed flow equation, Eq. (32), implies particular forms of the components of a species equation of motion. Express $\mathbf{u}_\alpha,\Omega_\alpha$ in terms of surface functions [Eqs. (3) and (16)]. The toroidal component of Eq. (32) is

$$\lambda_\alpha\frac{c}{4\pi q_\alpha}m_\alpha\Delta_n^*\psi_\alpha=-\lambda_\alpha\frac{c}{4\pi q_\alpha}\frac{q_\alpha}{c}\phi+\frac{q_\alpha}{c}\phi_\alpha$$

$$-n_\alpha r^2\Omega\frac{q_\alpha}{c},$$

comparing this with Eq. (23) shows that for minimum energy equilibria two of the surface functions must take the special forms

$$G_\alpha'=\lambda_\alpha, \tag{33}$$

$$H_\alpha'-\Theta(n)S_\alpha'=-\Omega\frac{q_\alpha}{c}. \tag{34}$$

The poloidal component of Eq. (32) is

$$\frac{\hat{\theta}}{r}\times G_\alpha'\nabla\Psi_\alpha=\lambda_\alpha\frac{\hat{\theta}}{r}\times\nabla\Psi_\alpha.$$

This is consistent with Eq. (33), which integrates at once to

$$G_\alpha(\Psi_\alpha)=\lambda_\alpha\Psi_\alpha+\text{const.} \tag{35}$$

In Eq. (34) the density appears; since H_α,S_α are functions of Ψ_α a separation of variables requires $S_\alpha'=0$, or

$$S_\alpha(\Psi_\alpha)=\text{const.} \tag{36}$$

Then the pressure relation [Eq. (13)] is $p_\alpha/n_\alpha^\gamma=S_\alpha=\text{const.}$, i.e., a global constant. In the isothermal case, $\gamma\rightarrow1$, this implies a globally uniform temperature. Finally then, the total enthalpy functions are

$$H_\alpha(\Psi_\alpha)=-\Omega\frac{q_\alpha}{c}\Psi_\alpha+\text{const.} \tag{37}$$

Finally then the system for minimum energy equilibria of an ideal two fluid [from Eqs. (29)–(31)] are

$$\lambda_i^2 n r^2 \nabla\cdot\left(\frac{\nabla\Psi_i}{nr^2}\right) = \frac{1}{l_c^2}\left[\frac{\psi-\Psi_i}{l_c^2}+\lambda_i(\lambda_i\Psi_i+\lambda_e\psi)-\Omega\frac{r^2}{l_c^2}\frac{m_i c}{e}\right], \tag{38}$$

$$\Delta^*\psi=\frac{\psi-\Psi_i}{l_c^2}-\lambda_e(\lambda_i\Psi_i+\lambda_e\psi)-\Omega\frac{r^2}{l_c^2}\frac{m_i c}{e}, \tag{39}$$

$$h_i(n,S_i)+h_e(n,S_e)+\frac{1}{8\pi n l_c^2 r^2}[(\psi-\Psi_i)^2 + \lambda_i^2(nl_c^2)^2|\nabla\Psi_i|^2]-\Omega\frac{e}{c}(\psi-\Psi_i)=\text{const.} \tag{40}$$

Equations (38) and (39) are a coupled pair of second-order equations for $\psi(=\Psi_e)$ and Ψ_i, and Eq. (40) is a algebraic Bernoulli equation for *n*. In the state of minimum energy, the surface functions are no longer arbitrary but must take the special forms [Eqs. (35)–(37)].

C. Magnetostatic system

In the case of a magnetostatic plasma (one fluid, no flow) the standard reduced Eq. (29)–(31) reduce to the familiar Grad–Shafranov system as follows. For no flow $\phi_i=0$, $\psi_i=G_i=0$. Then the ion surface function is $\Psi_i=\psi$. Replace $G_e=-\phi(\psi)$ using Eq. (25). The ion flow equation [Eq. (29)] reduces to

$$0=H_i'-\Theta(n)S_i'. \tag{41}$$

The magnetic stream function equation [Eq. (30)] reduces to

$$\Delta^*\psi=-\phi\phi'-4\pi n r^2[H_e'-\Theta(n)S_e']. \tag{42}$$

The Bernoulli law (no ion flow) is

$$h_i(p_i,\psi)+h_e(p_e,\psi)=H_i(\psi)+H_e(\psi).$$

Take the *total* derivative with respect to ψ and account for ideal fluids and the properties of the canonical form of the enthalpy, $\partial h_\alpha/\partial p_\alpha=1/n_\alpha$ and $\partial h_\alpha/\partial S_\alpha=\Theta_\alpha$. Then the term on the right side of Eq. (41) and the term in parenthesis in Eq. (42) are p_α'/n for $\alpha=i,e$, respectively (the prime denotes the derivative with respect to ψ). Then with $p=p_i+p_e$, the sum of Eqs. (41) and (42) is

$$\Delta^*\psi=-\phi\phi'-4\pi r^2 p', \tag{43}$$

which is the familiar Grad–Shafranov equation.

D. Flowing MHD system

The MHD system assumes the ideal Ohm's law, a reduced form of the electron equation of motion where electron inertia and pressure are neglected, and the ion velocity replaces the electron velocity in the Lorentz force (no Hall effect). These effects are retained in the general two-fluid model, although $m_e\to 0$ in the ''standard reduced model'' (Sec. III A). In terms of equations of state, MHD has only one (for the ions) since the electron pressure is neglected in Ohm's law. Here an inconsistency of sorts is sometimes tolerated since finite electron pressure is allowed in the MHD equation of motion (which is the sum of the electron and ion equations).

Reducing the two-fluid system to flowing MHD is much less straightforward than it was for magnetostatics. This reduction is examined in comparison with derivations of the flowing MHD system in the literature.[1,10] There the authors derive components of the equation of motion in the parallel (**B**) direction, i.e., a Bernoulli equation, and in the perpendicular ($\nabla\psi$) direction. In so doing an expression for the azimuthal magnetic field, B_θ, is derived; that arises from the θ component of the equation of motion. In a two fluid this equation can be placed in the form

$$\mathbf{B}\cdot\nabla\left(\frac{G_e'}{n}\phi_i+\frac{e}{c}\phi\right)=-\left(\frac{\hat\theta}{r}\times\nabla\phi\right)\cdot\nabla(ru_{i\theta}). \tag{44}$$

The left side is the MHD part, and equals zero in the MHD approximation [text equation just preceding Eq. (5) in Ref. 10]. The right side is the two-fluid correction. If two-fluid effects are neglected, then the parenthetic term on the left side is a function of ψ alone. This leads shortly to an expression for B_θ [Ref. 10, Eq. (5)]

$$B_\theta=\left[F(\psi)-\frac{c^2}{e^2}r^2G_e'H_e'\right]\Big/\left(1-\frac{c^2G_e'^2}{4\pi e^2 n}\right), \tag{45}$$

using the present terminology, where *F* is an arbitrary function. Herein emerges the importance of the MHD approximation. The denominator of Eq. (45) is equal $1-u_{ip}^2/c_s^2$ where u_{ip} is the ion poloidal flow speed, ν_A is the Alfven speed, and their ratio is the Mach number. The possibility of shocks in consequence was recognized in Refs. 1, 10, and elsewhere.

Two-fluid effects may modify this behavior. From the poloidal component of Ampere's law [Eq. (6)] and the definition of the ion surface function [Eq. (17)] the right side of Eq. (44) takes a new form

$$\mathbf{B}\cdot\nabla\left(G_e'ru_{i\theta}+\frac{e}{c}\phi\right)=(\mathbf{u}_{ip}-\mathbf{u}_{ep})\cdot\frac{\nabla(\psi-\Psi_i)}{l_c^2}. \tag{46}$$

Since the ion and electron velocities generally differ, the MHD approximation is valid if the term $(\psi-\Psi_i)/l_c^2$ is negligible. This quantity, which appears in all three of Eqs. (29)–(31) [as well as Eqs. (38)–(40)], represents the two-fluid effect. Thus the two-fluid system is reduced to MHD by dropping the $(\psi-\Psi_i)/l_c^2$ terms in these equations.

A peculiarity remains in the reduction to MHD. Dropping the $(\psi-\Psi_i)/l_c^2$ terms still leaves a coupled pair of second-order equations for ψ and ψ_i. [Since $\psi_i\sim G_i(\Psi_i)$ one can replace $G_i'\nabla\Psi_i$ can be replaced by $(4\pi e/c)\nabla\psi_i$.] If it were possible to eliminate one of these variables, say ψ_i, then a *fourth*-order equation for ψ would remain. Yet the MHD treatments in Refs. 1 and 10 produce a single *second*-order equation for ψ. The explanation of this is unknown, although it is certainly connected in some way with the MHD approximation.

IV. PROPERTIES OF TWO-FLUID EQUILIBRIA

A. Deviation between magnetic flux and ion surfaces

Expand the magnetic stream function in the expression for the ion surface variable, Eq. (17), as $\psi=\psi_0+|\nabla\psi|\delta$, where δ is a perpendicular displacement relative to a magnetic surface $\psi=\psi_0=\text{const}$. Since $\Psi_i=\text{const}$ on an ion drift surface and with $\phi_i=rnu_{i\theta}$ it follows that $\delta=m_icu_{i\theta}/eB_p$, where $B_p=|\nabla\psi|/r$ is the poloidal field. Introducing the Alfven speed based on the poloidal field, $v_{Ap}=B_p/(4\pi m_in_i)^{1/2}$, the displacement is

$$\delta=l_c(u_{i\theta}/v_{Ap}). \tag{47}$$

Thus if the toroidal flow speed is comparable to the poloidal Alfven speed, the excursion of an ion from a magnetic surface is comparable to the collisionless skin depth.

B. One- and two-fluid effects

The one-fluid model adopts the ideal Ohm's law. This leads[10] to the equilibrium relationship $\mathbf{u}_i\cdot\nabla\psi=0$ (using our terminology), which implies that the ions flow strictly on magnetic surfaces. Therefore, the one-fluid model becomes inaccurate whenever the two-fluid model predicts significant ion excursions [Eq. (46)] from magnetic surfaces. The two-fluid effect was identified with the $(\psi-\Psi_i)/l_c^2$ term. This is illuminates the physical meaning of the MHD approximation. Since $\psi=\text{const}$ is a magnetic surface and $\Psi_i=\text{const}$ is an ion drift surface, MHD assumes simply that deviation of ions from magnetic surfaces is ignored.

One consequence of this concerns the conjecture (Ref. 10 and elsewhere) that the equations of flowing equilibria may be hyperbolic in high flow cases. This contrasts with the usual elliptical form as appears; e.g., in the GS equation [Eq. (43)]. The transition to hyperbolic behavior arises because of the factor $1-u_{i\theta}^2/v_A^2$ in the equilibrium equation [Eq. (7) of Ref. 10] and also in Eq. (45) here. Hyperbolic behavior may arise for super-Alfvenic flow. However, according to the two-fluid model, if $u_{i\theta}/v_{Ap}$ is comparable to unity, the ion orbits depart significantly from magnetic surfaces; further, and since $v_{Ap}<v_A$, this threshold is reached before $u_{i\theta}/v_A=1$. Therefore, it is unclear whether hyperbolic behavior will arise when two-fluid effects are properly taken into account.

C. Implications of the Bernoulli relation

Pressure rises and falls on α-surfaces in more or less the same way as in a simple fluid, i.e., according to a Bernoulli relation. In this sense the plasma pressure decouples from the magnetic field. This agrees with intuition since, unlike the electric field, the magnetic field does not modify the energy of a particle. In effect, the Bernoulli equation is a statement of *parallel* energy flow. This should not be construed as meaning that the plasma is decoupled from the magnetic field, which would preclude the possibility of finite β (finite plasma pressure relative to the magnetic pressure). Instead, the plasma-magnetic field coupling is through the *perpendicular* component of the equation of motion. Thus, the plasma pressure may be higher on inner surfaces and lower on outer surfaces, as is the case in finite-β magnetically confined plasmas.

D. Minimum energy equilibria

In general, the total enthalpies, H_i,H_e, vary from one drift surface to another. However, in a minimum energy state, they vary from surface to surface in a very particular way, $H_\alpha\sim q_\alpha\Omega\Psi_\alpha+\text{const}$, with opposite signs for the two species. When the species Bernoulli equations are summed, this cancels, so that in effect the total enthalpy is globalized. The implication for a minimum energy state is that a finite-β configuration with peak pressure in the center and falling toward the edge *must* have rising velocity toward the edge. This implies the necessity of velocity shear in a finite-β minimum energy state.

A second property of minimum energy states is the uniformity of the entropy-like variable S_α. This is significant for magnetic fusion where isothermal drift surfaces are expected at fusion temperatures. Then $\gamma\approx1$, and $S_\alpha=kT_\alpha$ so that a minimum energy state has uniform temperature. Thus, for all the favorable attributes of a minimum energy state, a practical fusion plasma must depart from it at some point in order for the temperature to fall at the plasma edge.

V. SUMMARY

The multifluid description of a plasma has been applied to develop the system of equations governing flowing multifluid equilibria. Each species has its own characteristic surfaces (α surfaces) defined by the generalized vorticity $\mathbf{\Omega}_\alpha$ which accounts for the ''field'' as well as the fluid vorticity, and is the curl of the canonical momentum. The α surfaces, defined by $\Psi_\alpha=\text{const}$, are the drift surfaces of a species. Indeed $\Psi_\alpha=\text{const}$ is equivalent to the constancy of the canonical angular momentum. In defining an axisymmetric equilibrium, there are three arbitrary surface functions for each species: One giving the entropy structure; one giving the relationship between the stream function and the α surface variable Ψ_α; and one giving the enthalpy structure. In an important reduced case (two fluid of hydrogen ions and massless electrons, with quasineutrality and the ideal fluid equation of state), the equilibrium system reduces to a pair of second-order partial differential equations for the ion stream function and the magnetic stream function, plus a Bernoulli-like equation which determines the density. This system reduces to the Grad–Shafranov equation in the limit of no ion flow. An important subset of flowing equilibria are those representing a state of minimum energy. Here the three surface functions (per species) are no longer arbitrary but take special forms that depend on the Lagrange multipliers that enter the minimization procedure. Two important properties of minimum energy states are that they can only have finite β (pressure) if the flow speed rises toward the plasma edge, and that they are isothermal (assuming that the parallel thermal conductivity is large).

These results suggest several topics for future investigation. (1) A numerical solver is needed for computing 2D (two-dimensional) (r,z) equilibria. A possible approach is the successive over-relaxation technique. Another is an extension of the variational approach introduced by Shafranov for magnetostatic equilibria.[12] These computed equilibria will then be useful for testing stability and other properties. (2) Missing from the two-fluid model is allowance for finite orbit effects, such as might be represented by a gyroviscosity. This will alter the length scale by a factor of perhaps several compared with the collisionless skin depth, l_c. The length scale adjustment may be even more in the case of configurations with a large toroidal field where the neoclassical orbit size is set by the somewhat smaller poloidal field. (3) The possibility of minimum energy equilibria is intriguing because it suggests that equilibria may exist that are stable to all two-fluid instabilities ideal and nonideal, and that may consequently have very low transport. The fact that such states (with finite β) have strongly sheared flows suggests a connection between the stabilizing effect of flow shear, that has been investigated in several places. Further, the fact that minimum energy states (with large parallel thermal conduction) have uniform temperatures suggests a connection with ion and electron temperature gradient instabilities that play a major role in driving anomalous transport.

ACKNOWLEDGMENTS

Thanks are due to A. Ishida for helpful comments. This work was supported by U.S. Department of Energy Grant No. DE-FG03-98ER54480.

[1] H. Tasso and G. N. Throumoulopoulos, Phys. Plasmas **5**, 2378 (1998).
[2] A. Ishida, H. Momota, and L. C. Steinhauer, Phys. Fluids **31**, 3024 (1988).
[3] L. C. Steinhauer and A. Ishida, Phys. Rev. Lett. **79**, 3423 (1997).
[4] L. C. Steinhauer and A. Ishida, Phys. Plasmas **5**, 2609 (1998).
[5] L. Turner, IEEE Trans. Plasma Sci. **14**, 849 (1986).
[6] G. N. Throumoulopoulos and H. Tasso, Phys. Plasmas **4**, 1492 (1997).
[7] J. M. Finn and T. M. Antonsen, Jr., Phys. Fluids **26**, 3540 (1983).
[8] K. Avinash, Phys. Fluids B **4**, 3856 (1992).
[9] A. I. Morozov and L. S. Solov'ev, Rev. Plasma Phys. **8**, 1 (1980).
[10] E. Hameiri, Phys. Fluids **26**, 230 (1983).
[11] J. B. Taylor, Phys. Rev. Lett. **33**, 1139 (1974); Rev. Mod. Phys. **58**, 741 (1986).
[12] V. D. Shafranov, Rev. Plasma Phys. **2**, 103 (1966).