Unemployment and Inventories in the Business Cycle*

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Abstract

This paper investigates the role of inventories in the business cycle. The macroeconomic model is based on an overlapping-generations non-tâtonnement approach involving temporary equilibria with stochastic rationing in each period and price adjustment between successive periods. The presence of inventories reinforces the importance of spill-over effects between the goods market and the labor market. Starting from a stationary Walrasian equilibrium it is possible that the economy converges to a quasi-stationary Keynesian unemployment state following a restrictive monetary shock. Contrary to conventional wisdom, this is favored by sufficient downward flexibility of the nominal wage. Thus in that case money is non-neutral in the long run. Vice versa, to best avoid or overcome permanent unemployment, the nominal wage rate should be rigid downwards.

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1. Introduction

This paper builds on two previous articles (Bignami, Colombo and Weinrich, 2000 and Colombo and Weinrich, 2000) in which discrete-time dynamic non-tâtonnement macroeconomic models are analyzed. Within each period prices are fixed and an allocation is obtained by means of temporary equilibrium with rationing whereas prices are adjusted between successive periods according to the strength of rationing or disequilibrium on each market in the previous period.

The main cumulative results of the two papers are as follows. (i) Depending on the choice of parameter values there exist complex business cycles with consistent spells of Keynesian underemployment states followed by a phase of alternation between Classical underemployment and inflationary states followed by consistent spells of Keynesian underemployment states etc.; thus there is a structural regularity regarding the emergence of business cycles but the precise shape, size and length of these endogenous cycles is varying and unpredictable over time. (ii) There exist quasi-stationary states with permanent unemployment or permanent capacity underutilization, in spite of price adjustment between periods. (iii) There exists a Phillips Curve as (the image of) a chaotic attractor (of the state variables). Thus the Phillips Curve is a true long-run phenomenon but it is difficult to use it as a policy instrument.

A specific simplifying assumption in the above models is that there are no inventories, i.e. produced goods are perishable and thus cannot be transferred from one period to the next. The technical implication is that of the four conceptually possible equilibrium regimes only three (Keynesian underemployment, Classical underemployment, Repressed Inflation) actually occur, while states of Underconsumption are never observed.

In the present paper this assumption is abandoned, i.e. inventories are possible and stored goods may be sold in periods subsequent to the period of their production. This increases the dimensionality of the dynamical system from three to four.

The main features of the simpler models carry over to the model with inventories. In particular there exists, for any given set of parameter values and values of the state variables, a unique temporary equilibrium allocation. However, Underconsumption states are now a real possibility. We derive the partition of the price-wage-plane into the different equilibrium regimes. The Underconsumption regime now has positive measure. Moreover, the border line between the Underconsumption regime and the adjacent regime of Keynesian underemployment is positively sloped whereas the one between Underconsumption and Repressed Inflation may have positive or negative slope, depending on the size of stocks carried over from the previous period: for small stocks the borderline is upward sloping whereas for large stocks it slopes downward. Since stocks are an endogenous state variable, this means that the sign of the slope of the borderline between Underconsumption and Repressed Inflation may vary along trajectories. This increases further the complexity of the behavior of the dynamic system.

Since the dynamical system of the extended model is more complicated than the one of the simpler models without inventories, numerical simulations can produce all types of dynamic phenomena observed in the simpler models, including the emergence of the Phillips curve as an attractor. However, the presence of inventories allows us also to obtain new results. Specifically, due to the fact that excess supplies on the goods market are transferred through inventories to the next period, labor demand is more adversely affected than is the case without inventories. If the nominal wage rate is sufficiently flexible downwards, this fact implies that real wages and hence labor incomes fall. In turn this may result, after a restrictive monetary shock, in a permanent state of (Keynesian) unemployment. Moreover, and contrary to conventional thinking, downward rigidity of the nominal wage is a good recipe to avoid or overcome permanent underemployment.

The paper is organized as follows. In section two we present the model and describe the behavior of consumers, producers and the government. Section three focuses on temporary equilibria with rationing and proves the existence and uniqueness of equilibrium allocations. In section four we provide a representation of possible equilibrium regimes and in section five we set up the dynamic system. Section six presents numerical simulations and the main results mentioned above. Section seven concludes while the proofs of some technical results and the complete dynamic system are given in the appendices.

2. The Model

We consider an economy in which there are n OLG-consumers, n' firms and a government. Consumers offer labor inelastically when young and consume a composite consumption good in both periods. That good is produced by firms using an atemporal production function whose only input is labor. The government levies a proportional tax on firms' profits to finance its expenditure for goods. Nevertheless, budget deficits and surpluses may arise and are made possible through money creation or destruction.

2.1. Timing of the Model

The time structure of the model is depicted in Figure 2.1. In period $t-1$ producers obtain an aggregate profit of Π_{t-1} which is distributed at the beginning of period t in part as tax to the government ($tax\Pi_{t-1}$) and in part to young consumers ($(1-tax)\,\Pi_{t-1}$), where $0 \leq tax \leq 1$. Also at the beginning of period t old consumers hold a total quantity of money M_t, consisting of savings generated in period $t-1$. Thus households use money as a means of transfer of purchasing power between periods.

Let X_t denote the aggregate quantity of the good purchased by young consumers in period t, p_t its price, w_t the nominal wage and L_t the aggregate quantity of labor. Then

$$M_{t+1} = (1-tax)\,\Pi_{t-1} + w_t L_t - p_t X_t.$$

Denoting with G the quantity of goods purchased by the government and taking into account that old households want to consume all their money holdings in period t, the aggregate consumption of young and old households and the government is $Y_t = X_t + \frac{M_t}{p_t} + G$. Using that $\Pi_t = p_t Y_t - w_t L_t$, considering $\Pi_t - \Pi_{t-1} = \Delta M_t^P$ as the variation in the money stock held by producers before they distribute profits and denoting with $\Delta M_t^C = M_{t+1} - M_t$ the one referring to consumers, we obtain the usual accounting identity, i.e. $\Delta M_t^C + \Delta M_t^P = p_t G - tax\Pi_{t-1} =$ budget deficit.

Denoting with S_t the aggregate amount of inventories carried over by firms to period t and with Y_t^p the aggregate amount of goods produced in period t, there results $S_{t+1} = Y_t^p + S_t - Y_t$.

2.2. The Consumption Sector

In his first period of life each consumer born at t is endowed with labor ℓ^s and an amount of money $(1-tax)\,\Pi_{t-1}/n$ while his preferences are described by a utility function $u\,(x_t, x_{t+1})$. In taking any decision the young consumer has to meet the constraints

$$0 \leq x_t \leq \omega_t^i, 0 \leq x_{t+1} \leq \left(\omega_t^i - x_t\right) \frac{p_t}{p_{t+1}}, i = 0, 1 \tag{2.1}$$

where

$$\omega_t^1 = \frac{1-tax}{p_t} \frac{\Pi_{t-1}}{n} + \frac{w_t}{p_t} \ell^s$$

denotes his real wealth when he is employed and

$$\omega_t^0 = \frac{1-tax}{p_t} \frac{\Pi_{t-1}}{n}$$

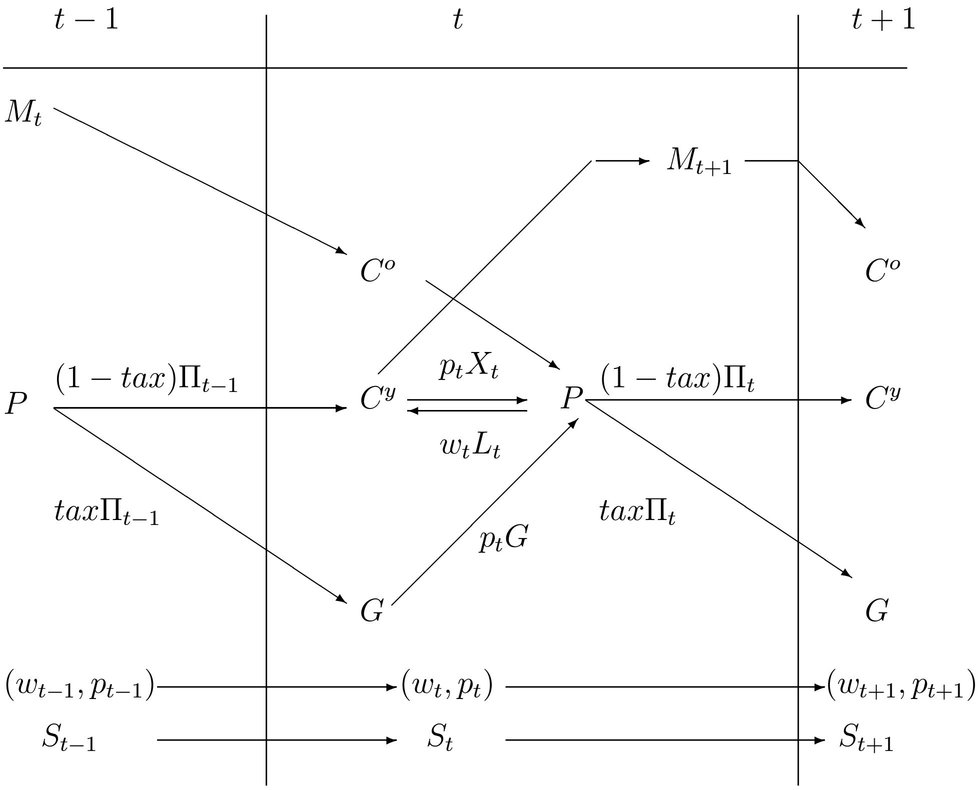

Figure 2.1: The time structure of the model

when he is unemployed. Implicit in this is the assumption that rationing on the labor market is of type all-or-nothing and that the labor market is visited before the goods market.

Regarding the goods market the young household may be rationed according to the stochastic rule

$$x_t = \begin{cases} x_t^d & \text{with prob.} \quad \rho\gamma_t^d \\ c_t x_t^d & \text{with prob.} \ 1 - \rho\gamma_t^d \end{cases}$$

where x_t^d is the quantity demanded, $\rho \in [0,1]$ a fixed structural parameter of the rationing mechanism, $\gamma_t^d \in [0,1]$ a rationing coefficient which the household perceives as given but which will be determined in equilibrium and

$$c_t = \frac{\gamma_t^d - \rho\gamma_t^d}{1 - \rho\gamma_t^d}.$$

These settings are chosen such that the expected value of x_t is $\gamma_t^d x_t^d$, that is, expected rationing is proportional and hence manipulable.[1]

Denoting with $\theta_t^e = p_{t+1}^e / p_t$ the expected relative price for period t, the effective demand $x_t^{di}, i = 0, 1$, is obtained by solving the agent's expected utility maximization problem

$$\max_{x_t} \rho\gamma_t^d u\left(x_t, \frac{\omega_t^i - x_t}{\theta_t^e}\right) + \left(1 - \rho\gamma_t^d\right) u\left(c_t x_t, \frac{\omega_t^i - c_t x_t}{\theta_t^e}\right)$$

subject to the constraints (2.1). The resulting first-order condition yields

$$\frac{\rho u_1\left(x_t, \frac{\omega_t^i - x_t}{\theta_t^e}\right) + (1-\rho)\, u_1\left(c_t x_t, \frac{\omega_t^i - c_t x_t}{\theta_t^e}\right)}{\rho u_2\left(x_t, \frac{\omega_t^i - x_t}{\theta_t^e}\right) + (1-\rho)\, u_2\left(c_t x_t, \frac{\omega_t^i - c_t x_t}{\theta_t^e}\right)} = \frac{1}{\theta_t^e}. \tag{2.2}$$

For a generic utility function it is hard to solve this equation for x_t but it is possible under the following assumption:

(A1) $u(x_t, x_{t+1}) = x_t^h x_{t+1}^{1-h}$ and $\rho = 1$ (i.e. 0/1-rationing).

[1] As has been shown by Green [1980] and Weinrich [1982], in case of rationing where the quantity signals are given by means of the aggregate values of demand and supply, the only mechanisms compatible with equilibrium are those for which the expected realization is proportional to the transaction offer.

In this case we can prove that $x_t^{di} = h\omega_t^i, i = 0, 1$ (Lemma 1 in Appendix 1). In particular the young consumer's effective demand is independent of both γ_t^d and p_{t+1}^e.

The aggregate supply of labor is $L^s = n\ell^s$. Denoting with L_t^d the aggregate demand of labor and with $\lambda_t^s = \min\left\{\frac{L_t^d}{L^s}, 1\right\}$ the fraction of young consumers that will be employed, the aggregate demand of goods of young consumers is

$$\begin{aligned} X_t^d &= \lambda_t^s n x_t^{d1} + (1 - \lambda_t^s)\, n x_t^{d0} \\ &= h\,(1 - tax)\,\frac{\Pi_{t-1}}{p_t} + h\frac{w_t}{p_t}\lambda_t^s L^s \equiv X^d\left(\lambda_t^s; \frac{w_t}{p_t}, \frac{(1 - tax)\,\Pi_{t-1}}{p_t}\right). \quad (2.3) \end{aligned}$$

The total aggregate demand of the consumption sector is then obtained by adding old consumers' aggregate demand M_t/p_t and government demand G:

$$Y_t^d = X^d\left(\lambda_t^s; \alpha_t, (1 - tax)\,\pi_t\right) + m_t + G_t$$

where $\alpha_t \equiv w_t/p_t, \pi_t \equiv \Pi_{t-1}/p_t$ and $m_t \equiv M_t/p_t$.

2.3. The Production Sector

Each of the n' identical firms uses an atemporal production function $y_t^p = f(\ell_t)$. Having transferred stocks from the previous period and being thus endowed with inventories s_t at the beginning of period t, the total amount supplied by a firm is $y_t^s = y_t^p + s_t$. As with consumers, firms too may be rationed, by means of a rationing mechanism analogue to that assumed for the consumption sector.

Denoting the single firm's effective demand of labor by ℓ_t^d, the quantity of labor effectively transacted is

$$\ell_t = \begin{cases} \ell_t^d, & \text{with prob. } \lambda_t^d \\ 0, & \text{with prob. } 1 - \lambda_t^d \end{cases}$$

where $\lambda_t^d \in [0, 1]$. It is obvious that $E\ell_t = \lambda_t^d \ell_t^d$. On the goods market the rationing rule is assumed to be

$$y_t = \begin{cases} y_t^s, & \text{with prob. } \sigma\gamma_t^s \\ d_t y_t^s, & \text{with prob. } 1 - \sigma\gamma_t^s \end{cases},$$

where $\sigma \in (0, 1), \gamma_t^s \in [0, 1]$ and $d_t = (\gamma_t^s - \sigma\gamma_t^s) / (1 - \sigma\gamma_t^s)$. σ is a fixed parameter of the mechanism whereas λ_t^d and γ_t^s are perceived rationing coefficients taken as given by the firm the effective value of which will be determined in equilibrium. The definition of d_t implies that $Ey_t = \gamma_t^s y_t^s$; in particular it is independent of σ. It is obvious that $E\ell_t = \lambda_t^d \ell_t^d$.

The firm's effective demand $\ell_t^d = \ell^d(\gamma_t^s; \alpha_t)$ is obtained from the expected profit maximization problem

$$\max_{\ell_t^d} \gamma_t^s \left[f\left(\ell_t^d\right) + s_t \right] - \alpha_t \ell_t^d$$

subject to

$$0 \leq \ell_t^d \leq \frac{d_t}{\alpha_t} \left[f\left(\ell_t^d\right) + s_t \right]$$

while its effective supply is $y_t^s = f\left(\ell_t^d\right) + s_t$. The upper bound on labor demand reflects the fact that the firm must be prepared to finance labor service purchases even if rationed on the goods market (since the labor market is visited first it will know whether it is rationed on the goods market only after it has hired labor). In general the solution depends on this constraint but it is not binding (Appendix, Lemma 2) if we make the following assumption:

(A2) $f(\ell) = a\ell^b$, $a > 0$, $0 < b \leq (1 - \sigma)$.

In this case labor demand is

$$\ell_t^d = \ell^d(\gamma_t^s; \alpha_t) = \left(\frac{\gamma_t^s ab}{\alpha_t} \right)^{\frac{1}{1-b}} \tag{2.4}$$

Notice that labor demand is independent of s_t. The aggregate labor demand then is $L_t^d = n'\ell^d(\gamma_t^s; \alpha_t) \equiv L^d(\gamma_t^s; \alpha_t)$ and, because only a fraction λ_t^d of firms can hire workers, the aggregate supply of goods is

$$Y_t^s = \lambda_t^d n' f\left(\ell^d(\gamma_t^s; \alpha_t)\right) + S_t \equiv Y^s\left(\lambda_t^d, \gamma_t^s; \alpha_t, S_t\right) \tag{2.5}$$

3. Temporary Equilibrium Allocations

For any given period t we can now describe a feasible allocation as a temporary equilibrium with rationing as follows.

Definition 3.1. : *Given a real wage α_t, a real profit level π_t, real money balances m_t, inventories S_t, a level of public expenditure G and a tax rate tax, a list* *of rationing coefficients* $\left(\gamma_t^d, \gamma_t^s, \lambda_t^d, \lambda_t^s, \delta_t, \varepsilon_t\right) \in [0,1]^6$ *and an aggregate allocation* $\left(\overline{L}_t, \overline{Y}_t\right)$ *constitute* a temporary equilibrium *if the following conditions are fulfilled:*

(1) $\overline{L}_t = \lambda_t^s L^s = \lambda_t^d L^d\left(\gamma_t^s; \alpha_t\right)$;

(2) $\overline{Y}_t = \gamma_t^s Y^s\left(\lambda_t^d, \gamma_t^s; \alpha_t, S_t\right) = \gamma_t^d X^d\left(\lambda_t^s; \alpha_t, (1 - tax)\,\pi_t\right) + \delta_t m_t + \varepsilon_t G$;

(3) $\left(1 - \lambda_t^s\right)\left(1 - \lambda_t^d\right) = 0; \left(1 - \gamma_t^s\right)\left(1 - \gamma_t^d\right) = 0$;

(4) $\gamma_t^d\left(1 - \delta_t\right) = 0;\ \delta_t\left(1 - \varepsilon_t\right) = 0$.

Conditions (1) and (2) require that expected aggregate transactions balance. This means that all agents have correct perceptions of the rationing coefficients $\gamma_t^d, \gamma_t^s, \lambda_t^d$ and λ_t^s. Equations (3) formalize the short-side rule according to which at most one side on each market is rationed. The meaning of the coefficients δ_t and ε_t is that also old households and/or the government can be rationed. However, according to condition (4) this may occur only after young households have been rationed (to zero).

As shown in the table below it is possible to distinguish different types of equilibrium according to which market sides are rationed: excess supply on both markets is called *Keynesian Unemployment* $[K]$, excess demand on both markets *Repressed Inflation* $[I]$, excess supply on the labor market and excess demand on the goods market *Classical Unemployment* $[C]$ and excess demand on the labor market with excess supply on the goods market *Underconsumption* $[U]$.

	K	I	C	U
λ_t^s	<1	$=1$	<1	$=1$
λ_t^d	$=1$	<1	$=1$	<1
γ_t^s	<1	$=1$	$=1$	<1
γ_t^d	$=1$	<1	<1	$=1$
δ_t	$=1$	≤ 1	≤ 1	$=1$
ε_t	$=1$	≤ 1	≤ 1	$=1$

Of course there are further intermediate cases which, however, can be considered as limiting cases of the above ones. In particular, when all the rationing coefficients are equal to one, we are in a Walrasian Equilibrium.

Existence and uniqueness of temporary equilibrium are established by the following proposition.

Proposition 3.2. *Under assumptions (A1) and (A2) there exists, for any quadruple of variables* $(\alpha_t, m_t, \pi_t, S_t)$ *with* α_t *strictly positive and* m_t, π_t *and* S_t *non-*

negative, and any non-negative pair of policy parameters (G, tax)*, a unique temporary equilibrium allocation* $\left(\overline{L}_t, \overline{Y}_t\right)$*.* $\overline{L}_t$ *is given by*

$$\overline{L}_t = \min\left\{\widetilde{L}\left(\alpha_t, \pi_t, m_t, S_t, G, tax\right), L^d\left(1, \alpha_t\right), L^s\right\} \equiv \mathcal{L}\left(\alpha_t, \pi_t, m_t, S_t, G, tax\right) \tag{3.1}$$

where $\widetilde{L}\left(\alpha_t, \pi_t, m_t, S_t, G, tax\right)$ *is the unique solution in* L *of*

$$\alpha_t\left(\frac{1}{b} - h\right)L + \frac{\alpha_t}{ab}\left(\frac{L}{n'}\right)^{1-b} S_t = h\left(1 - tax\right)\pi_t + m_t + G \tag{3.2}$$

and

$$L^d\left(1, \alpha_t\right) = n'\left(\frac{ab}{\alpha_t}\right)^{\frac{1}{1-b}}. \tag{3.3}$$

$\overline{Y}_t \equiv \mathcal{Y}\left(\alpha_t, \pi_t, m_t, S_t, G, tax\right)$ *is determined as follows. If* $\overline{L}_t = \widetilde{L}\left(\cdot\right)$*, then* $\overline{Y}_t = \frac{\alpha_t}{b}\overline{L}_t + \frac{\alpha_t}{ab}\left(\frac{\overline{L}_t}{n'}\right)^{1-b} S_t$*, and if* $\overline{L}_t = L^d\left(1, \alpha_t\right)$*, then* $\overline{Y}_t = \frac{\alpha_t}{b}L^d\left(1, \alpha_t\right) + S_t$*. Finally, if* $\overline{L}_t = L^s$*, then* $\overline{Y}_t = \min\left\{\frac{\alpha_t}{b}L^s + S_t, h\left(1 - tax\right)\pi_t + h\alpha_t L^s + m_t + G\right\}$*.*

Proof. Since we hold $\{\alpha_t, m_t, \pi_t, S_t\}$and (G, tax) fixed, we omit them whenever possible as arguments in the subsequent functions. Define the set

$$\overline{H} = \left\{\left(\lambda^s L^s, \gamma^d X^d\left(\lambda^s\right)\right) \mid \left(\lambda^s, \gamma^d\right) \in [0,1]^2\right\}$$

and its subsets $\overline{H}^K = \overline{H}\mid_{\gamma^d=1,\lambda^s<1}$, $\overline{H}^I = \overline{H}\mid_{\gamma^d<1,\lambda^s=1}$, $\overline{H}^C = \overline{H}\mid_{\gamma^d<1,\lambda^s<1}$ and $\overline{H}^U = \overline{H}\mid_{\gamma^d=1,\lambda^s=1}$. Using the terminology introduced by Honkapohja and Ito (1985), we derive from these the consumption sector's *trade curves*

$$\overline{H}_0^K = \overline{H}^K + \{(0, m_t + G)\} = \left\{\left(\lambda^s L^s, X^d\left(\lambda^s\right) + m_t + G\right) \mid \lambda^s \in [0,1)\right\},$$

$$\overline{H}_0^I = \left\{\left(L^s, \gamma^d X^d\left(1\right) + m_t + G\right) \mid \gamma^d \in (0,1)\right\} \cup \left\{\left(L^s, \delta m_t + G\right) \mid \delta \in (0,1]\right\}$$
$$\cup \left\{\left(L^s, \varepsilon G\right) \mid \varepsilon \in [0,1]\right\},$$

$$\overline{H}_0^C = \left\{\left(\lambda^s L^s, \gamma^d X^d\left(\lambda^s\right) + m_t + G\right) \mid \left(\lambda^s, \gamma^d\right) \in [0,1) \times (0,1)\right\}$$
$$\cup \left\{\left(\lambda^s L^s, \delta m_t + G\right) \mid \left(\lambda^s, \delta\right) \in [0,1) \times (0,1]\right\} \cup \left\{\left(\lambda^s L^s, \varepsilon G\right) \mid \left(\lambda^s, \varepsilon\right) \in [0,1) \times [0,1]\right\}.$$

and

$$\overline{H}_0^U = \overline{H}^U + \{(0, m_t + G)\} = \left\{\left(L^s, X^d\left(1\right) + m_t + G\right)\right\}.$$

Similarly, starting from

$$\overline{F} \equiv \left\{ \left(\lambda^d L^d \left(\gamma^s \right), \gamma^s Y^s \left(\lambda^d, \gamma^s \right) \right) \mid \left(\lambda^d, \gamma^s \right) \in [0,1]^2 \right\}$$

we define the production sector's trade curves as $\overline{F}^K = \overline{F} \mid_{\lambda^d=1,\gamma^s<1}$, $\overline{F}^I = \overline{F} \mid_{\lambda^d<1,\gamma^s=1}$, $\overline{F}^C = \overline{F} \mid_{\lambda^d=1,\gamma^s=1}$and $\overline{F}^U = \overline{F} \mid_{\lambda^d<1,\gamma^s<1}$.

To derive these curves, we start with noticing that

$$\gamma^s Y^s \left(\lambda^d, \gamma^s; \alpha_t, S_t \right) = \frac{\alpha_t}{b} \lambda^d L^d \left(\gamma^s_t; \alpha_t \right) + \gamma^s S_t. \tag{3.4}$$

Indeed, by (2.5)

$$\gamma^s Y^s \left(\lambda^d, \gamma^s; \alpha_t, S_t \right) = \gamma^s \left[\lambda^d n' f \left(\ell^d \left(\gamma^s_t; \alpha_t \right) \right) + S_t \right]$$

whereas from $f(\ell) = a\ell^b$ follows $f'(\ell) = b\frac{f(\ell)}{\ell}$, which implies $f(\ell) = \frac{1}{b} f'(\ell)\,\ell$. Therefore

$$\gamma^s Y^s \left(\lambda^d, \gamma^s; \alpha_t, S_t \right) = \gamma^s \left[\lambda^d n' \frac{1}{b} f' \left(\ell^d \left(\gamma^s_t; \alpha_t \right) \right) \ell^d \left(\gamma^s; \alpha_t \right) + S_t \right].$$

But $\gamma^s f' \left(\ell^d \left(\gamma^s; \alpha_t \right) \right) = \alpha_t$ from any producer's optimizing behavior, and thus

$$\gamma^s Y^s \left(\lambda^d, \gamma^s; \alpha_t, S_t \right) = \frac{\alpha_t}{b} \lambda^d n' \ell^d \left(\gamma^s; \alpha_t \right) + \gamma^s S_t = \frac{\alpha_t}{b} \lambda^d L^d \left(\gamma^s_t; \alpha_t \right) + \gamma^s S_t.$$

This implies immediately that

$$\overline{F}^C = \left\{ \left(L^d (1; \alpha_t), \frac{\alpha_t}{b} L^d (1; \alpha_t) + S_t \right) \right\}.$$

Consider now

$$\overline{F}^K = \left\{ \left(L^d \left(\gamma^s; \alpha_t \right), \gamma^s Y^s \left(1, \gamma^s; \alpha_t, S_t \right) \right) \mid \gamma^s \in [0,1) \right\}.$$

Then (3.4) yields

$$\gamma^s Y^s \left(1, \gamma^s; \alpha_t, S_t \right) = \frac{\alpha_t}{b} L^d \left(\gamma^s_t; \alpha_t \right) + \gamma^s S_t.$$

On the other hand, (2.4) implies

$$\gamma^s = \frac{\alpha_t}{ab} \left(\ell^d \left(\gamma^s_t; \alpha_t \right) \right)^{1-b} = \frac{\alpha_t}{ab} \left(\frac{L^d \left(\gamma^s_t; \alpha_t \right)}{n'} \right)^{1-b}$$

and therefore

$$\gamma^s Y^s(1,\gamma^s;\alpha_t,S_t) = \frac{\alpha_t}{b} L^d(\gamma_t^s;\alpha_t) + \frac{\alpha_t}{ab}\left(\frac{L^d(\gamma_t^s;\alpha_t)}{n'}\right)^{1-b} S_{t.}$$

Since $L^d(\gamma_t^s;\alpha_t)$ is strictly increasing in γ_t^s, this yields

$$\overline{F}^K = \left\{ \left(L, \frac{\alpha_t}{b} L + \frac{\alpha_t}{ab}\left(\frac{L}{n'}\right)^{1-b} S_{t.} \right) \mid 0 \le L < L^d(1;\alpha_t) \right\}. \tag{3.5}$$

Consider next

$$\overline{F}^I = \left\{ \left(\lambda^d L^d(1;\alpha_t), Y^s\left(\lambda^d, 1; \alpha_t, S_t\right)\right) \mid \lambda^d \in [0,1) \right\}.$$

By (3.4) $Y^s\left(\lambda^d, 1; \alpha_t\right) = \frac{\alpha_t}{b}\lambda^d L^d(1;\alpha_t) + S_{t-1}$ and therefore

$$\overline{F}^I = \left\{ \left(L, \frac{\alpha_t}{b} L + S_t \right) \mid 0 \le L < L^d(1;\alpha_t) \right\}.$$

Since $\frac{\alpha_t}{ab}\left(\frac{L}{n'}\right)^{1-b} = \gamma^s \le 1$, $\overline{F}^K$ is positioned below $\overline{F}^I$.

Finally consider $\overline{F}^U$. It is given by

$$\overline{F}^U = \left\{ \left(\lambda^d L^d(\gamma^s;\alpha_t), \frac{\alpha_t}{b}\lambda^d L^d(\gamma_t^s;\alpha_t) + \frac{\alpha_t}{ab}\left(\frac{L^d(\gamma_t^s;\alpha_t)}{n'}\right)^{1-b} S_t \right) \mid \left(\lambda^d, \gamma^s\right) \in [0,1)^2 \right\} \tag{3.6}$$

Comparing with $\overline{F}^K$ and $\overline{F}^I$, it is clear that $\overline{F}^U$ is the set of points contained between $\overline{F}^K$ and $\overline{F}^I$. Figure 3.1 illustrates the producers' trade curves.

Using the consumption sector's and the production sector's trade curves and indicating with S^c the closure of the set S, we now note that a pair $\left(\overline{L}, \overline{Y}\right) \in R_+^2$ is a temporary equilibrium allocation if and only if it is an element of the set

$$Z = \left(\left(\overline{H}_0^K\right)^c \cap \left(\overline{F}^K\right)^c\right) \cup \left(\left(\overline{H}_0^I\right)^c \cap \left(\overline{F}^I\right)^c\right) \cup \left(\left(\overline{H}_0^C\right)^c \cap \left(\overline{F}^C\right)^c\right) \cup \left(\left(\overline{H}_0^U\right)^c \cap \left(\overline{F}^U\right)^c\right).$$

(Here S^c indicates the closure of the set S.) To show existence of an equilibrium is equivalent to showing that Z is not empty. To this end consider first the locus

$$\left(\overline{H}_0^K\right)^c = \left\{ \left(\lambda_t^s L^s, X^d\left(\lambda_t^s\right) + m_t + G\right) \mid \lambda_t^s \in [0,1] \right\}$$

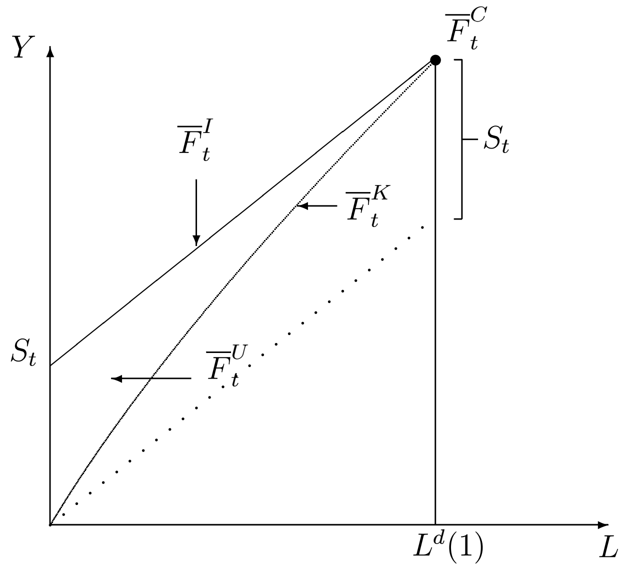

Figure 3.1: The producers' trade curves

and recall that

$$X^d(\lambda_t^s) = nh\left(\lambda_t^s \omega_t^1 + (1 - \lambda_t^s)\,\omega_t^0\right) = h(1 - tax)\,\pi_t + h\alpha_t \lambda_t^s L^s.$$

Defining the function

$$\Gamma_t(L) = h(1 - tax)\,\pi_t + h\alpha_t L + m_t + G,\ \ L \geq 0,$$

we see that $\left(\overline{H}_0^K\right)^c$ is the part of the graph of Γ_t for which $L \leq L^s$.

Next consider again the production sector's trade curves. From (3.5) we conclude that the locus $\left(\overline{F}^K\right)^c$ is the part of the graph of the function

$$\Delta_t(L) = \frac{\alpha_t}{b} L + \frac{\alpha_t}{ab}\left(\frac{L}{n'}\right)^{1-b} S_t,\ \ L \geq 0,$$

for which $L \leq L^d(1)$. Notice that the graphs of the functions Γ_t and Δ_t always intersect. Indeed, $\Gamma_t'(L) = h\alpha_t$ and $\Gamma_t(0) = h(1 - tax)\,\pi_t + m_t + G > 0$,whereas

$\Delta'_t(L) \geq \frac{\alpha_t}{b} > h\alpha_t$ (since $1/b > 1 > h$) and $\Delta_t(0) = 0$. Setting $\Delta_t(L) = \Gamma_t(L)$ yields (3.2) with the unique solution denoted $\widetilde{L}(\alpha_t, \pi_t, m_t, G, tax)$. Therefore the equilibrium level on the labor market is

$$\overline{L}_t = \min\left\{\widetilde{L}(\alpha_t, \pi_t, m_t, G, tax), L^d(1, \alpha_t), L^s\right\} = \mathcal{L}(\alpha_t, \pi_t, m_t, S_t, G, tax).$$

whereas the on the goods market is, by definition of the function $\mathcal{Y}(\cdot)$,

$$\overline{Y}_t = \mathcal{Y}(\alpha_t, \pi_t, m_t, S_t, G, tax).$$

This shows that the equilibrium allocation

$$\left(\overline{L}_t, \overline{Y}_t\right) = \left(\mathcal{L}(\alpha_t, \pi_t, m_t, S_t, G, tax), \mathcal{Y}(\alpha_t, \pi_t, m_t, S_t, G, tax)\right)$$

exists and is uniquely defined. ■

Equation (3.1) allows us to characterize the type of equilibrium defined in Table 1: if $\overline{L}_t = \widetilde{L}(\alpha_t, \pi_t, m_t, S_t, G, tax)$, the resulting equilibrium is of type K or a limiting case of it. If $\overline{L}_t = L^d(1, \alpha_t)$, type C or a limiting case of it occurs. Finally, if $\overline{L}_t = L^s$, an equilibrium of type I or a limiting case results if $\frac{\alpha_t}{b}L^s + S_t \leq h(1 - tax)\pi_t + h\alpha_t L^s + m_t + G$; otherwise the equilibrium is of type U.

The above discussion and Proposition 3.2 allow us to determine the expressions of those rationing coefficients which are possibly smaller than one. This is summarized in the following corollary.

Corollary 3.3. *In case K, $\lambda_t^s = \frac{\overline{L}_t}{L^s}$ and $\gamma_t^s = \frac{\alpha_t}{ab}\left(\frac{\overline{L}_t}{n'}\right)^{1-b}$. In case C, $\lambda_t^s = \frac{\overline{L}_t}{L^s}$ and, in case I, $\lambda_t^d = \frac{L^s}{L^d(1,\alpha_t)}$. Moreover, in both these latter cases,*

$$\left(\gamma_t^d, \delta_t, \varepsilon_t\right) = \begin{cases} \left(\frac{\overline{Y}_t - m_t - G}{h(1-tax)\pi_t + h\alpha_t\overline{L}_t}, 1, 1\right) & \text{if } \overline{Y}_t \geq G + m_t \\ \left(0, \frac{\overline{Y}_t - G}{m_t}, 1\right) & \text{if } G + m_t > \overline{Y}_t \geq G \\ \left(0, 0, \frac{\overline{Y}_t}{G}\right) & \text{if } \overline{Y}_t < G \end{cases}$$

Finally, in case U $\gamma_t^s = \frac{1}{S_t}\left(\overline{Y}_t - \frac{\alpha_t}{b}\overline{L}_t\right)$ and $\lambda_t^d = \overline{L}_t / L^d(\gamma_t^s; \alpha_t)$.

Proof: By (3.6) in case U it must be true that

$$\left(\overline{L}_t, \overline{Y}_t\right) = \left(\lambda^d L^d(\gamma^s; \alpha_t), \frac{\alpha_t}{b}\lambda^d L^d(\gamma_t^s; \alpha_t) + \frac{\alpha_t}{ab}\left(\frac{L^d(\gamma_t^s; \alpha_t)}{n'}\right)^{1-b} S_t\right).$$

Moreover by (2.4)

$$L^d(\gamma^s;\alpha_t) = n'\left(\frac{\gamma_t^s ab}{\alpha_t}\right)^{\frac{1}{1-b}}.$$

Therefore

$$\frac{\alpha_t}{b}\lambda^d L^d(\gamma_t^s;\alpha_t) + \frac{\alpha_t}{ab}\left(\frac{L^d(\gamma_t^s;\alpha_t)}{n'}\right)^{1-b} S_t = \overline{Y}_t$$

$\Leftrightarrow$

$$\frac{\alpha_t}{b}\lambda_t^d L^d(\gamma_t^s;\alpha_t) + \gamma_t^s S_t = \overline{Y}_t$$

Recalling that $\lambda^d L^d(\gamma^s;\alpha_t) = \overline{L}_t$ and solving for γ_t^s yields the claimed expression. The values of λ_t^s and λ_t^d are immediate by definition; γ_t^s can be obtained using assumption (A2) and equation (2.4). Finally, γ_t^d, δ_t, ε_t are determined by means of (2.3). ■

Using the consumption and the production sectors' trade and offer curves it is possible to analyze the various equilibrium regimes in more detail. We do this here for the case of Keynesian Unemployment only. This type of equilibrium involves rationing of households on the labor market and of firms on the goods market. It is given by a pair $(\lambda_t^s, \gamma_t^s)$ such that

$$\begin{aligned} \overline{L}_t &= \lambda_t^s L^s = L^d(\gamma_t^s) \\ \overline{Y}_t &= \gamma_t^s Y^s(1,\gamma^s) = X^d(\lambda_t^s) + m_t + G \end{aligned}$$

(where we have suppressed all arguments that are not rationing coefficients). Recalling the definition of the trade curves $\overline{H}^K$ and $\overline{F}^K$ the pair $(\overline{L}_t, \overline{Y}_t)$ is a Keynesian equilibrium allocation if

$$\begin{aligned} (\overline{L}_t, \overline{Y}_t) &\in \left\{\left(\lambda^s L^s, X^d(\lambda^s) + m_t + G\right) \mid \lambda^s \in [0,1)\right\} \\ &\quad \cap \left\{\left(L^d(\gamma^s), \gamma^s Y^s(1,\gamma^s)\right) \mid \gamma^s \in [0,1)\right\} \\ &= \left[\overline{H}_t^K + \{(0, m_t + G)\}\right] \cap \overline{F}_t^K. \end{aligned}$$

Thus $(\overline{L}_t, \overline{Y}_t)$ is given by the intersection of the trade curves $\overline{H}_t^K + \{(0, m_t + G)\}$ and $\overline{F}_t^K$, as shown in Figure 3.2. There an equilibrium of type Keynesian unemployment is shown, where $F_t^K \equiv \left\{\left(L^d(\gamma^s;\alpha_t), Y^s(1,\gamma^s;\alpha_t,S_t)\right) \mid \gamma^s \in [0,1)\right\}$.

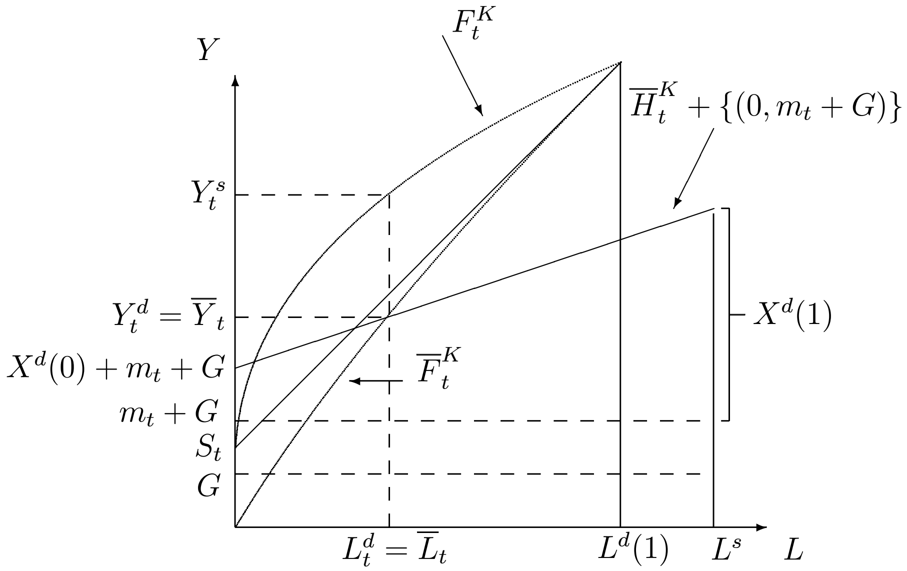

Figure 3.2: Keynesian Unemployment Equilibrium

The consumption sector supplies the amount of labor $L^s > \overline{L}_t$ and demands the quantity of goods $Y_t^d = \overline{Y}_t$ whereas firms demand labor $L_t^d = \overline{L}_t$ and supply $Y_t^s > \overline{Y}_t$ of goods. It follows that $\lambda_t^s = \overline{L}_t / L^s, \gamma_t^s = \overline{Y}_t / Y_t^s$ and $\lambda_t^d = \gamma_t^d = 1 \; (= \delta_t = \varepsilon_t)$, which are just the values that led households and firms to express their respective transaction offers. Thus their expectations regarding these rationing coefficients are confirmed. Nevertheless, due to the randomness in rationing at an individual agent's level, effective aggregate demands and supplies of rationed agents exceed their actual transactions. Moreover, as indicated earlier, these excesses can be used to get an indicator of the strength of rationing. Since there is zero-one rationing on the labor market, $1 - \lambda_t^s = (L^s - \overline{L}_t)/L^s$ is the ratio of the number of unemployed workers and the total number of young households. Regarding the goods market, in a K-equilibrium $\overline{Y}_t - \gamma_t^s Y^s(1, \gamma_t^s) = 0$, and therefore

$$\frac{d(1-\gamma_t^s)}{d\overline{Y}_t} = -\frac{1}{Y_t^s + \gamma_t^s \frac{\partial Y^s}{\partial \gamma_t^s}} < 0$$

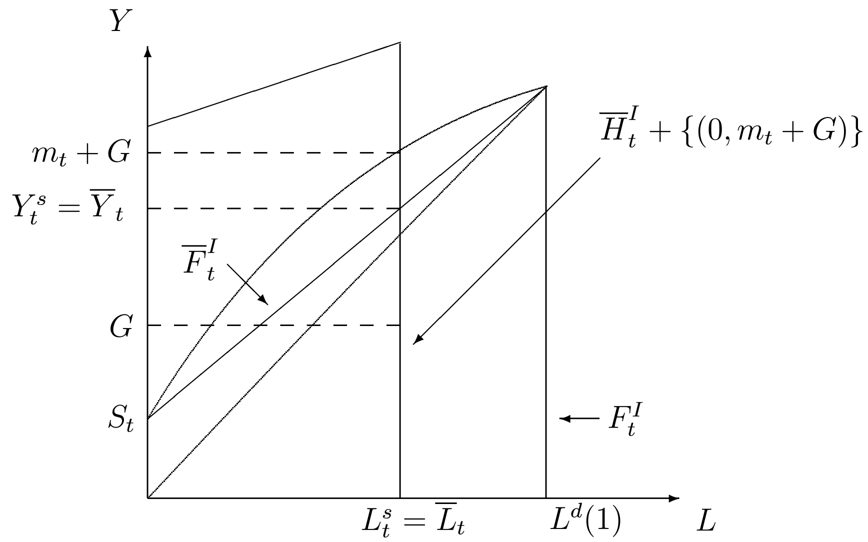

Figure 3.3: Repressed Inflation Equilibrium

since $\frac{\partial Y^s}{\partial \gamma_t^s}(1, \gamma_t^s) = n' f'\left(\ell^d\left(\gamma_t^s\right)\right) \frac{d\ell^d}{d\gamma_t^s} > 0$. So a decrease in $\overline{Y}_t$ (for example due to a reduction of government spending), and thus an aggravation of the shortage of aggregate demand for firms' goods, is unambiguously related to an increase in $1-\gamma_t^s$ which can therefore be interpreted as a measure of the strength of rationing on the goods market. A similar reasoning justifies the use as rationing measures of the terms $1 - \lambda_t^d$ and $1 - \gamma_t^d$ in the other equilibrium regimes.

The illustration of the other temporary equilibrium regimes works similarly except for the fact that under repressed inflation and classical unemployment old agents and/or the government may be rationed, too. This is shown in Figure 3.3 for the case of repressed inflation and rationing of old agents.

4. Representation of Equilibrium Regimes

Given the existence and uniqueness of temporary equilibrium we can, holding all other variables fixed, partition the set R_+^4 of all combinations of real wage α_t, real profits π_t, real money stock m_t and inventories S_t according to the type of

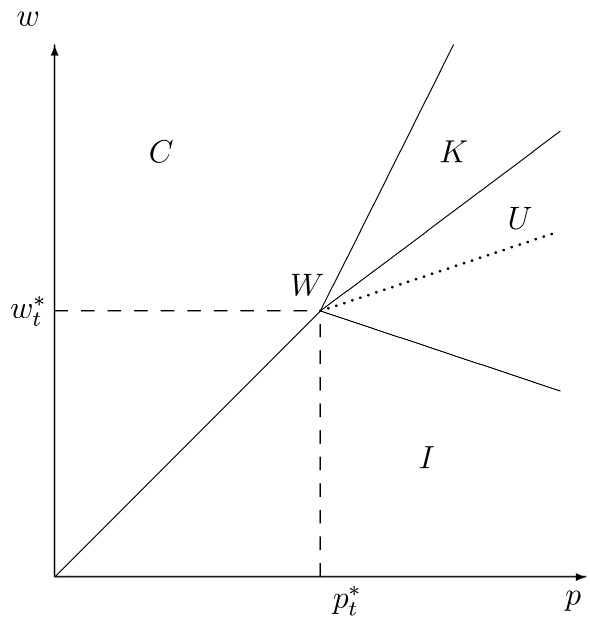

Figure 4.1: Temporary Equilibrium Regimes in the $p - w$ plane.

equilibrium they give rise to. Formally, we have a map $(\alpha_t, \pi_t, m_t, S_t) \mapsto T \in \{K, I, C, U\}$. Holding also nominal money M, nominal profits Π and inventories S parametrically fixed, we can furthermore derive from this a map

$$(p_t, w_t) \mapsto (w_t/p_t, \Pi/p_t, M/p_t, S) \mapsto T$$

which is illustrated in Figure 4.1 and shows the partitioning of $p_t - w_t-$plane in different regimes of types of equilibrium. From this diagram, in principle familiar from the literature,[2] it can be seen that too high a goods price and a nominal wage give rise to a state of Keynesian unemployment and hence excess supply on both markets, even if the real wage is at its Walrasian level. If the real wage is too high, Classical unemployment occurs whereas in the opposite case a situation of repressed inflation.

The figure differs from what is shown in the literature with respect to the slope of the borderline between regimes U and I: there it is negative whereas here it may be positive. To see this, consider (p_t, w_t) such that $T(w_t/p_t, \Pi/p_t, M/p_t, S) =$

[2] See for instance Malinvaud [1977] and Muellbauer and Portes [1978].

$U^c \cap I^c$. Then consumers are not rationed while producers are rationed only on the labor market. Writing $(w_t/p_t, \Pi/p_t, M/p_t, S) = (\alpha, \pi, m, S)$, the corresponding equilibrium must thus satisfy

$$\left(\overline{L}_t, \overline{Y}_t\right) = \left(\lambda^d L^d (1; \alpha), Y^s \left(\lambda^d, 1; \alpha, S\right)\right) = \left(L^s, h(1 - tax)\pi + h\alpha L^s + m + G\right).$$

By (3.4) $Y^s \left(\lambda^d, 1; \alpha, S\right) = \frac{\alpha_t}{b}\lambda^d L^d (1; \alpha_t) + S_t = \frac{\alpha}{b} L^s + S$, and therefore we obtain the condition

$$\frac{\alpha}{b} L^s + S = h(1 - tax)\pi + h\alpha L^s + m + G.$$

Multiplying by p and solving for w yields

$$w = \frac{h(1 - tax)\Pi + M}{\left(\frac{1}{b} - h\right) L^s} + \frac{G - S}{\left(\frac{1}{b} - h\right) L^s} p.$$

From this it is obvious that this function, which describes the borderline between the regimes U and I, is downward sloping iff $S > G$.

5. Dynamics

So far our analysis has been essentially static. For any given vector $(\alpha_t, \pi_t, m_t, S_t, G, tax)$ we have described a feasible allocation in terms of a temporary equilibrium with rationing. To extend now our analysis to a dynamic one we must link successive periods one to another. This link will of course be given by the adjustment of prices but also by the changes in the stock of money and in profits. Regarding the latter, this is automatic by definition of these variables and equations (3.1) to (3.3), i.e.

$$\Pi_t = p_t \mathcal{Y}(\alpha_t, \pi_t, m_t, S_t, G, tax) - w_t \mathcal{L}(\alpha_t, \pi_t, m_t, G, tax),$$

$$\begin{aligned} M_{t+1} &= (1 - tax)\Pi_{t-1} + w_t \overline{L}_t - p_t \overline{Y}_t + \delta_t M_t + \varepsilon_t p_t G \\ &= (1 - tax)\Pi_{t-1} - \Pi_t + \delta_t M_t + \varepsilon_t p_t G. \end{aligned}$$

$$S_{t+1} = Y^s \left(\lambda_t^d, \gamma_t^s; \alpha_t, S_t\right) - \mathcal{Y}(\alpha_t, \pi_t, m_t, S_t, G, tax).$$

Regarding the adjustment of prices and wages we follow the standard hypothesis that, whenever an excess of demand (supply) is observed, the price rises (falls). In terms of the rationing coefficients observed in period t, this amounts to

$$p_{t+1} < p_t \Leftrightarrow \gamma_t^s < 1; \; p_{t+1} > p_t \Leftrightarrow \gamma_t^d < 1,$$

$$w_{t+1} < w_t \Leftrightarrow \lambda_t^s < 1;\ w_{t+1} > w_t \Leftrightarrow \lambda_t^d < 1.$$

More precisely, in our simulation model we have specified these adjustments as follows:[3]

$$p_{t+1} = \begin{cases} \left[1 - \mu_1 \left(1 - \gamma_t^s\right)\right] p_t & \text{if } \gamma_t^s < 1 \\ \left[1 + \mu_2 \left(1 - \frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right)\right] p_t & \text{if } \gamma_t^d < 1 \end{cases} \tag{5.1}$$

$$w_{t+1} = \begin{cases} \left[1 - \nu_1 \left(1 - \lambda_t^s\right)\right] w_t & \text{if } \lambda_t^s < 1 \\ \left[1 + \nu_2 \left(1 - \lambda_t^d\right)\right] w_t & \text{if } \lambda_t^d < 1 \end{cases}. \tag{5.2}$$

Then the adjustment equations for the real wage are

$$\alpha_{t+1} = \begin{cases} \frac{1-\nu_1(1-\lambda_t^s)}{1-\mu_1(1-\gamma_t^s)} \alpha_t & \text{if } \left(\overline{L}_t, \overline{Y}_t\right) \in K \\ \frac{1-\nu_1(1-\lambda_t^s)}{1+\mu_2\left(1-\frac{\gamma_t^d+\delta_t+\varepsilon_t}{3}\right)} \alpha_t & \text{if } \left(\overline{L}_t, \overline{Y}_t\right) \in C \\ \frac{1+\nu_2\left(1-\lambda_t^d\right)}{1+\mu_2\left(1-\frac{\gamma_t^d+\delta_t+\varepsilon_t}{3}\right)} \alpha_t & \text{if } \left(\overline{L}_t, \overline{Y}_t\right) \in I \\ \frac{1+\nu_2\left(1-\lambda_t^d\right)}{1-\mu_1(1-\gamma_t^s)} \alpha_t & \text{if } \left(\overline{L}_t, \overline{Y}_t\right) \in U \end{cases} \tag{5.3}$$

whereas θ_t is given by

$$\theta_t = \begin{cases} 1 - \mu_1 \left(1 - \gamma_t^s\right) & \text{if } \left(\overline{L}_t, \overline{Y}_t\right) \in K \cup U \\ 1 + \mu_2 \left(1 - \frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right) & \text{if } \left(\overline{L}_t, \overline{Y}_t\right) \in C \cup I \end{cases}. \tag{5.4}$$

The dynamics of the model in real terms is given by the sequence $\{(\alpha_t, m_t, \pi_t, S_t)\}_{t=1}^{\infty}$, where α_{t+1} is as in (5.3) and, using equations (3.1) to (3.3),

$$\pi_{t+1} = \begin{cases} \frac{1-b}{\theta_t(1-hb)} \left[h \left(1 - tax\right) \pi_t + m_t + G\right] & \text{if } \left(\overline{L}_t, \overline{Y}_t\right) \in K \\ \frac{1-b}{\theta_t} n' \left(\frac{\alpha_t}{ab}\right)^{\frac{b}{b-1}} \left(\frac{1}{a}\right)^{\frac{1}{b-1}} & \text{if } \left(\overline{L}_t, \overline{Y}_t\right) \in C \\ \frac{\alpha_t}{\theta_t} \frac{1-b}{b} L^s & \text{if } \left(\overline{L}_t, \overline{Y}_t\right) \in I \\ \frac{1}{\theta_t} \left[h \left(1 - tax\right) \pi_t + m_t + G - \alpha_t \left(1 - h\right) L^s\right] & \text{if } \left(\overline{L}_t, \overline{Y}_t\right) \in U \end{cases}.$$

The case U is derived as follows:

$$\begin{aligned} \pi_{t+1} &= \frac{\Pi_t}{p_{t+1}} = \frac{p_t \left[h \left(1 - tax\right) \pi_t + h\alpha_t L^s + m_t + G\right] - w_t L^s}{p_{t+1}} \\ &= \frac{1}{\theta_t} \left[h \left(1 - tax\right) \pi_t + h\alpha_t L^s + m_t + G - \alpha_t L^s\right]. \end{aligned}$$

[3] We employ a linear rule to avoid that one might suspect that the complex dynamics be generated by a nonlinear adjustment mechanism. Experimenting with some other specifications of the adjustment mechanism has revealed that our subsequent simulation results are not limited to the one presented here.

Finally,

$$m_{t+1} = \frac{1}{\theta_t}\left[\delta_t m_t + \varepsilon_t G + (1 - tax)\,\pi_t\right] - \pi_{t+1},$$

and

$$S_{t+1} = \lambda_t^d n' a \left(\frac{\gamma_t^s ab}{\alpha_t}\right)^{\frac{b}{1-b}} + S_t - \overline{Y}_t.$$

6. Simulations

The economic model introduced in the previous sections represents a non-linear dynamical system that cannot be studied with analytical tools only. This is due to the fact that the system is four-dimensional, with state variables α_t, m_t, π_t and S_t. Moreover, since there are four nondegenerate equilibrium regimes, the overall dynamic system can be viewed as being composed of four subsystems each of which may become effective through endogenous regime switching. (The complete equations of these systems are given in Appendix 2.)

In order to get some insights in these dynamics we are reporting numerical simulations using programs developed for this paper's purposes based on the packages GAUSS and MACRODYN [4]. The basic parameter set specifies values for the technological coefficients (a and b), the exponent of the utility function (h), the labor supply (L^s) and the total number of producers in the economy (n'), for the price adjustment speeds downward and upward (respectively μ_1 and μ_2) and the corresponding wage adjustment speeds (ν_1 and ν_2). We also have to specify initial values for the real wage, real money stock, real profit level and inventories (α_0, m_0, π_0 and S_0), and values for the government policy parameters (G and tax). Choosing in addition an initial value p_0 for the goods price, we can moreover keep track of the development of the nominal variables by using (5.1) to determine p_t for any t from which follow $w_t = \alpha_t p_t$ and $M_t = m_t p_t$.

Assuming the parameter values $a = 1$, $b = 0.85$, $h = 0.5$, $L^s = 100$ and $n' = 100$, a stationary Walrasian equilibrium is obtained for

$$\alpha^* = 0.85, \quad m^* = 46.25, \quad \pi^* = 15, \quad S^* = 0, \quad G^* = 7.5, \quad tax^* = 0.5,$$

with trading levels $L^* = Y^* = 100$. For the adjustment speeds of prices out of Walrasian equilibrium we set $\mu_1 = \mu_2 = \nu_2 = 0.1$ whereas ν_1, the downward

[4] MACRODYN has been developed at the University of Bielefeld. See Böhm,V., Lohmann, M. and U. Middelberg [1999], MACRODYN – a dynamical system's tool kit, version x99 and Böhm and Schenk-Hoppé [1998].

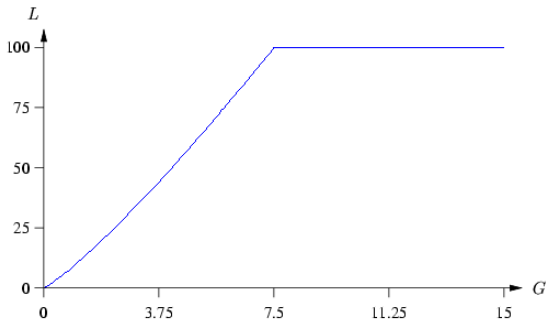

Figure 6.1: Stationary employment values when $\nu_1 = 0$.

speed of wage adjustment, will be varied between 0 and 0.1. This includes the case $\nu_1 = 0$ in which the wage rate is rigid downwards.

6.1. Fiscal Shocks

Our first investigation regards a change in G. Starting from $(\alpha_0, m_0, \pi_0, S_0) = (\alpha^*, m^*, m^*, S^*)$, the bifurcation diagram in Figure 6.1 shows, for $\nu_1 = 0$, the stationary values of employment to which the system converges in dependence of values of G between 0 and 15. From this it is evident that $\overline{L} < L^*$ for $G < G^*$ and $\overline{L} = L^*$ for $G = G^*$. What happens in case $G < G^*$ is that aggregate demand Y^d is diminished which creates an excess supply on the goods market. Consequently firms reduce their production and cut back on employment. The result is an excess supply on the labor market, too, and the economy enters in a state of Keynesian unemployment. The imbalance on the goods market gives rise to a price decrease whereas on the labor market the nominal wage cannot decrease as $\nu_1 = 0$. As a result the real wage increases. This is illustrated in Figure 6.2 which shows the time series for employment L, inventories S, the real money stock m and the real wage α for the first 200 periods where $G = 7$. The real wage is rising until approximately period 30 at which point it has become large enough so that the system enters into the regime of Classical unemployment. Here the goods price decreases and the real wage falls until at around period 50 it settles at a stationary value $\overline{\alpha} > \alpha^*$. Since the nominal wage rate does not change, the constant real wage implies that the goods price does not change either beyond period 50, and

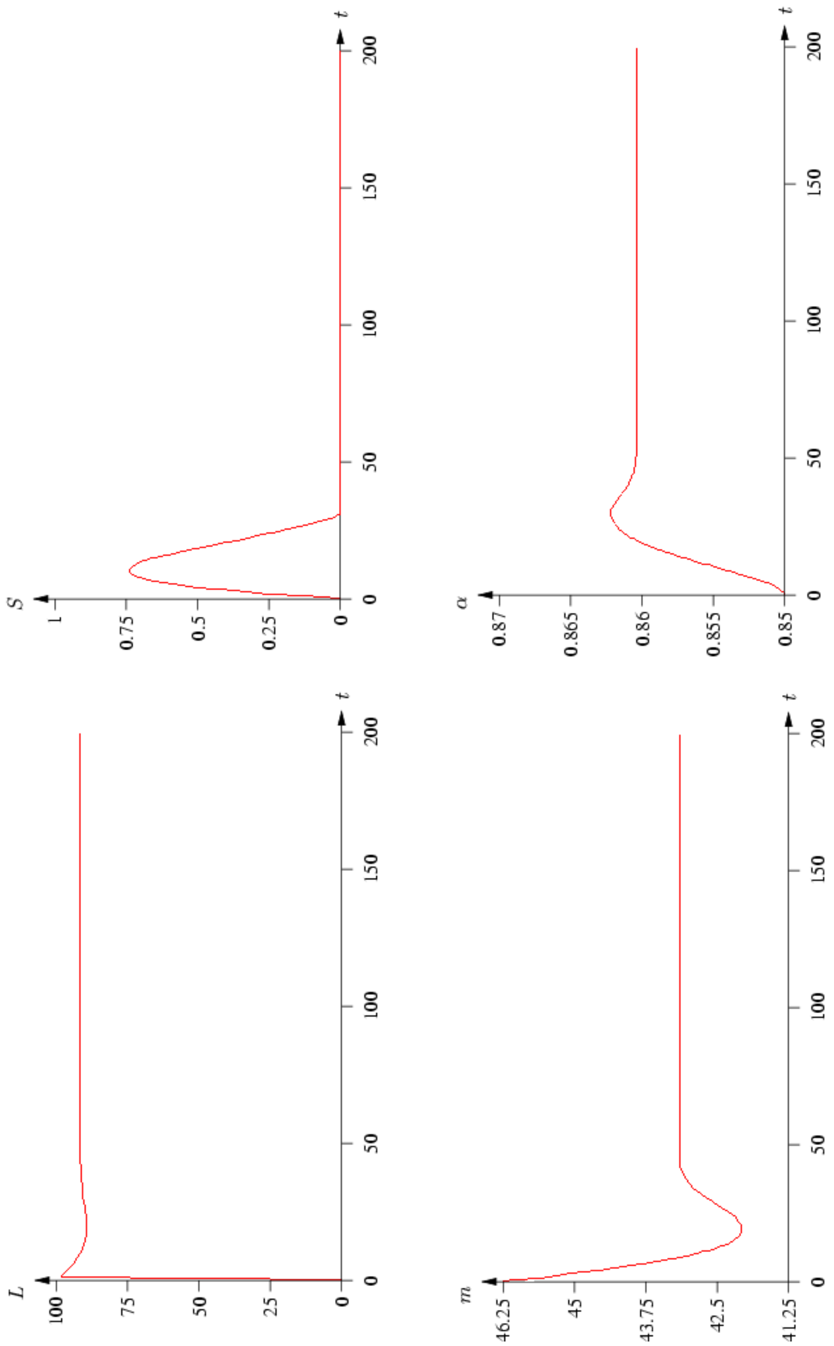

Figure 6.2: Time series when $\nu_1 = 0$ and $G = 7$.

the economy has reached a stationary state at the frontier between Keynesian and Classical unemployment. In that state there is market clearing on the goods market but excess supply on the labor market.

Next consider, again for $G < G^*$, what happens when $\nu_1 > 0$. The charts in Figure 6.3 show, analogously to Figure 6.1, stationary values of employment for various values of G. The top and the middle chart refer to downward wage flexibilities of $\nu_1 = 0.025$ and $\nu_1 = 0.1$, respectively. The striking result is that a little downward wage flexibility has an enormous effect on the impact of fiscal restraint as is documented by the discontinuity of the graphs at $G = G^*$. Note that this does not happen in the model without inventories, as is shown by the lower chart in Figure 6.3 where $\nu_1 = 0.1$ but S_t is exogenously set to zero at the beginning of each period.

Why inventories have such a dramatic effect is easily explained. When aggregate demand is diminished due to a decrease in G, inventories become positive and rise further as excess supply on the goods market builds up. As $\gamma^s = \overline{Y}_t/Y^s\left(\lambda_t^d, \gamma_t^s; \alpha_t, S_t\right)$ by (2) of Definition 1 and S_t influences Y^s positively by (2.5), an increase in S_t reduces the sales expectation ratio γ^s which by (2.4) diminishes the labor demand of firms and thus increases further the excess supply on the labor market. Therefore the downward flexible wage rate decreases more than would be the case without inventories. If the decrease in the wage rate is larger than the decrease in the goods price, the real wage decreases, and it may continue to decrease permanently approaching a limit level below the Walrasian real wage. The lower real wage diminishes labor income of workers which diminishes aggregate goods demand which in turn keeps employment below full employment. The dynamical system converges to a quasi-stationary Keynesian state with permanent deflation of all nominal variables but constant real magnitudes.[5] The nominal money stock shrinks because, due to the small government spending, the government is permanently realizing a budget surplus. These facts are illustrated in Figure 6.4 which shows time series for $\nu_1 = 0.025$.

When $G > G^*$, one can similarly show that the economy converges to a quasi-stationary state of Repressed inflation with permanent increase of all nominal variables and full employment with constant excess demands on the labor and the goods market.

[5] A state is *stationary* if all variables are constant; it is *quasi-stationary* if all real variables are constant but the nominal variables may change.

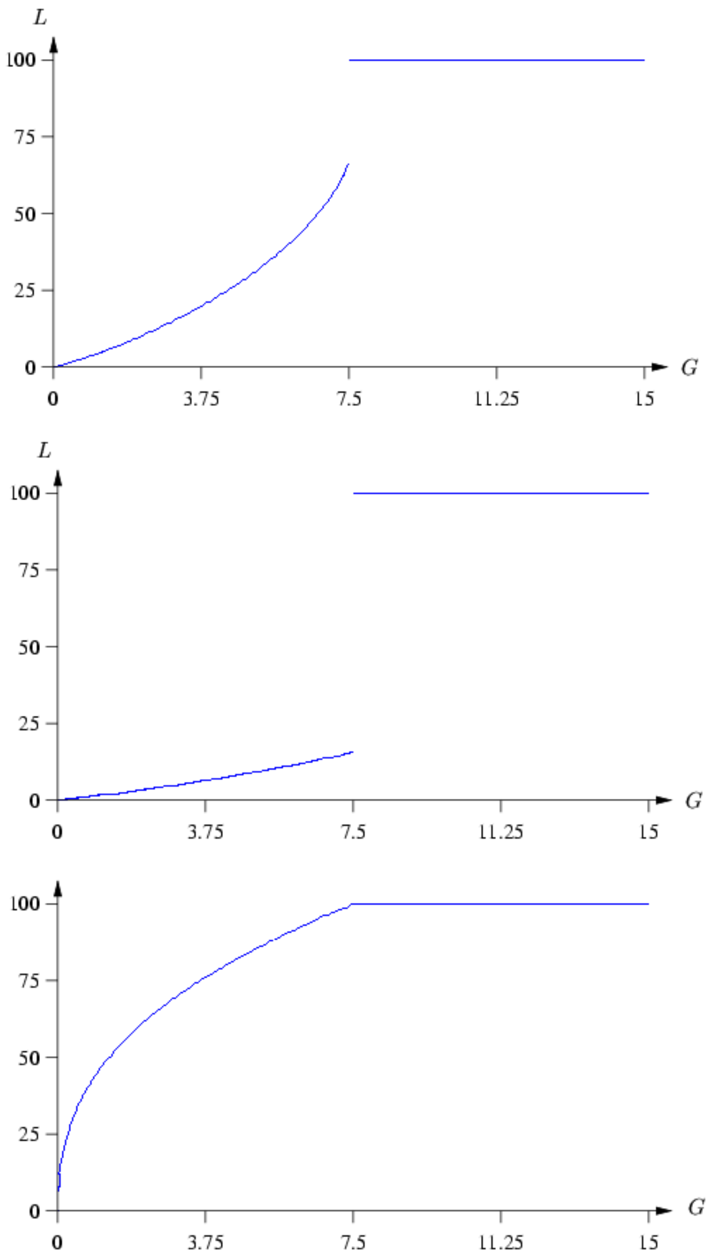

Figure 6.3: Stationary employment values for $\nu_1 = 0.025$ (top chart), $\nu_1 = 0.1$ (mid chart), and $\nu_1 = 0.1$ and $S_t \equiv 0$.

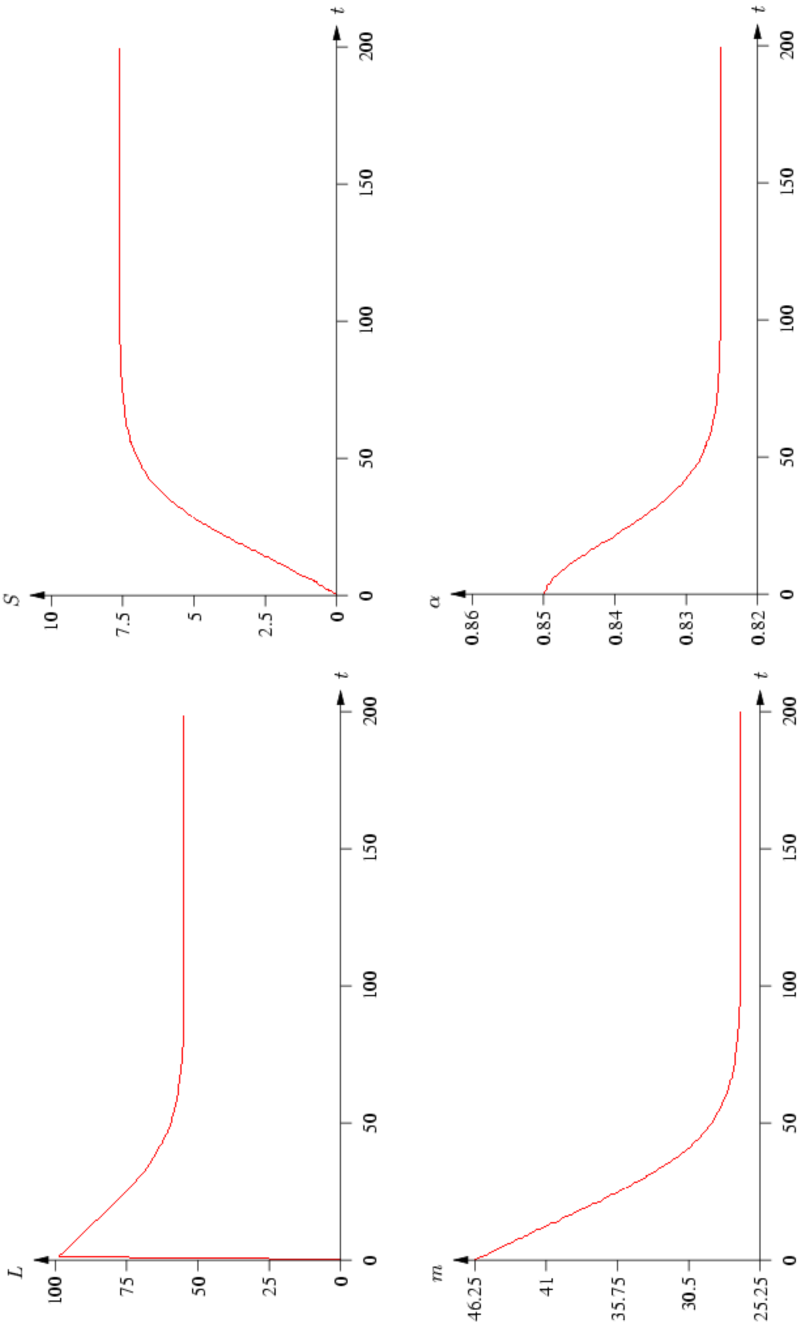

Figure 6.4: Time series when $\nu_1 = 0.025$ and $G = 7$.

6.2. A Restrictive Monetary Shock

We consider a reduction in the initial money stock to $m_0 = 40$, keeping all other parameters and initial values at their Walrasian levels. Having set $p_0 = 1$, this is equivalent to a reduction in the nominal money stock from $M_0 = 46.25$ to $M_0 = 40$. Since m_0 is the demand of old agents at time $t = 0$, aggregate demand is reduced. Consequently there is excess supply on the goods market and, since firms adjust to the reduced transaction level on the goods market, they reduce their labor demand. Thus there is excess supply on the labor market, too, and the economy enters in a state of Keynesian unemployment. What happens next depends on whether the nominal wage is flexible downwards. If not, the real wage and the real money stock increase - as shown in Figure 6.5 - until the economy reaches a state of Classical unemployment. Thereafter the price increase reduces the real wage until the system is back at the Walrasian equilibrium. With nominal wage rigid downwards the restrictive money shock has had a temporary but not lasting effect on economic activity.

The picture changes when downward wage flexibility is allowed. This is shown in Figure 6.6 for $\nu_1 = 0.025$, where employment, real wage and real money all converge to values lower than the respective Walrasian values. The reason is similar to that already discussed in the context of fiscal shocks: the presence of inventories increases the fall of labor demand by firms which in turn depresses labor income and aggregate demand. The system tends to a quasi-stationary Keynesian state with permanent deflation of nominal variables. The restrictive monetary shock has caused a permanent decrease in employment and output.

As in the case of a fiscal shock, setting $S_t \equiv 0$ changes the outcome also in the scenario of a monetary shock: this is shown in Figure 6.7 where again $\nu_1 = 0.025$. The real wage decreases initially but then the decrease in the goods price dominates the one in the nominal wage, and the real wage moves back to its Walrasian level, as do all the other variables.

At this point the natural question is which downward wage flexibility is needed to drive the economy into a permanent recession or even depression. The answer is given in the bifurcation diagram of Figure 6.8. From that it can be seen that approximately until $\nu_1 = 0.02$ the economy is capable of returning to the full employment after the monetary shock, whereas for speeds of wage adjustment larger than this the economy gets trapped in underemployment.

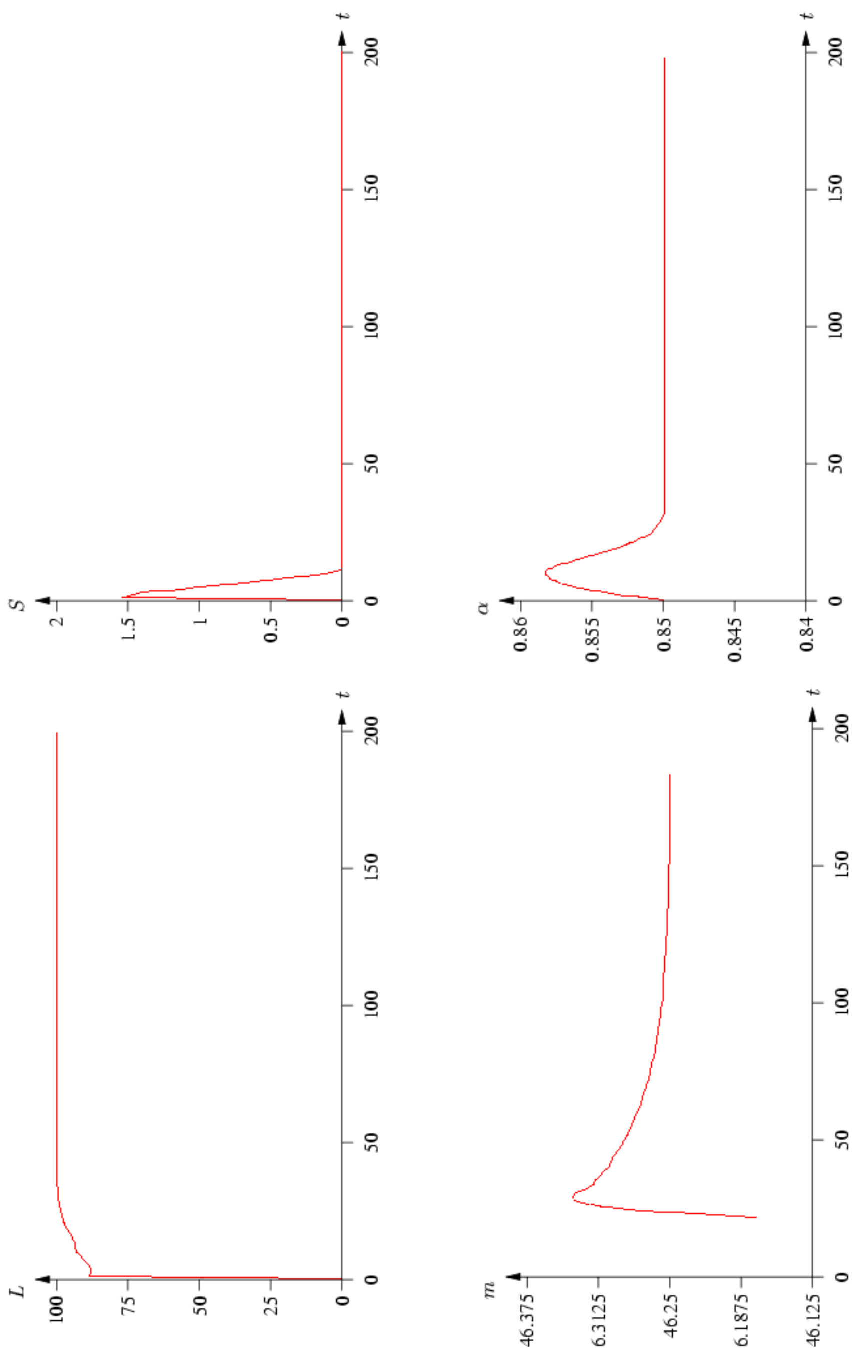

Figure 6.5: Time series when $\nu_1 = 0$ and $m_0 = 40$.

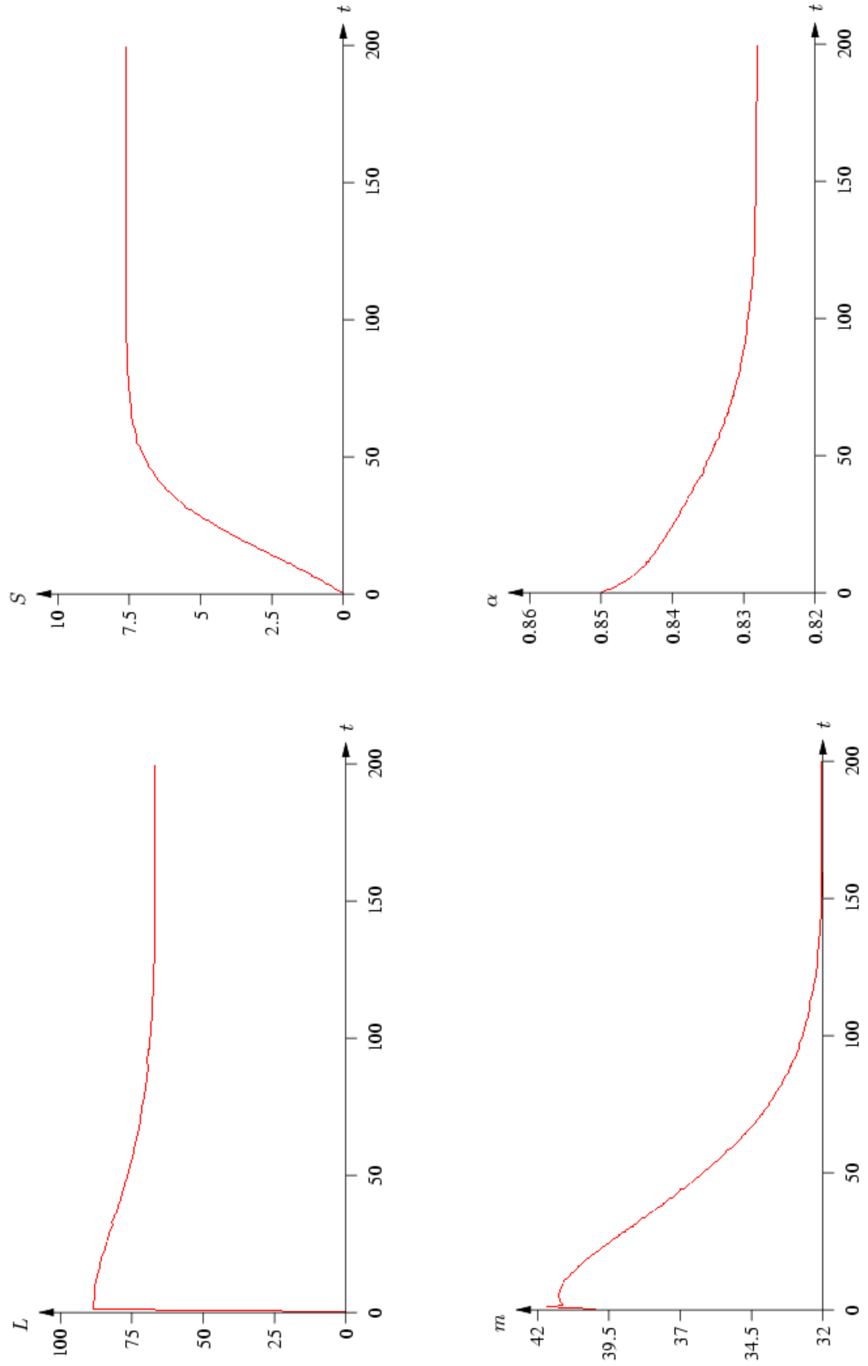

Figure 6.6: Time series when $\nu_1 = 0.025$ and $m_0 = 40$.

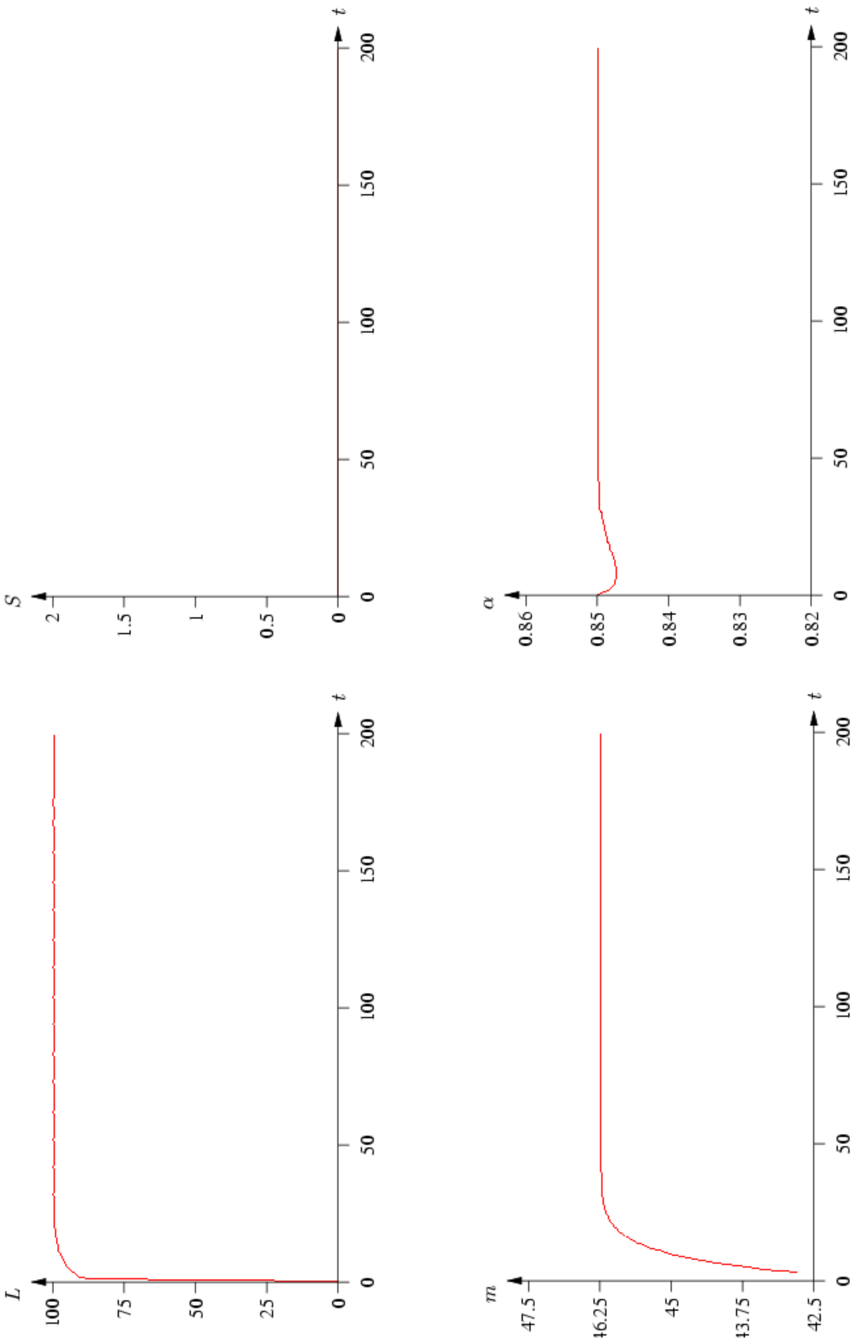

Figure 6.7: Time series when $\nu_1 = 0.025$, $m_0 = 40$ and $S_t \equiv 0$.

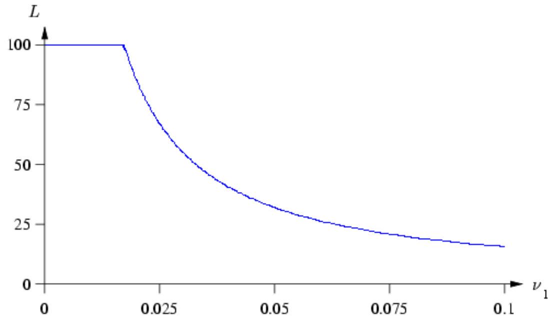

Figure 6.8: Stationary employment values when $m_0 = 40$.

7. Conclusions

In this paper we have developed a non-tâtonnement dynamic macroeconomic model involving temporary equilibria with fixprices and stochastic rationing in each period, and price adjustment between periods. The model allows for trade also when prices are not at their market clearing levels, and consistent allocations are described in every period, obeying at the same time a well defined dynamics.

The analysis builds on our previous research that, using a similar analytical framework, has shown both the existence of business cycles with complex dynamics and of quasi-stationary states characterized by permanent unemployment or capacity underutilization (Bignami, Colombo and Weinrich, 2000), as well as the emergence of the Phillips curve as a chaotic attractor (Colombo and Weinrich, 2000).

This paper adds to the previous ones by explicitly introducing inventories. On modelling grounds, this enriches the structure of the economy, by allowing for the emergence of the underconsumption disequilibrium regime, that was absent in the previous studies. Technically, moreover, inventories are an additional state variable that further complicates the dynamic system describing the behavior of the economy, making it four-dimensional. As it has been already the case with the three-dimensional dynamic systems studied in our previous papers, the dynamics is too complex to be fully understood by means of analytical tools only, requiring the use of simulations to shed light on it.

Our simulations, in addition to confirming all the results already obtained in the previous papers, allow for novel economic insights, whose emergence is indeed related to the explicit consideration of inventories. In particular, it emerges that money is not necessarily neutral in the long run. Starting from a Walrasian equilibrium, a restrictive monetary shock can cause the economy to end up in a permanent recession, i.e. in a quasi-stationary state in which employment and output are permanently below their Walrasian levels and there is permanent deflation of nominal variables (price, wage and the nominal money stock). Such a situation resembles what we have been witnessing for Japan since the end of the Nineties, with increasing unemployment rates and decreasing prices and wages.[6]

Our approach, by allowing us to characterize the economic forces behind these scenarios can help in shedding light on some possible remedies to them. Besides the stabilizing effect of (temporary) increases in public expenditure (viable when the shock hitting the economy has been originated by a monetary restriction or an increase in inventories), our numerical analysis shows that the downward speed of adjustment of wages between periods plays a crucial role in determining the impact of a restrictive monetary shock on the economy. In particular, downward wage rigidity proves capable to overcome long-term unemployment, restoring the Walrasian equilibrium. When nominal wages are rigid downwards, a restrictive monetary shock can have a temporary, but not a lasting effect on the economy. However, when downward wage flexibility is allowed, employment, real wages and real money can converge to values lower than the respective Walrasian levels. The reason is that the presence of inventories amplifies the fall of labor demand following the decrease in aggregate demand originated by the restrictive shock, further depressing real labor income and aggregate demand. As long as the downward wage flexibility is below a certain threshold level, the economy is capable of returning to full employment; above it, however, it gets stuck in a permanent recession.

It is worth emphasizing that these results depend crucially on the possibility of modelling the quantity spillover effects between markets, which in turn is rendered possible using as modelling strategy the non-tâtonnement approach and the adoption of the concept of equilibrium with quantity rationing.

[6]According to OECD statistics (OECD Main Economic Indicators, May 2003), the Japanese standardized unemployment rates increased steadily from 4.7 in 2000 to 5.4 in the first quarter 2003. At the same time, the consumer price index (base 1995=100) fell from 101.5 in 2000 to 99.4 in the first quarter 2003. Similarly, the producer price index (base 1995=100) fell from 96.1 to 91.6 in the same period. Finally, hourly earnings (base 1995=100) in the manufacturing sector decreased from 105 in 2000 to 104 in 2002.

Appendix 1: Lemma 2.

Lemma 2. *When the production function is* $f(\ell) = a\ell^b$, *with* $a > 0$ *and* $0 < b \leq 1 - \sigma$, *the solution to the firm's maximization problem is independent of the constraint* $\ell_t^d \leq \frac{d_t}{\alpha_t}\left[f\left(\ell_t^d\right) + s_t\right]$.

Proof. The first order condition for an interior solution of the firm's problem is

$$\gamma^s f'(\ell) = \alpha \Leftrightarrow \gamma^s \frac{bf(\ell)}{\ell} = \alpha \Leftrightarrow \ell = \gamma^s \frac{bf(\ell)}{\alpha}.$$

Moreover the inequalities $\frac{1}{b} \geq \frac{1}{1-\sigma} \geq \frac{1-\gamma^s\sigma}{1-\sigma}$ yield $1 \leq \frac{1-\sigma}{b(1-\gamma^s\sigma)}$. From this follows

$$\ell \leq \frac{\gamma^s(1-\sigma)}{1-\gamma^s\sigma}\frac{1}{\gamma^s}\frac{1}{b}\ell = d\frac{1}{\gamma^s}\frac{1}{b}\ell = d\frac{1}{\gamma^s}\frac{1}{b}\gamma^s\frac{bf(\ell)}{\alpha} = \frac{d}{\alpha}f(\ell),$$

which proves our claim. ■

Appendix 2: The complete dynamic system

The dynamic system is given by four different subsystems, one for each of the equilibrium types K, I, C and U and endogenous regime switching. For given (G, tax), any list $(\alpha_t, \pi_t, m_t, S_t)$ gives rise to a uniquely determined equilibrium allocation $\left(\overline{L}_t, \overline{Y}_t\right)$ being of one of the above types (or of an intermediate one). This type is determined according to the procedure described in section 3. More precisely,

$$\overline{L}_t = \min\left\{\widetilde{L}(\alpha_t, \pi_t, m_t, S_t, G, tax), L^d(1, \alpha_t), L^s\right\}$$

where $\widetilde{L}(\alpha_t, \pi_t, m_t, S_t, G, tax)$ is the unique solution in L of

$$\alpha_t\left(\frac{1}{b} - h\right)L + \frac{\alpha_t}{ab}\left(\frac{L}{n'}\right)^{1-b}S_t = h(1 - tax)\pi_t + m_t + G$$

and

$$L^d(1, \alpha_t) = n'\left(\frac{\alpha_t}{ab}\right)^{\frac{1}{b-1}}.$$

If $\overline{L}_t = \widetilde{L}(\cdot)$, the K-subsystem applies whereas if $\overline{L}_t = L^d(\cdot)$ type C occurs. Finally, when $\overline{L}_t = L^s$ an equilibrium of type I occurs if $\frac{\alpha_t}{b}L^s + S_t \leq h(1 - tax)\pi_t + h\alpha_t L^s + m_t + G$; otherwise the equilibrum is of type U. Regime switching may occur because $\left(\overline{L}_t, \overline{Y}_t\right)$ may be of type $T \in \{K, I, C, U\}$ and $\left(\overline{L}_{t+1}, \overline{Y}_{t+1}\right)$ of type $T' \neq T$. Regarding the subsystems, they are the following.

Keynesian unemployment system

Employment level: $\overline{L}_t = \widetilde{L}\left(\alpha_t, \pi_t, m_t, S_t, G, tax\right).$
Output level: $\overline{Y}_t = \frac{\alpha_t}{b}\overline{L}_t + \frac{\alpha_t}{ab}\left(\frac{\overline{L}_t}{n'}\right)^{1-b} S_t.$
Rationing coefficients: $\lambda_t^s = \frac{\overline{L}_t}{L^s}$, $\lambda_t^d = 1$, $\gamma_t^s = \frac{\alpha_t}{ab}\left(\frac{\overline{L}_t}{n'}\right)^{1-b}$, $\gamma_t^d = 1$, $\delta_t = \varepsilon_t = 1.$
Price inflation: $\theta_t = 1 - \mu_1\left(1 - \gamma_t^s\right).$
Real wage adjustment: $\alpha_{t+1} = \frac{1-\nu_1(1-\lambda_t^s)}{1-\mu_1(1-\gamma_t^s)}\alpha_t.$
Real profit: $\pi_{t+1} = \frac{1}{\theta_t}\left(\overline{Y}_t - \alpha_t\overline{L}_t\right) = \frac{1-b}{\theta_t(1-hb)}\left[h\left(1-tax\right)\pi_t + m_t + G\right].$
Real money stock: $m_{t+1} = \frac{1}{\theta_t}\left[m_t + G + \left(1-tax\right)\pi_t\right] - \pi_{t+1}.$
Inventories: $S_{t+1} = n'a\left(\frac{ab\gamma_t^s}{\alpha_t}\right)^{\frac{b}{1-b}} + S_t - \overline{Y}_t.$

Repressed inflation system
(*It applies when* $\frac{\alpha_t}{b}L^s + S_t \leq h\left(1-tax\right)\pi_t + h\alpha_t L^s + m_t + G$)

$\overline{L}_t = L^s.$
$\overline{Y}_t = \frac{\alpha_t}{b}\overline{L}_t + S_t.$
$\lambda_t^s = 1$, $\lambda_t^d = \frac{L^s}{L^d(1,\alpha_t)}$; $\gamma_t^s = 1.$
If $\overline{Y}_t \geq G + m_t$, then $\gamma_t^d = \frac{\overline{Y}_t - m_t - G}{h(1-tax)\pi_t + h\alpha_t\overline{L}_t}$, $\delta_t = \varepsilon_t = 1;$
if $G + m_t > \overline{Y}_t \geq G$, then $\gamma_t^d = 0$, $\delta_t = \frac{\overline{Y}_t - G}{m_t}$, $\varepsilon_t = 1;$
if $\overline{Y}_t < G$, then $\gamma_t^d = \delta_t = 0$, $\varepsilon_t = \frac{\overline{Y}_t}{G}.$
$\theta_t = 1 + \mu_2\left(1 - \frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right).$
$\alpha_{t+1} = \frac{1+\nu_2\left(1-\lambda_t^d\right)}{1+\mu_2\left(1-\frac{\gamma_t^d+\delta_t+\varepsilon_t}{3}\right)}\alpha_t.$
$\pi_{t+1} = \frac{1}{\theta_t}\left(\overline{Y}_t - \alpha_t\overline{L}_t\right) = \frac{\alpha_t}{\theta_t}\frac{1-b}{b}L^s.$
$m_{t+1} = \frac{1}{\theta_t}\left[\delta_t m_t + \varepsilon_t G + \left(1-tax\right)\pi_t\right] - \pi_{t+1}.$
$S_{t+1} = \lambda_t^d n'a\left(\frac{ab}{\alpha_t}\right)^{\frac{b}{1-b}} + S_t - \overline{Y}_t.$

Classical Unemployment System

$\overline{L}_t = L^d(1, \alpha_t)$.
$\overline{Y}_t = \frac{\alpha_t}{b}\overline{L}_t + S_t$.
$\lambda_t^s = \frac{\overline{L}_t}{L^s}$, $\lambda_t^d = 1$, $\gamma_t^s = 1$;
if $\overline{Y}_t \geq G + m_t$, then $\gamma_t^d = \frac{\overline{Y}_t - m_t - G}{h(1-tax)\pi_t + h\alpha_t\overline{L}_t}$, $\delta_t = \varepsilon_t = 1$;
if $G + m_t > \overline{Y}_t \geq G$, then $\gamma_t^d = 0$, $\delta_t = \frac{\overline{Y}_t - G}{m_t}$, $\varepsilon_t = 1$;
if $\overline{Y}_t < G$, then $\gamma_t^d = \delta_t = 0$, $\varepsilon_t = \frac{\overline{Y}_t}{G}$.
$\theta_t = 1 + \mu_2\left(1 - \frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right)$.
$\alpha_{t+1} = \frac{1-\nu_1(1-\lambda_t^s)}{1+\mu_2\left(1-\frac{\gamma_t^d+\delta_t+\varepsilon_t}{3}\right)}\alpha_t$
$\pi_{t+1} = \frac{1}{\theta_t}\left(\overline{Y}_t - \alpha_t\overline{L}_t\right) = \frac{1-b}{\theta_t}n'\left(\frac{\alpha_t}{ab}\right)^{\frac{b}{b-1}}\left(\frac{1}{a}\right)^{\frac{1}{b-1}}$.
$m_{t+1} = \frac{1}{\theta_t}\left[\delta_t m_t + \varepsilon_t G + (1-tax)\pi_t\right] - \pi_{t+1}$.
$S_{t+1} = n'a\left(\frac{ab}{\alpha_t}\right)^{\frac{b}{1-b}} + S_t - \overline{Y}_t$.

Underconsumption
(*It applies when* $\frac{\alpha_t}{b}L^s + S_t > h(1-tax)\pi_t + h\alpha_t L^s + m_t + G$)

$\overline{L}_t = L^s$.
$\overline{Y}_t = h(1-tax)\pi_t + h\alpha_t L^s + m_t + G$.
$\lambda_t^s = 1$, $\lambda_t^d = \frac{L^s}{L^d(\gamma_t^s, \alpha_t)} = \frac{(ab\gamma_t^s)^{1/(1-b)}L^s}{n'\alpha_t^{1/(1-b)}}$;
$\gamma_t^s = \frac{\alpha_t}{ab}\left(\frac{\overline{L}_t}{n'}\right)^{1-b}$, $\gamma_t^d = 1$, $\delta_t = \varepsilon_t = 1$.
$\theta_t = 1 - \mu_1(1 - \gamma_t^s)$.
$\alpha_{t+1} = \frac{1+\nu_2\left(1-\lambda_t^d\right)}{1-\mu_1(1-\gamma_t^s)}\alpha_t$.
$\pi_{t+1} = \frac{1}{\theta_t}\left(\overline{Y}_t - \alpha_t\overline{L}_t\right) = \frac{1}{\theta_t}\left[h(1-tax)\pi_t + m_t + G - \alpha_t(1-h)L^s\right]$.
$m_{t+1} = \frac{1}{\theta_t}\left[m_t + G + (1-tax)\pi_t\right] - \pi_{t+1}$.
$S_{t+1} = \lambda_t^d n'a\left(\frac{\gamma_t^s ab}{\alpha_t}\right)^{\frac{b}{1-b}} + S_t - \overline{Y}_t$.

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