# Economic Growth and the Evolution of Preferences under Uncertainty

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**Abstract:**•In this paper we show how it is possible to construct a theory of preference evolution from a model of economic growth under uncertainty. Our starting point is the Solow-Merton model of growth under uncertainty. We use stochastic dynamic equations that we derive from this growth model to construct a model of preference evolution from a von Neumann-Morgenstern expected utility model of consumption for a representative agent. We show that the dynamics that describe the change in the representative agent's preferences, lead to a canonical representation of preference, which is given by the Fokker-Planck equation for consumption. We then provide a characterization of possible boundary conditions for the Fokker-Planck equation. We then show how it possible to use some of these boundary conditions to derive a steady state distribution for consumption from this evolution equation for preferences. This steady state distribution of consumption can then be used to solve the Ramsey problem of optimal savings under uncertainty, giving an equivalent result to that derived in Merton's paper. Hence, the Solow-Merton Neo-Classical growth model is a special case of our model.

**Keywords**: Evolutionary models of economic growth, evolution of preferences, economic growth under uncertainty, Solow-Merton growth model.

**JEL Classification**: O41, E21.

## 1. Introduction

In an evolving economy new commodities are created with the passage of time. Preferences for novel commodities will not be defined, unless economic agents have previously imagined these commodities. These commodities will not appear from nowhere. They must be produced and they must be produced by labour, which will in turn eventually consume some of these novel commodities. Consequently the creation of new commodities will depend upon and lead to the evolution of preferences.

This paper begins by re-examining the stochastic extension of the Solow's (1956) exogenous growth model (Bourguigon 1974, Merton 1975, Chiang and Malliaris 1987). The usual path taken by the deterministic exogenous growth literature is to characterise which values of the growth equation's parameters lead to stable growth paths. However, when uncertainty is incorporated into the exogenous growth model, the possible growth paths are realizations given by a sequence of random variables. Each realization in the growth path is determined by a distribution that is conditional on the prior history of the growth path. This conditional distribution is given by the Fokker-Planck equation for the capital-labour ratio.

For example in the stochastic extension of the Solow's (1956) exogenous growth model (Bourguigon 1974, Merton 1975, Chiang and Malliaris 1987), the solution of Fokker-Planck equation is used to derive the steady state condition for economic growth, i.e. the distribution on which the probabilities converge asymptotically. The standard approach is to derive a stochastic differential equation describing the dynamics of capital accumulation in terms of the capital-labour ratio. The Fokker-Planck equation that provides equivalent representation of this stochastic differential equation can then be used to derive asymptotic distributions for income and consumption.

In this paper, we begin with a one-sector stochastic exogenous growth model as formulated by Merton (1975). From this model, we derive a stochastic differential equation describing the dynamics of capital accumulation in terms of the capital-labour ratio as a function of savings and stochastic changes in labour productivity. This changing nature of labour is captured as a branching process. The intent is to capture two effects: a human capital effect and composition of labour effect.

It is argued that labour is embodied with human capital. In this paper certain classes of labour are "hardwired'' with particular skills, new commodities can only be generated if the composition of labour changes in order to imagine novel products. Innovations to the type of labour employed in production then lead to a stochastic representation of production and consumption and an interpretation of preferences as a probability of consuming a particular consumption bundle.

From this stochastic differential equation describing per-capita change in capital, we then derive a stochastic differential equation for consumption. The equivalent representation is then the Fokker-Planck equation describing the probabilistic change in the consumption path. One interpretation of this Fokker-Planck equation is that in a dynamic, ever-changing world the probability density of consumption, captures the likelihood of consuming a particular consumption bundle in the next time period. The change in the likelihood of this consumption is in turn driven by changes income via changes in the composition of production.

However this explanation does not explain the relationship between preference evolution and the likelihood following a particular consumption path. Our starting point for this is the derivation of a von Neumann-Morgenstern expected utility model of consumption for the representative consume. Our main result is a representation theorem showing that the partial differential equation describing the change in utility with respect to time is equal to the Fokker-Planck equation for consumption.

Hence the Fokker-Planck equation for consumption provides a canonical representation for the evolution of preferences. In this representation, current consumption behaviour is conditioned on previous purchases. Our interpretation of preference evolution in this representation ascribes the change in current consumption behaviour as being due to changes in likelihood to consume, with respect to changes in key underlying variables. The change in preferences over time will therefore describe the likely path of consumption.

This path of consumption should be subject to boundary conditions on the feasible consumption set. These boundary conditions will be determined by physical, behavioural, economic and institutional constraints. In this paper we provide a characterization of the Fokker-Planck equation subject to different boundary conditions on the feasible consumption set. We explain how these boundary conditions can be used to place consumption constraints on the consumer. In doing this the possible impact on economic growth of behavioural phenomenon like fashion cycles or the impact of resource constraints on consumption and growth.

In this paper we use the boundary conditions that are proposed by Merton (1975) to secure a unique solution for the stationary distribution for the Fokker-Planck equation of the capital-labour ratio. As is shown by Merton (1975) this asymptotic distribution on the capital-labour ratio can be used to provide a solution to the stochastic Ramsey problem. Moreover, because the solution to the Fokker-Planck equation is a steady state solution, it does not depend on time. As a consequence, the dynamic optimal savings problem can be solved using static optimisation techniques.

We show that the steady state solution of the Fokker-Planck equation, when substituted into the infinite horizon Ramsey model, gives an identical solution to that provided by Merton (1975). This leads to our conjecture that the consumption/savings ascribed by the Ramsey (1928) model under the Solow-Merton exogenous growth dynamics, is a special case of the consumption behaviour that can be described under our theory of growth with preference evolution.

## 2 Solow-Merton Model of Growth Under Uncertainty

We begin by deriving a one-sector stochastic Neo-Classical growth model. Production $Y(t) = F(K(t), N(t))$ at time $t \in [0, \infty)$ in the one-sector economy is modelled as a function of capital $K(t)$ and labour productivity $L(t)$. Production technology is defined by the production function $F(.,.) \in C^2(\mathbb{R}^2{}_+ \times [0, \infty))$. We follow Merton (1975) and assume the following:

i) $F_K, F_N > 0, F_{KK}, F_{NN} < 0, F_{KN} > 0,$

ii) $F(l K, l N) = l F(K, N), \quad l > 0.$

Based on Merton (1975), we assume that the labour force $L(t)$ consists of a finite number of individual workers, $i = 1,..., N(t)$. A branching process is used to model labour productivity of each worker

$$L_i(t+h) = nh + s\,h(t;h) + n_i e_i(t;h), \quad i = 1,..., N(t), t \in (0, \infty]$$

where $n$, $s$ and $n_i$ are constants and $h(t;h)$ and $e_i(t;h)$ are random variables. We assume that $E_t(h) = E_t(n_i) = 0$ and $E_t(h^2) = E_t(n_i{}^2) = h$ and that

$$E_t(hn_i) = E_t(n_i n_j) = 0, \quad i \neq j, i = 1,..., m.$$

and $h(t;h)$ is covariance stationary

$$E_t(h(t;h)\,h(t+kh;h)) = 0, \quad k = 1,2,....$$

The stochastic differential equation for labour productivity can be derived as follows

$$N(t+h) - N(t) = \sum_{i=1}^{N(t)} L_i(t+h) = N(t)(nh + s\,h(t;h)) + \sum_{i=1}^{N(t)} n_i e_i(t;h), \quad t \in (0, \infty].$$

The conditional mean and variance for this equation is given as follows:

$$E_t[N(t+h) - N(t)] = N(t)\,nh$$

and

$$\mathrm{var}_t[N(t+h) - N(t)] = \left(N(t)^2 s^2 + N(t)\left(\sum_{i=1}^{m} n_i{}^2 \Big/ N(t)\right)\right)h, \quad t \in (0, \infty].$$

We assume that the $n_i$ are bounded and approximately the same size so that $\sum_{i=1}^{N(t)} n_i{}^2 \big/ N(t) \doteq O(1)$.

We can see that under these assumptions labour productivity can be approximated by the stochastic differential equation (SDE)

$$dN(t) = N(t)(ndt + s\,dW(t)),$$

where $n$ is the deterministic trend component, $s$ governs the rate of diffusion and the noise term $W(t)$ is a Wiener process $dW(t) \sim N(0, dt)$, $t \in [0, \infty)$.

We note that under this framework, a number of alternative distributions could be used to model the branching process governing individual worker productivity. Two possible examples that could be used are a Poisson birth-death-migration process and an Ehrenberg urn process. The Poisson birth-death-migration process would capture the churning effects due to job creation and destruction and worker migration. The urn process would model path dependence in the hiring process of firms. Both of these processes could be used to provide a new type of endogenous growth theory.

With respect to the assumptions we have employed in this paper, our interpretation of the behaviour of the branching process is as follows. This branching process for individual labour productivity may be interpreted as follows. Labour is embodied with human capital. If labour is "hardwired'' to imagine particular commodities, new commodities can only be generated if labour somehow changes in order to imagine novel products. Hence labour productivity will vary over time for each individual and we can associate each individual with a particular skill set and ability to imagine technological production possibilities

We model the dynamics of capital accumulation as follows

$$dK(t) = \left(I(t) + d\ K(t)\right)\ dt, \quad t \in [0, \infty)$$

where $I(t)\,dt$ is the change in the stock of investment and $d\ K(t)\,dt$ is the depreciation on capital stock.

As the production function $F$ is linear homogeneous, per capita output can be defined as

$$y(t) = \frac{1}{N(t)} F\left(K(t), N(t)\right) = f\left(k(t)\right), \quad t \in [0, \infty)$$

where $k(t) = K(t)/N(t)$ is the capital labour ratio. The stochastic dynamics of $k(t)$, $t \in [0, \infty)$ can be derived using Ito's lemma to yield the following SDE

$$dk(t) = \left(sf\left(k(t)\right) - \left((d-n) - s^2\right)k(t)\right)dt + s\,k(t)\,dW(t), \quad t \in [0, \infty)$$

where $s \in (0,1]$ is the marginal propensity to save. This is the Solow-Merton growth equation. Here we let

$$m\left(k(t); s\right) = \left(sf\left(k(t)\right) - \left((d-n) - s^2\right)k(t)\right)$$

and

$$s\left(k(t); s\right) = s\,k(t), \quad s \in (0,1], t \in [0, \infty).$$

Per capita consumption is defined via the Keynesian demand identity as follows,

$$c(t) = c\left(k(t); s\right) = (1-s)\, f\left(k(t)\right), \quad t \in [0, \infty).$$

When taken together the vector process $\{(k_t, c_t); t \in [0,\infty)\}$ defines the feasible growth path for the economy. Combined with the initial conditions $(k_0, c_0)$, the vector process gives defines the admissible growth path subject to $\Pr\{k(0) = k_0\} > 0$.

## 3 A Canonical Representation for the Evolution of Preferences

In this section we derive a canonical representation for the evolution of preferences. This result gives equivalence between the time derivative of expected utility and the Fokker-Planck equation of the underlying stochastic process. Hence the Fokker-Planck equation for consumption literally gives the evolution of preferences in this stochastic economy.

We now state our result in the form of a representation theorem for preference evolution:

**Theorem 3.1.(Canonical Representation Theorem)** *The von Neumann-Morgenstern expected utility function for the representative agent is defined as follows:*

$$U(c(t)) = E_t\left[u(c(t))\right] = \int_{x\in C} u(x)\, p(c(t) = x)\, dx,$$

*where* $\Pr\{c(t) = x\} = p(c(t) = x)$, $x \in C$ *with* $C$ *as the feasible consumption set. If* $u(.) \in C^2(C \subset \mathbb{R}_+)$, *then*

$$\frac{\partial}{\partial t} U(c_t) = \frac{\partial}{\partial t}\int_{x\in C} u(x)\, p(c_t = x \mid c_s)\, dx$$

*is equal to the Fokker-Planck equation of the stochastic process* $\{c(t); t \in [0,\infty)\}$:

$$\frac{\partial}{\partial t} p(c_t = z \mid c_s) = -\left(\left(a(k(t);s)\, m(k(t);s) + \frac{1}{2} b(k(t);s)\, s^2(k(t);s)\right)\frac{\partial}{\partial z} p(c_t = z \mid c_s)\right.$$

$$\frac{1}{2}\left(a(k(t);s)\, s(k(t);s)\right)^2 \frac{\partial^2}{\partial z^2} p(c_t = z \mid c_s).$$

**Proof:** We begin by applying Ito's lemma to the equation for per capita consumption. We express this SDE for per capita consumption as follows:

$$dc(t) = \left(a(k(t);s)\, m(k(t);s) + \frac{1}{2} b(k(t);s)\, s^2(k(t);s)\right) dt$$

$$+ a(k(t);s)\, s(k(t);s)\, dW(t), \quad t \in [0,\infty).$$

The coefficients in this SDE are respectively defined as

$$a(k(t),q) = \left(\left(1 - s(k(t),q)\right) f'(k(t)) - f(k(t))\, s_k\right)$$

and

$$b(k(t),q) = \left(\left(1 - s(k(t),q)\right) f''(k(t)) - \left(2 f'(k(t))\, s_{kk} + f(k(t))\, s_{kk}\right)\right).$$

We assume the existence of a representative agent. We define an equation for the representative agent's von Neumann-Morgenstern expected utility function as follows:

$$U\left(c\left(t\right)\right)=E_{t}\left[u\left(c\left(t\right)\right)\right]=\int_{x\in C}u\left(x\right)p\left(c\left(t\right)=x\right)dx,$$

where $\Pr\left\{c\left(t\right)=x\right\}=p\left(c\left(t\right)=x\right)$, $x\in C$ with $C$ as the feasible consumption set.

The differential equation describing the evolution of preferences can now be derived as follows:

$$\begin{aligned}\frac{\partial}{\partial t}\int_{x\in C}u\left(x\right)p\left(c_{t}=x\mid c_{s}\right)dx&=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{x\in C}u\left(x\right)\left\{p\left(c_{t+\Delta t}=x\mid c_{s}\right)-p\left(c_{t}=x\mid c_{s}\right)\right\}dx\\&=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{x\in C}\int_{z\in C}u\left(x\right)p\left(c_{t+\Delta t}=x\mid c_{t}=z\right)p\left(c_{t}=z\mid c_{s}\right)dzdx\right.\\&\qquad\left.-\int_{z\in C}u\left(z\right)p\left(c_{t}=z\mid c_{s}\right)dz\right],\end{aligned}$$

where $p\left(c_{t}=x\mid c_{s}\right)=\Pr\left\{c_{t}=x\mid c_{s}\right\}$, $x\in C$, is the condition probability of consuming $c_{t}=x$ at time *t* given that $c_{s}$ was consumed at time *s*, $s\le t$.

We now divide the integral over $x$ into two regions, $\left|x-z\right|\ge e$ and $\left|x-z\right|<e$ We assume that the utility function $u\left(.\right)$ is twice continuously differentiable with respect to *c*. Hence for the region $\left|x-z\right|<e$ this allows us to construct a second-order Taylor expansion of $u\left(.\right)$ with respect to *z*

$$u\left(x\right)=u\left(z\right)+u'\left(z\right)\left(x-z\right)+u''\left(z\right)\left(x-z\right)^{2}+\left|x-z\right|^{2}R\left(x,z\right),$$

where $R\left(x,z\right)$ denotes the higher order remainder term. We note that $\left|R\left(x,z\right)\right|\to 0$ as $\left|x-z\right|\to 0$.

Substituting this into the preference evolution equation, we arrive at the following expanded version of this equation:

$$\begin{aligned}\frac{\partial}{\partial t}\int_{x\in C}u\left(x\right)p\left(c_{t}=x\mid c_{s}\right)dx=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\Bigg[&\iint_{\left|x-z\right|<e}\left(u'\left(z\right)\left(x-z\right)+u''\left(z\right)\left(x-z\right)^{2}\right)\\&\qquad\times p\left(c_{t+\Delta t}=x\mid c_{t}=z\right)p\left(c_{t}=z\mid c_{s}\right)dzdx\\&+\iint_{\left|x-z\right|<e}\left|x-z\right|^{2}R\left(x,z\right)p\left(c_{t+\Delta t}=x\mid c_{t}=z\right)p\left(c_{t}=z\mid c_{s}\right)dzdx\\&+\iint_{\left|x-z\right|<e}u\left(z\right)p\left(c_{t+\Delta t}=x\mid c_{t}=z\right)p\left(c_{t}=z\mid c_{s}\right)dzdx\\&+\iint_{\left|x-z\right|\ge e}u\left(x\right)p\left(c_{t+\Delta t}=x\mid c_{t}=z\right)p\left(c_{t}=z\mid c_{s}\right)dzdx\\&-\int_{z\in C}u\left(z\right)p\left(c_{t}=z\mid c_{s}\right)dz\Bigg]\end{aligned}$$

We note that by assuming uniform convergence we can take the limit inside the integral. As a consequence, we get the following

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\iint_{|x-z|<e} u'(z)(x-z)\,p\left(c_{t+\Delta t}=x\,|\,c_t=z\right)p\left(c_t=z\,|\,c_s\right)dzdx\right]$$

$$=\int u'(z)\left(a\left(k(t);s\right)m\left(k(t);s\right)+\frac{1}{2}b\left(k(t);s\right)s^{2}\left(k(t);s\right)\right)\frac{\partial}{\partial z}p\left(c_t=z\,|\,c_s\right)dz$$

and

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\iint_{|x-z|<e} u''(z)(x-z)^2\,p\left(c_{t+\Delta t}=x\,|\,c_t=z\right)p\left(c_t=z\,|\,c_s\right)dzdx\right]$$

$$=\frac{1}{2}\int u''(z)\left(a\left(k(t);s\right)s\left(k(t);s\right)\right)^2\frac{\partial^2}{\partial z^2}p\left(c_t=z\,|\,c_s\right)dz.$$

We also note that the remainder term on the third line of the consumption evolution equation vanishes as $e\to 0$ because

$$\left|\lim_{\Delta t\to 0}\frac{1}{\Delta t}\iint_{|x-z|<e}|x-z|^2\,R(x,z)\,p\left(c_{t+\Delta t}=x\,|\,c_t=z\right)p\left(c_t=z\,|\,c_s\right)dzdx\right|$$

$$\leq\left[\lim_{\Delta t\to 0}\frac{1}{\Delta t}\iint_{|x-z|<e}|x-z|^2\,p\left(c_{t+\Delta t}=x\,|\,c_t=z\right)dzdx\right]\max_{|x-z|<e}\left|R(x,z)\right|\to\left(b(k,t)+O(e)\right)\max_{|x-z|<e}\left|R(x,z)\right|$$

The remaining three lines of the expanded preference evolution equation can now be given as

$$\int_{z\in C}u(z)\,dz\int_{x\in C}\left(w(x\,|\,z)\,p(x\,|\,c_s)-w(x\,|\,z)\,p(z\,|\,c_s)\right)dzdx.$$

Upon integrating by parts, the preference evolution equation then reduces to

$$\frac{\partial}{\partial t}\int_{x\in C}u(x)\,p\left(c_t=x\,|\,c_s\right)dx=\int_{x\in C}\left(\left(a\left(k(t);s\right)m\left(k(t);s\right)+\frac{1}{2}b\left(k(t);s\right)s^{2}\left(k(t);s\right)\right)\frac{\partial}{\partial z}u(z)\right.$$

$$\left.-\frac{1}{2}\left(a\left(k(t);s\right)s\left(k(t);s\right)\right)^2\frac{\partial^2}{\partial z^2}u(z)\right)p\left(c_t=z\,|\,c_s\right)dz$$

$$+\int_{z\in C}u(z)\,dz\int_{x\in C}\left(w(x\,|\,z)\,p(x\,|\,c_s)-w(x\,|\,z)\,p(z\,|\,c_s)\right)dzdx.$$

We then differentiate the preference evolution equation with respect to *z* and cancel the utility functions. Following the terminology of Gardiner (1985), we then arrive at a differentiable Chapman-Kolmogorov equation

$$\frac{\partial}{\partial t} p\left(c_t = z \mid c_s\right) = -\int_{x \in \mathrm{C}} \left( \left( a\left(k(t);s\right) m\left(k(t);s\right) + \frac{1}{2} b\left(k(t);s\right) s^{2}\left(k(t);s\right) \right) \frac{\partial}{\partial z} p\left(c_t = z \mid c_s\right) \right.$$

$$+ \frac{1}{2} \left( a\left(k(t);s\right) s\left(k(t);s\right) \right)^{2} \frac{\partial^2}{\partial z^2} p\left(c_t = z \mid c_s\right)$$

$$+ \int_{x \in \mathrm{C}} \left( w(z \mid x)\, p\left(x \mid c_s\right) - w(x \mid z)\, p\left(z \mid c_s\right) \right) dx.$$

The third line of the differentiable Chapman-Kolmogorov equation is a Martingale difference equation and as the noise $w$ is Brownian, by definition this will be equal to zero

$$\int_{x \in \mathrm{C}} \left( w(z \mid x)\, p\left(x \mid c_s\right) - w(x \mid z)\, p\left(z \mid c_s\right) \right) dx = 0.$$

Hence we arrive at the Fokker-Planck equation for consumption

$$\frac{\partial}{\partial t} p\left(c_t = z \mid c_s\right) = -\left( \left( a\left(k(t);s\right) m\left(k(t);s\right) + \frac{1}{2} b\left(k(t);s\right) s^{2}\left(k(t);s\right) \right) \frac{\partial}{\partial z} p\left(c_t = z \mid c_s\right) \right.$$

$$\frac{1}{2} \left( a\left(k(t);s\right) s\left(k(t);s\right) \right)^{2} \frac{\partial^2}{\partial z^2} p\left(c_t = z \mid c_s\right).$$

?

**4. Boundary and Barrier Conditions on the Feasible Consumption Set**

We recall that for the definition of the von Neumann-Morgenstern utility function we defined a feasible consumption set

$$x \in \mathrm{C} \subset \mathbb{R}_+.$$

This set defines a region of state space that is determined by cultural, legal, economic physical and behavioural constraints. In other words, "You can only consume so much."

Given that this is the important question to ask is "How much do we consume?" The nature of the barriers on the consumption set will determine this. Hence the barriers on the consumption set will determine steady state preferences.

The Fokker-Planck equation for consumption is expressed as

$$\frac{\partial}{\partial t} p\left(c_t = z \mid c_s\right) = -\left( \left( a\left(k(t);s\right) m\left(k(t);s\right) + \frac{1}{2} b\left(k(t);s\right) s^{2}\left(k(t);s\right) \right) \frac{\partial}{\partial z} p\left(c_t = z \mid c_s\right) \right.$$

$$\frac{1}{2} \left( a\left(k(t);s\right) s\left(k(t);s\right) \right)^{2} \frac{\partial^2}{\partial z^2} p\left(c_t = z \mid c_s\right).$$

We note that Fokker-Planck equation can also be written as

$$\frac{\partial}{\partial t} p\left(c_t = z \mid c_s\right) + \frac{\partial}{\partial z} V\left(z,t\right) = 0$$

where

$$V(z,t) = \left(\left(a\left(k(t);s\right)m\left(k(t);s\right) + \frac{1}{2}b\left(k(t);s\right)s^2\left(k(t);s\right)\right)p\left(c_t = z \mid c_s\right)\right.$$
$$\left. - \frac{1}{2}\left(a\left(k(t);s\right)s\left(k(t);s\right)\right)^2 \frac{\partial}{\partial z} p\left(c_t = z \mid c_s\right)\right).$$

is a potential difference equation.

The integral form of the Fokker-Planck equation is equivalent to a surface integral on the feasible consumption set $\mathsf{C}$ (see Gardiner 1985, p.119)

$$\frac{\partial}{\partial t} p\left(R \mid c_s\right) = -\int_{x \in S} n \cdot V\left(z,t\right) dS ,$$

where $S$ is the boundary set of the set $R \subset \mathsf{C}$ and $n \perp S$. Hence another way of understanding the Fokker-Planck equation is as a net flow of probability across the consumption set capturing the evolution of preferences.

Some examples of boundary conditions that would be of interest to economics are now given.

**Example 4.1. (Reflecting Barrier)** *Consider the situation where z cannot leave a region* $R \subset \mathsf{C}$. *Let S define the barrier. Thus we require that*

$$n \cdot V\left(z,t\right) = 0, \quad \forall z \in S, n \perp S.$$

*This would the case where there was a budget constraint restricting inter-temporal consumption. In this case, the orthogonal vector n gives the vector of prices. A similar condition to this underlies the selection of prices in static consumer choice theory. Alternatively, a reflecting barrier could be used to find a minimum sustainable consumption level.*

**Example 4.2. (Absorbing Barrier)** *In this situation, when z reaches the barrier S, it is removed from the consumption set, i.e.*

$$p\left(z,t\right) = 0, \quad \forall z \in S .$$

*This would be the when the bundle of goods is outside the consumption space.*

**Example 4.3. (Periodic Barrier)** *This is the case of periodic consumption cycles. A typical example is seasonal fluctuations associated with seasonal goods. A subtler example is fad goods with long period cycles, such as the yo-yo. These boundary conditions are expressed as follows*:

$$\text{i).} \lim_{x \to b^-} p\left(x,t\right) = \lim_{x \to a^+} p\left(x,t\right)$$
$$\text{ii).} \lim_{x \to b^-} V\left(x,t\right) = \lim_{x \to a^+} V\left(x,t\right),$$

where $\left[a,b\right] \subset \mathsf{C}$ define the interval on which the boundary is defined.

## 5. Evolution of Preferences and the Ramsey Model of Optimal Savings

As in the previous section, we define the expected utility derived from consumption at time $t$ by

$$U\left(c\left(t\right)\right)=E\left[u\left(c\left(t\right)\right)\right]=\int_{x\in C}u\left(x\right)p\left(c_t=x\right)dx,$$

where $u'\left(c\right)\geq 0$ and $u''\left(c\right)<0$ for all $c\in C$. The stochastic Ramsey problem (Ramsey 1929, Merton 1975) can be stated as follows:

$$J\left[k\left(t\right),t,T\right]=\max_{c(t)\in C}E_0\left[\int_0^T U\left(c\left(k\left(t\right);s\right)\right)dt\right],$$

subject to a budget constraint

$$dk\left(t\right)=\left(f\left(k\left(t\right)\right)-\left(\left(d-n\right)-s^2\right)k\left(t\right)-c\left(t\right)\right)dt+s\,k\left(t\right)dW\left(t\right),$$

where $k\left(t\right)>0$ for all $t\in\left[0,T\right)$ and the initial and boundary conditions: $k\left(0\right)=k_0>0$ and $k\left(T\right)\geq 0$ with probability 1 respectively.

Following Merton (1975) we solve this problem using the Bellman's equation from dynamic programming. The derivation of the Bellman function for this optimisation problem is given as follows:

$$\begin{aligned}J\left[k,t,T\right]&=\max_{c(t)\in C}E_t\left[\int_t^{t+\Delta t}U\left(c\left(k\left(t\right),s\right)\right)dt\right]+\max_{c(t)\in C}E_{t+\Delta t}\left[\int_{t+\Delta t}^{T}U\left(c\left(k\left(t\right);s\right)\right)dt\right]\\&=\max_{c(t)\in C}\left\{E_t\left[U\left(c\left(k\left(t\right),s\right)\right)\Delta t\right]+J\left[k,t+\Delta t,T\right]\right\}.\end{aligned}$$

At this point, we would like to focus the reader's attention on the second term in the last line of the Bellman equation. We can Taylor expand this equation with respect to $t$ as follows:

$$J\left[k,t+\Delta t,T\right]=J\left[k,t,T\right]+J_k\Delta k+J_t\Delta t+\frac{1}{2}\left(J_{kk}\left(\Delta k\right)^2+J_{tt}\left(\Delta t\right)^2\right)+J_{kt}\Delta k\Delta t+O\left(\Delta t\right),$$

where

$$\Delta k\left(t\right)=s\left(k_t,q\right)f\left(k_t\right)-\left(\left(d-n\right)-s^2\right)k\left(t\right)\Delta t+s\,k_t\Delta W\left(t\right)$$

and $\Delta W\left(t\right)\sim N\left(0,\Delta t\right)$. Upon substitution of this term into $J\left[k,t,T\right]$, we arrive at the following equation

$$\begin{aligned}J\left[k,t,T\right]=\max_{c(t)\in C}\Big\{E_t\Big[&U\left(c\left(k\left(t\right),s\right)\right)e^{-dt}\Delta t+J\left[k,t,T\right]+J_k\Delta k\\&+J_t\Delta t+\frac{1}{2}\left(J_{kk}\left(\Delta k\right)^2+J_{tt}\left(\Delta t\right)^2\right)J_{kt}\Delta k\Delta t+O\left(\Delta t\right)\Big]\Big\}.\end{aligned}$$

Upon rearranging this equation, we arrive at the representation of the Bellman optimality condition

$$0=\max_{c(t)\in C}E_t\left[U\left(c\left(k_t,s\right)\right)e^{-dt}\Delta t+J_c\Delta c+J_t\Delta t+\frac{1}{2}\left(J_{cc}\left(\Delta c\right)^2+J_{tt}\left(\Delta t\right)^2\right)+J_{ct}\Delta c\Delta t+O\left(\Delta t\right)\right].$$

We now take the limit of the Bellman optimality condition as $\Delta t \to 0$:

$$0 = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \max_{c(t)\in C} E_t \left[ U\left(c\left(k(t), s\right)\right) e^{-dt} \Delta t + J_k \Delta k + J_t \Delta t + \frac{1}{2}\left( J_{kk} (\Delta k)^2 + J_{tt} (\Delta t)^2 \right) + J_{kt} \Delta k \Delta t + O(\Delta t) \right]$$

$$= \lim_{\Delta t \to 0} \max_{c(t)\in C} \left\{ E_t \left[ U\left(c\left(k(t), s\right)\right) e^{-dt} \right] + J_t + J_k \frac{1}{\Delta t} E_t \left[ k(t+\Delta t) - k(t) \right] \right.$$

$$\left. + \frac{1}{2}\left( J_{kk} \frac{1}{\Delta t} E_t \left[ k(t+\Delta t) - k(t) \right]^2 + J_{tt} (\Delta t) \right) + J_{kt} E_t \left[ k(t+\Delta t) - k(t) \right] + O(\Delta t) \right\}$$

$$= \max_{c(t)\in C} \left\{ E_t \left[ U\left(c\left(k(t), s\right)\right) e^{-dt} \right] + J_t + J_k \lim_{\Delta t \to 0} \frac{1}{\Delta t} E_t \left[ k(t+\Delta t) - k(t) \right] \right.$$

$$\left. + \frac{1}{2} J_{kk} \left( \lim_{\Delta t \to 0} \frac{1}{\Delta t} E_t \left[ k(t+\Delta t) - k(t) \right]^2 \right) \right\}.$$

Upon rearranging the last two terms in this equation HJB equation, we arrive at the following optimality condition

$$-J_t = \max_{c\in C} \left\{ U\left(c\left(k(t), s\right)\right) + J_k m\left(k(t), t\right) + \frac{1}{2} s^2\left(k(t), t\right) J_{kk} \right\},$$

where

$$m(k,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E_t \left[ k(t+\Delta t) - k(t) \right] = s\left(k(t), s\right) f\left(k(t)\right) - \left((d-n) - s^2\right) k(t)$$

and

$$s^2(k,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E_t \left[ k(t+\Delta t) - k(t) \right]^2 = \left( s\, k(t) \right)^2.$$

To obtain the dynamic programming solution to the stochastic Ramsey problem, we the first order necessary condition is give by

$$U'\left(c\left(k^*, s^*\right)\right) - J_k = 0,$$

where $U'(c) = dU/dc$ and $s^* = s\left(k^*, T-t\right)$ is the optimal savings rate as a function of $k^*$ and $T-t$.

In order to solve for $s^*$, we substitute the first order condition back into the HJB equation, to obtain the following PDE

$$J_t = -\left\{ U\left(c\left(k^*, s^*\right)\right) + m\left(k^*, t\right) U'\left(c\left(k^*, s^*\right)\right) + \frac{1}{2} s^2\left(k^*, t\right) J_{kk} \right\}.$$

We then solve this PDE to obtain a solution for $k^*$ which is then substituted back into the first order condition to solve for $s^*$.

Following Merton (1975), we now examine the limiting case of the Ramsey problem where $T \to \infty$. As the stochastic process $k(t)$ is time homogenous and $U(.)$ is not time dependent, we have that

$$J_t = -U\left(1 - s^*(k, T-t) f\left(k(T-t)\right)\right).$$

Mirrlees (1965, 1973) has shown that for an optimal policy to exists under uncertainty, $f(.)$ must satisfy the Inada conditions and $((d-n)-s^2)>0$. If this is the case then

$$\lim_{T\to\infty} s^*(k,T-t)=s^*(k)$$

and, as Merton (1975) as shown, there exists an associated steady state distribution for $k$. This implies that

$$\lim_{T\to\infty} J_t = -U\left(1-s^*(k)\,f(k)\right)=-B,$$

where $B$ is the "bliss-point".

Substituting this into the HJB equation, we have that $J$ must satisfy the following ODE as $T\to\infty$

$$0=U\left(1-s^*(k)\,f(k)\right)-B+J'm\left(k,s^*\right)+\frac{1}{2}s^2k^2J'',$$

where

$$m\left(k,s^*\right)=s^*f(k)-\left((d-n)-s^2\right)k$$

and the primes denote derivatives with respect to $k^2$. The first order condition is given by

$$J''=U''\left(1-s^*f(k)\right)\left(\left(1-s^*\right)f'(k)+\frac{\partial s^*}{\partial k}f(k)\right).$$

Substituting for $J'$ and $J''$ in the above ODE, we now arrive at the following ODE

$$0=\left(\tfrac{1}{2}s^2k^2fU''\right)\frac{ds^*}{dk}+\left(fU'-\tfrac{1}{2}s^2k^2U''f'\right)s^*+\tfrac{1}{2}s^2k^2U''f'-U'bk+U-B,$$

where the coefficient $b=n-d-s^2$. We can solve this ODE to get $s^*$. We note that under certainty (i.e. when $s^2=0$) this ODE reduces to

$$\left(s^*f-(n-d)k\right)=\frac{B-U}{U'}$$

which is the "Ramsey Rule" for optimal savings.

**6. Conclusion**

In this paper, we set out to show how changes in consumer behaviour can impact on economic growth. Our question differs from the standard approach that has been used by growth theory is embedded in a tradition of supply-side change impacting on demand. In both the Neo-Classical and endogenous growth literatures, preferences over consumables are taken to be stationary over time.

In contrast, we begin by providing a theorem for preference evolution in terms of the Fokker-Planck equation for consumption. We also showed that by changing the boundary conditions on the Fokker-Planck equation we could account for different demand behaviour like the impact of fashion cycles on preference formation.

Furthermore, we showed that by employing the boundary conditions used by Merton (1975) we are able to derive the steady-state distribution for consumption that he used to predict the outcome the infinite horizon Ramsey savings model. Hence our model of preference evolution and economic growth encompasses the Merton's model Neo-Classical growth under uncertainty.

In addition, by changing the type of noise exhibited by the labour dynamics, we capture the types of growth patterns due to other types of production/factor relationships. Hence, we can drive the growth dynamics for other models attributable to various patterns of job creation and destruction or path dependence in hiring. This could lead to a new endogenous growth theory based on the microstructure of the economy via the Theory of the Firm.

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