Building Confidence intervals for the Band-Pass and Hodrick-Prescott Filters: An Application using Bootstrapping *

Francisco A. Gallego‡ Christian A. Johnson†

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Abstract

This article generates innovative confidence intervals for two of the most popular de-trending methods: the Hodrick-Prescott and Band-Pass filters. The confidence intervals are obtained using block-bootstrapping techniques for dependent data. As an example, we present GDP trend growth and output gap intervals for the G7 economies. This new concept can improve the utility and applications of these filters by overcoming on the most cited limitations: not having confidence intervals.

Keywords: Hodrick-Prescott Filter, Band-Pass Filter, Output Gap, Block Bootstrapping

JEL classification: C15, C22, E32

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‡ Corresponding author, Department of Economics, Massachusetts Institute of Technology, Cambridge, MA, USA. E-mail: fgallego@mit.edu

† Business School, Universidad Adolfo Ibáñez, Santiago, Chile. E-mail: chjohnson@uai.cl

1 Introduction

However, most modern macroeconomic theories emphasize the relevance of distinguishing the effects of permanent and transitory changes in economic behavior, only with the popularity of RBC literature, the partition of (macro) economic series between their trend (permanent) and cyclical (temporary) components became a common place in empirical analysis. Many economic applications can be mentioned, however, one of the most common are related with decomposing GDP between its trend and cyclical components.

One of the most popular methods to estimate trend and cyclical components of time series is the use of statistical filters, in either its univariate or multivariate form. Among the statistical filters one of the most common is the Hodrick and Prescott (1997)[1] procedure (HP) developed as a mechanical and statistical procedure to extract very low and very high frequencies (less than two years and more than eight years approximately) from time series such as GDP. Although this data window seems arbitrary, it comes from the seminal paper by Burns and Mitchell (1944) carried out at the National Bureau of Economic Research, which concluded that the US economy presents very clear business cycles lasting up to eight years.[2]

Following a different but related approach, other papers, such as Baxter and King (1995, 1999) and Christiano and Fitzgerald (1999) have designed and implemented specific band-pass filters which try to isolate business-cycle fluctuations in macroeconomic time series. This sort of filters is an approximation of the "ideal band pass filter" that can not be computed because requires infinite observations. The band pass filters are designed to isolate fluctuations in the data which persist for periods of 1.5 or 2 through 8 years. Both papers apply the filter to several macroeconomic time series, and, as Hodrick and Prescott, describe the picture of the U.S. postwar business cycle that emerges from their results. The authors claim that this kind of filter can improve two practical problems encounter using the HP filter: unusual behavior of cyclical

[1] Although this study was published in 1997, the original draft, which is widely cited, was written in 1980.

[2] Recent studies present sophisticated methods for finding the optimal window for any time series, adjusting for data frequency (Ravn and Uhlig, 2001).

components at the end and beginning of the sample, and the choice of smoothing parameter for non-quarterly data. In this paper we use the Christiano and Fitzgerald (1999) (CF) version of the band pass filter. While the HP procedure was included in widely used econometric software, which helped to spread the filter's popularity and applicability, several versions of the band pass filter have been used in a number of papers and programs to implement it area available in several web-pages. Hence, nowadays, studies by most academic and official institutions (treasury departments and central banks) that mention any potential output measurements make some reference to some of the two above-mentioned filters.

This paper presents a new methodology for creating confidence intervals around the cyclical component (that is, the difference between the original series and its estimated trend component) and the series trend growth rate, generated by applying the HP and CF filters to the original series, based on block bootstrap sampling techniques. The procedure is used, as an illustration, to construct confidence intervals for the output gap and trend growth rate. The output gap is an approximation for the cycle of the economy, and is defined as the difference between actual output and the HP or BK output trend generated time series (as a proportion of the filtered series). An application of this new methodology is performed using quarterly GDP data from G7 economies (Canada, France, Germany, Italy, Japan, United Kingdom and United States).

2 The Hodrick-Prescott and Christiano-Fitzgerald Filters

In this section we briefly present a review of the two filters to illustrate the way they are implemented.

2.1 The Hodrick-Prescott Filter

The HP is a linear filter that creates artificial time series minimizing departures from the original, limiting the volatility of this artificially generated vector to some upper bound. This limit is decided based on data frequency (monthly, quarterly, or annual), and in equations (1) and (2) is represented by the parameter λ, using values from 100 to 14,400.[3] The expression to minimize is:

$$\frac{1}{T}\sum_{t=1}^{T}\left(\ln y_t - \ln y_t^*\right)^2 + \frac{\lambda}{T}\sum_{t=1}^{T}\left[\left(\ln y_{t+1}^* - \ln y_t^*\right) - \left(\ln y_t^* - \ln y_{t-1}^*\right)\right]^2 \qquad (1)$$

where $y_t^\star$ represents the HP filtered time series generated from the actual y_t. Its counterpart matrix can be written as:

$$y^* = \left(I_{(T)} + \lambda\left(P'P\right)\right)^{-1} y \qquad (2)$$

where y and $y^\star$ are column vectors both with dimension (Tx1), and P is the TxT matrix represented by a series of concatenated identity and zero matrices:

$$P = \left\lfloor I_{(T)} \mid O_{(t-2)\times 2} \right\rfloor - 2\left\lfloor O_{(t-2)\times 1} \mid I_{(t-2)} \mid O_{(t-2)\times 1} \right\rfloor + \left\lfloor O_{(t-2)\times 2} \mid I_{(t-2)} \right\rfloor$$

2.2 The Christiano-Fitzgerald Filter

The CF is also a linear filter, as the HP, but in this case it recovers the trend as the component of the series with periodicity between a lower and an upper bound (the default case for CF is 1.5 and 8 years). The main practical problem with this sort of filter is that the *ideal* filter requires an infinite amount of data. Hence, the proposed filter is a linear approximation of the ideal filter.

The way they approximate the filter is the following. First, it is assumed that the data are generated by a pure random walk (or, if there is a drift, it was previously removed). The authors argue this approximation (likely false) is cost-effective, in the sense it approximates quiet well most macroeconomic series (in particular, output which is used in the illustrations presented in section 5). Second, to isolate the component of y with period of oscillation between p_l and p_u, they proposed $\hat{y}$ as an optimal approximation of y^* (the series derived from applying the ideal band-pass filter).

$$\hat{y} = B_0 y_t + B_1 y_{t+1} + \ldots + B_{T-1-t} y_{T-1} + \tilde{B}_{T-t} y_T + B_1 y_{t-1} + \ldots + B_{t-2} y_2 + \tilde{B}_{t-1} y_1 , \qquad (4)$$

[3] For an analysis of adjusting the HP Filter depending on the data frequency, see Ravn and Uhlig (2001).

for $t = 3,4,...,T\text{-}2$. Where

$$B_j = \frac{sin(jb) - sin(ja)}{\pi j}, j \geq 1$$

$$B_0 = \frac{b-a}{\pi}, a = \frac{2\pi}{p_u}, b = \frac{2\pi}{p_l}.$$

$$\widetilde{B}_{T-t} = -\frac{1}{2}B_0 - \sum_{j=1}^{T-t-1} B_j \text{ , for } t = 3,..., T\text{-}2.$$

And $\widetilde{B}_{t-1}$ solves $0 = B_0 + B_1 + ... + B_{T-1-t} + \widetilde{B}_{T-t} + B_1 + ... + B_{t-2} + \widetilde{B}_{t-1}$

It is worth to notice that the CF filter varies with time, and it is not symmetric in terms of lead and lagged *y*.

4 Confidence Intervals and Bootstrapping

To generate confidence intervals we employ a bootstrapping methodology (Hamilton, 1994) described in the following diagram:

INSERT FIGURE 1

This methodology is very appealing for the problem we are trying to assess in this paper –to construct confidence intervals for statistical filters. This approach allows evaluating many sources of uncertainty surrounding the estimation of trend components of the series. To do that, the block bootstrapping procedure re-samples the cyclical component of the series using blocks of observations. Next, it is possible to re-compute the trend component. Hence, it is possible to build a big group of estimations of the estimated trend and cyclical series.

Given the autocorrelated characteristic of the T-vector elements, the drawing procedure must consider this restriction by sampling sectors or blocks of this time series in such a way that the generated sampled vector preserves the autocorrelation characteristics present in the data. These blocks may be non-overlapping or overlapping, with the bootstrap sample created by sampling random blocks using replacement. The literature of optimal block length for dependent data suggests a size defined by $T^{\frac{1}{3}}$ where T represents the original sample length (Hall *et al.*, 1995).

5 An illustration: confidence intervals for output gap and trend growth in the G7.

This procedure was applied to G7 countries' GDP. We use quarterly information for GDP taken from *International Financial Statistics* of the IMF. The data cover periods ranging from 1957-2001 in the case of Japan, UK, and USA; to 1986-2001 in the case of Canada.

The output gap defined by $\left(\frac{y_t - y_t^*}{y_t^*}\right) \bullet 100$ makes up the generated vector using expression the HP and the CF filters, on which the bootstrapping sampling is done.

Figure 2 presents the estimated output gap generated for these economies by applying the HP filter to quarterly time series data (λ=1,600). The corresponding confidence intervals are defined with upper and lower limits, efficiently reflecting (through wide confidence intervals) the popular and widespread criticism that the HP filter is very end-of-sample data-dependent. This point is crucial for several purposes. For example, the policymakers, who are often forced to make on-the-spot evaluations of current economic conditions, most of the time without a solid tool to build a credible and time consistent argument that won' t vanished two or three months into the future. Confidence intervals generated using bootstrapping techniques can resolve current limitations of not having real-time data and a non-dependent end-of-sample tool to evaluate the current economic stance at least partially.

INSERT FIGURE 2

The interpretation from figure 2 is straightforward. Once the output gap exceeds any upper or lower limit, we can say that the economy faces a high risk of GDP misalignment with respect to trend GDP. Alternatively, if the output gap indicator rises past the upper bound of the interval and if one believes in a form of Phillips curve, inflationary risks become likely in the near future, whereas if it falls below the lower bound, the risk of a recession or deflationary situation rises. However, it is interesting to notice that confidence intervals are very wide in most cases in the whole sample, implying that, in most cases, output-gaps of –1% to +1% are not different from 0.

In addition, shaded areas in Figure 2 represent the NBER definition for recessionary episodes. It is interesting to notice that in most cases the periods when the negative cycle is different from 0 coincide with periods when the NBER definition implies a recession (namely, in the case of the US). In addition the proposed methodology allows identifying not only recessionary, but also expansionary periods.

Figure 3 presents the results of the same exercise but now applying the CF filter. Again, the confidence intervals are very wide, especially at the end of the sample. In terms of the width of the confidence intervals, in average, they move in the range (-1, 1). However, in the case of the CF filter, the volatility of the intervals is smaller in all cases in comparison to the HP filter. Once more, only in small cases output gap estimations are statistically different from 0.

INSERT FIGURE 3

Formal comparisons of confidence intervals for both filters are presented in Table 1. As it is possible to see, tests of equality of means and medians are rejected in most cases (but for Canada and France), implying that in general the confidence intervals of CF filters are wider than the intervals generated using a HP filter. However, on the other hand, test for equality of variance show that, in all cases, confidence intervals generated using the CF filter produce less volatile intervals. In a sense, there would be a trade-off involved in the choice between the two filters: the HP filter presents more precise estimates (in the sense of wide intervals), but at a cost of more volatile interval bands. While the opposite is true for the CF filter.

Now, we will move to the application of our block-bootstrapping procedure to define trend output growth, figures 4 and 5 present the changes in this indicator with its corresponding confidence interval, again for the G7 economies. In this case we present the median of the bootstrapping estimations for each filter. The picture is quite similar in both filters; there are some slow movements in the trend growth, in general with smoother changes in the case of the HP filter.

The case of the US is interesting because that country has experienced a steady increase in trend growth during the 1990s, which has risen from rates between 1 and 3% in the early 1990s to a current interval between 3 and 5%. The economic interpretation of this phenomenon can be attributed to productivity gains, possibly related to the "new economy", and prosperity arising from information and communications technology (Lansing, 2000). However, it is interesting to highlight that both filters present some differences, while the HP filter implies an almost steady increase with an increase in the band at the end of the sample, the trend growth estimated using the CF filter implies an increase up to 1995 with a subsequent decrease.

INSERT FIGURES 4 AND 5

Table 2 compares the confidence intervals generated for both filters. In this case, the results are quite similar to the case of the bands for the output gap: the CF-bands are wider and less volatile than the HP-bands.

5 Conclusion

This paper presents a simple methodology that for the first time generates confidence intervals for output gaps and output trend growth based on the application of block bootstrapping techniques to the results of two of the most popular statistical filters. This new procedure provides a measure of the uncertainty surrounding the estimation of trend growth and output gap. This application also has potential application for policymaking, as it provides authorities with a powerful tool to be considered when evaluating current economic conditions. Instead of only having output gap and trend growth point estimates, now they may include confidence intervals in order to have an improve macroeconomic appreciation as they consider future modifications to monetary policies.

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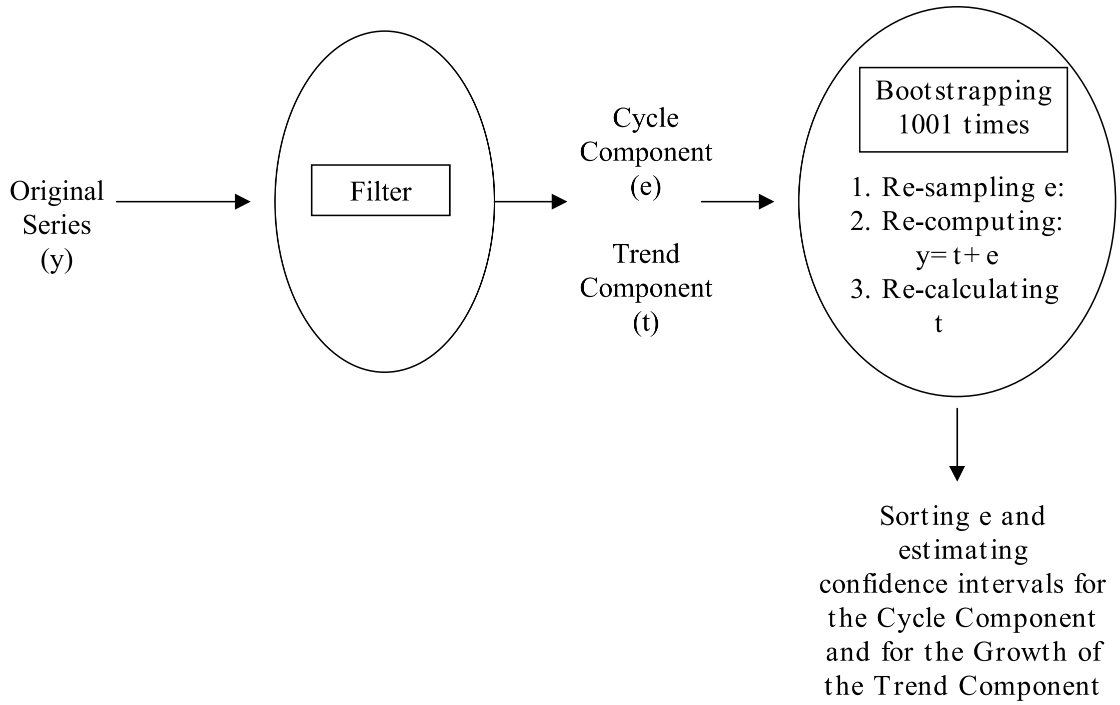

Figure 1. Algorithm to Generate Filter Confidence Intervals

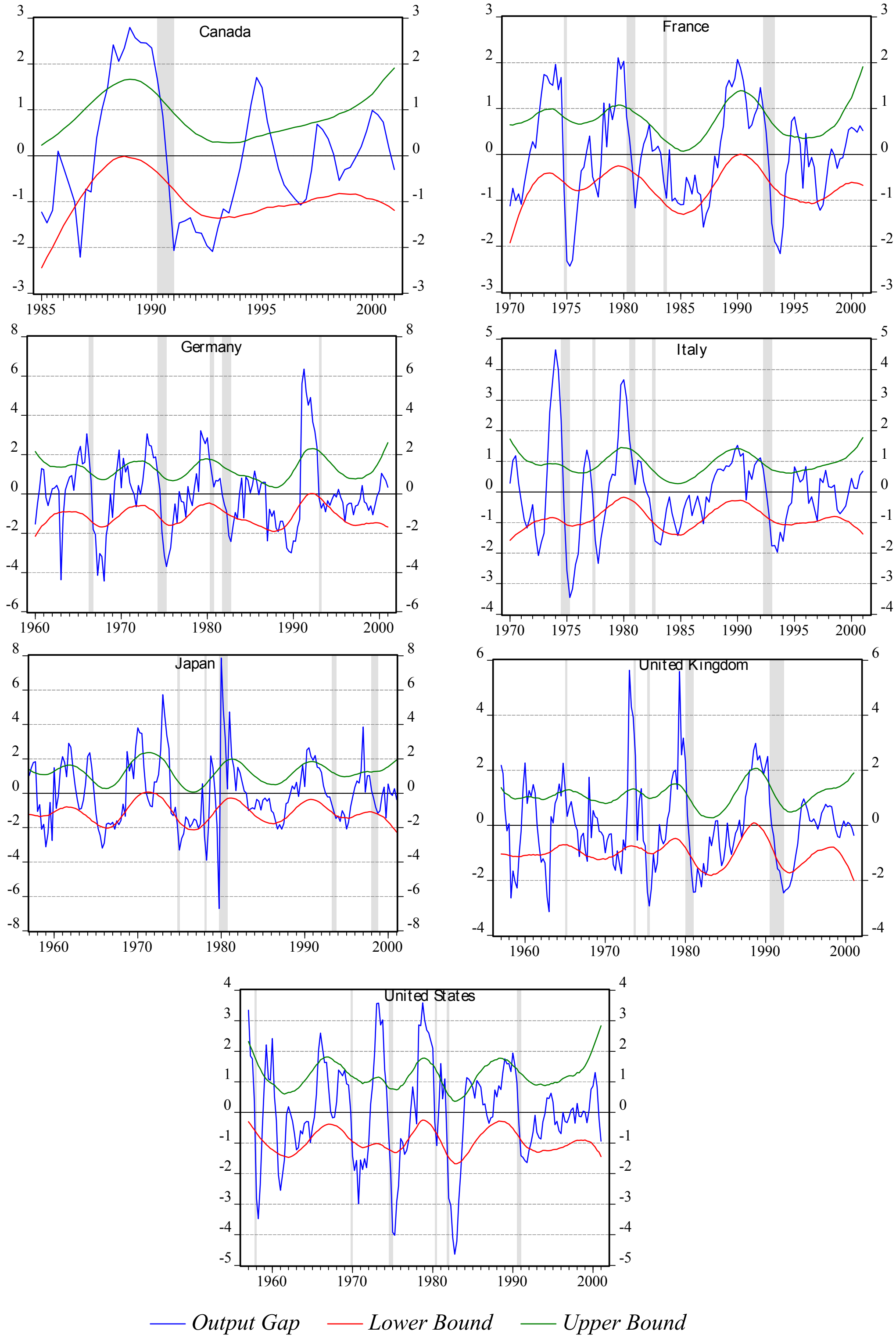

Figure 2. Output Gap Confidence Intervals using the HP filter for G7 Economies

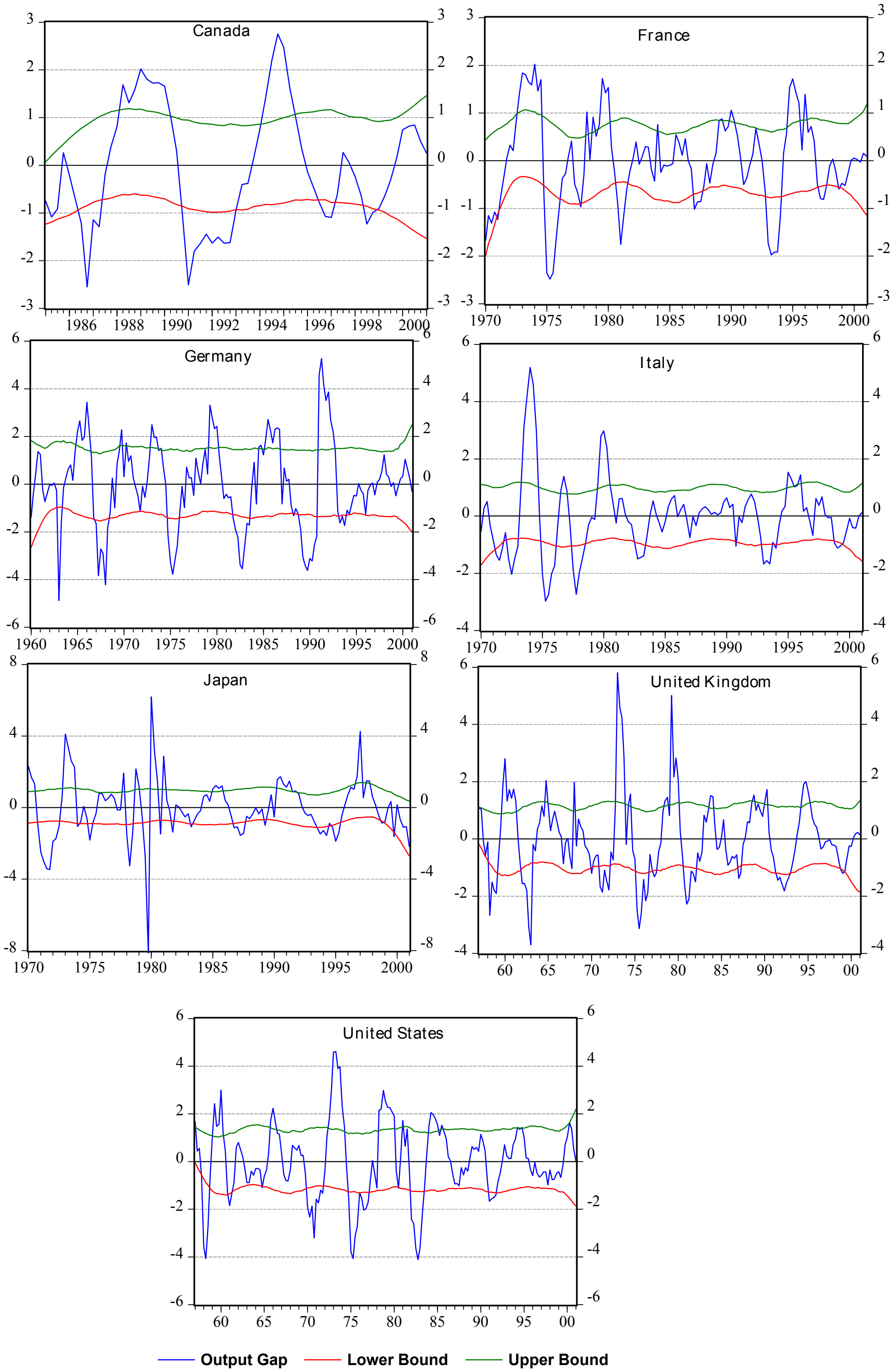

Figure 3. Output Gap Confidence Intervals using the CF filter for G7 Economies

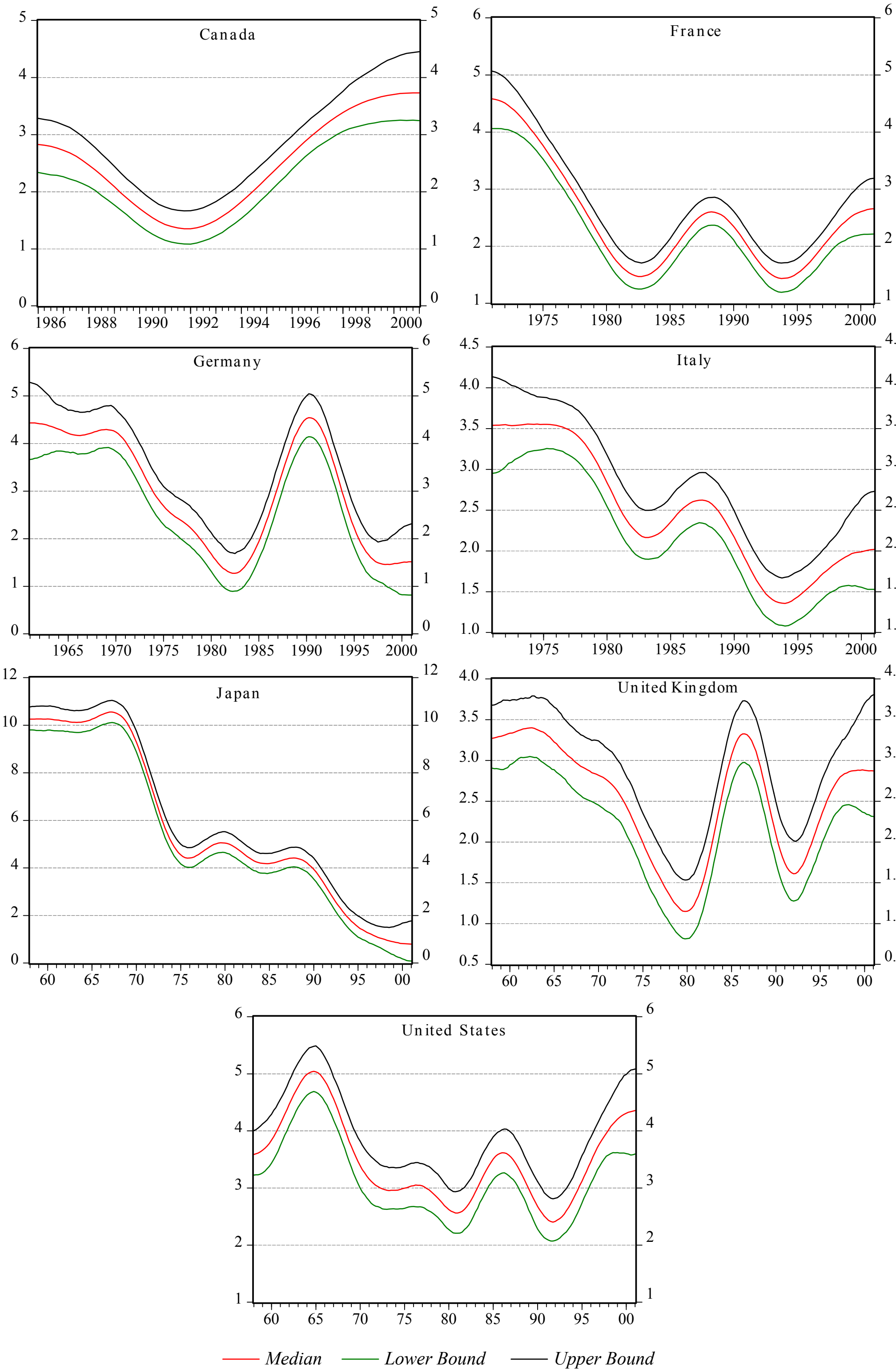

Figure 4. GDP Trend Growth for G7 Economies: Block Bootstrapping Simulations using the HP filter

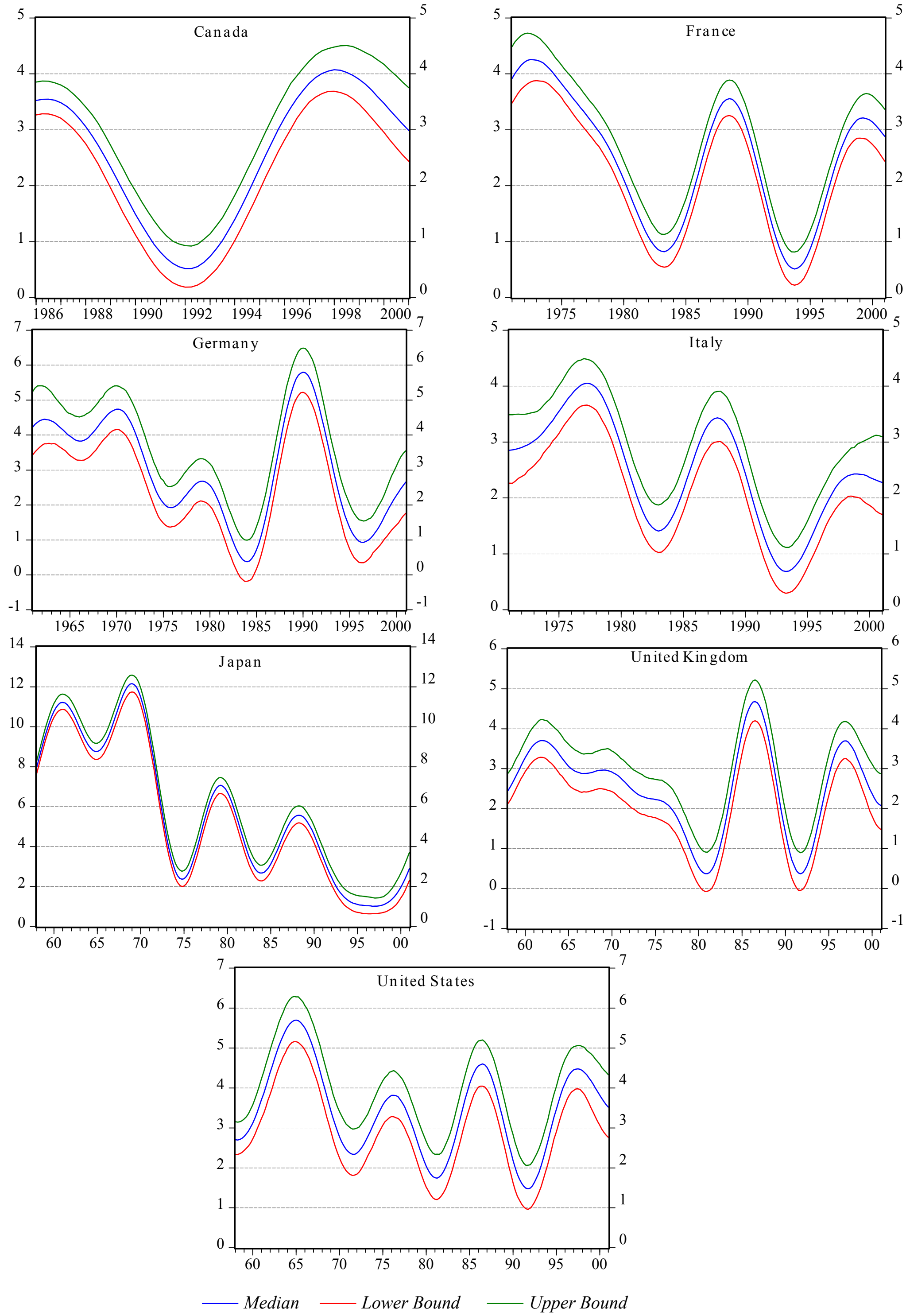

Figure 5. GDP Trend Growth for G7 Economies: Block Bootstrapping Simulations using the CF filter

Table 1. Statistics for Confidence Intervals for Output Gap (in percentage points)

	Canada	France	Germany	Italy	Japan	United Kingdom	United States
HP Filter							
Mean	1.84	1.46	2.40	1.80	2.36	2.12	2.18
Median	1.69	1.39	2.29	1.69	2.31	2.08	2.11
Minimum	3.10	2.59	4.31	3.31	4.21	3.91	4.28
Maximum	1.60	1.31	2.16	1.61	2.19	1.92	2.00
St. Deviation	0.34	0.24	0.37	0.32	0.26	0.25	0.30
CF Filter							
Mean	1.85	1.45	2.85	1.94	1.84	2.21	2.53
Median	1.81	1.40	2.78	1.90	1.82	2.19	2.50
Minimum	2.99	2.42	4.48	2.81	3.07	3.20	4.06
Maximum	1.31	1.30	2.62	1.76	1.47	1.74	1.95
St. Deviation	0.27	0.19	0.29	0.17	0.17	0.15	0.20
Tests of Equality (p-values)							
Mean (t-test)	0.87	0.62	0.00***	0.00***	0.00***	0.00***	0.00***
Median (Chi-square)	0.00***	0.13	0.00***	0.00***	0.00***	0.00***	0.00***
Variance (F-test)	0.08*	0.01***	0.00***	0.00***	0.00***	0.00***	0.00***

Table 2. Statistics for Confidence Intervals for Trend growth (in percentage points)

	Canada	France	Germany	Italy	Japan	United Kingdom	United States
HP Filter							
Mean	0.74	0.56	0.91	0.68	0.91	0.78	0.81
Median	0.63	0.50	0.85	0.61	0.87	0.75	0.78
Minimum	1.21	1.01	1.62	1.20	1.68	1.49	1.49
Maximum	0.58	0.45	0.77	0.58	0.80	0.69	0.72
St. Deviation	0.19	0.14	0.19	0.16	0.15	0.13	0.14
CF Filter							
Mean	0.81	0.65	1.28	0.88	0.84	0.98	1.13
Median	0.77	0.61	1.24	0.84	0.83	0.99	1.13
Minimum	1.32	1.01	1.83	1.39	1.40	1.39	1.57
Maximum	0.58	0.57	1.07	0.77	0.63	0.72	0.79
St. Deviation	0.18	0.11	0.15	0.13	0.11	0.10	0.12
Tests of Equality (p-values)							
Mean (t-test)	0.00***	0.00***	0.00***	0.00***	0.00***	0.00***	0.00***
Median (Chi-square)	0.00***	0.00***	0.00***	0.00***	0.00***	0.00***	0.00***
Variance (F-test)	0.00***	0.00***	0.00***	0.62	0.10	0.16	0.00***