CALCULATING AND USING SECOND ORDER ACCURATE SOLUTIONS OF DISCRETE TIME DYNAMIC EQUILIBRIUM MODELS

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I. INTRODUCTION

It is now widely understood how to obtain first-order accurate approximations to the solution to a dynamic, stochastic general equilibrium model (DSGE model). Such solutions are fairly easy to construct and useful for a wide variety of purposes. They are likely to be accurate enough to be a basis for fitting the models to data, for example.

However, for some purposes first-order accuracy is not enough. This is true in particular for comparing welfare across policies that do not have first-order effects on the model's deterministic steady state, for example. It is also true for attempts to study asset pricing in the context of DSGE models. It is possible to assume directly that nonlinearities are themselves small in certain dimensions as a justification for use of first-order approximations in these contexts; Woodford (2002) is an example of making the necessary auxiliary assumptions explicit. But the usual reliance on local approximation being generally locally accurate does not apply to these contexts.

It is therefore of some interest to have an algorithm available that will produce second-order accurate approximations to the solutions to DSGE's from a straightforward second-order expansion of the model's equilibrium equations, and this is an active area of recent research.

Kenneth Judd pioneered this field by using perturbation methods in solving various types of economic models[1]. Jin and Judd (2002) describe how to compute approximations of arbitrary order in discrete-time rational expectations models.

[1]Judd (1998). For continuous time models see Gaspar and Judd (1997) as well.

They aim at providing a complete set of regularity conditions justifying the local approximations, and they discuss methods for checking the validity of the approximations. Others also have studied perturbation methods of higher than first order including Collard and Juillard (2000), Anderson and Levin (2002), and Schmitt-Grohé and Uribe (2002).

Kim and Kim (forthcoming) and Sutherland (2002) have developed a bias correction method that produce the same results as the second order perturbation method for certain welfare calculations, while requiring less computational effort than a full perturbation solution.

Several papers have applied the second-order perturbation method to dynamic general equilibrium models. Kim and Kim (2003) used the second-order solution method to analyze welfare effects of tax policies in a two-country framework. In particular, they calculate the optimal degree of response for various tax rates against TFP shocks faced by each country. Welfare gains of tax policies are measured by conditional welfare changes from the benchmark case. Kollmann (2002) has analyzed the welfare effects of monetary policies in open economies using the software that has been developed along with this paper, and Bergin and Tchakarov (2002) have used it to examine the welfare effects of exchange rate risk.

This paper describes the algorithm for computing a second order approximation and shows how to apply it to calculating forecasts and impulse responses in dynamic models and to evaluating welfare in DSGE models. It points out some necessary regularity conditions for application of the method and discusses the sense in which the approximate solutions are locally accurate.

While much of the paper parallels others in this rapidly growing literature, this paper makes some new contributions. The rest of the literature in most cases begins from a formulation of the problem in which a partition of variables in the model into "states" and "controls" or "co-states" is assumed known. While in smaller models such a partition is often obvious, in larger models it can be unclear how to partition the variables into states and controls. The Matlab program `gensys.m`, implementing the approach described in Sims (2002), accepts model specifications that do not partition the variable list into states and co-states, constructing its own version of such a partition automatically. This paper extends that approach to second-order approximations.

The "state-free" approach of `gensys.m` has the disadvantage that its output, while completely characterizing the dynamics in terms of the original variables, includes only its own artificial decomposition into states and co-states, which may be opaque. For some purposes it is important to have an intuitively appealing decomposition into states and co-states. We discuss how to do this, with the aid

of another program, `gstate.m`, that uses the output of `gensys.m` or `gensys2.m` to test proposed state vectors and and to provide guidance as to what a valid state vector must look like.

Where the sense in which accuracy of local expansions is claimed has been made explicit in the literature, it has for the most part (Jin and Judd, 2002, most prominently) focused on accuracy of the function mapping state variables to co-states. It has also tended to assert as regularity conditions almost-sure boundedness of stochastic disturbances and stationarity of the dynamic model being studied. These assumptions allow strong claims to be made about approximation accuracy, but they are disquieting for most DSGE modeling applications. Models with unit roots, or even mild explosiveness, are not uncommon in macroeconomics, and models with near-unit roots are the rule. Often disturbance distributions with unbounded support seem more realistic than any particular truncation to bounded support. If perturbation methods break down, or are at the edge of their domain of applicability, for such models, they might seem to be unattractive for many of the models to which they have in fact been applied.

In this paper we argue that boundedness of shocks and stationarity of the model are not essential to the validity of perturbation methods. For their main applications so far, perturbation methods can be shown to produce results that are in a natural sense locally accurate, without the invocation of the dubious stationarity and boundedness assumptions.

There is little explicit discussion in the literature of how to use higher order perturbation approximations in constructing simulations, forecasts, and welfare evaluations. We show that some apparently obvious approaches to these tasks in fact result in an accumulation of "garbage" high-order terms that can make accuracy deteriorate. We lay out an algorithm that always produces stationary second-order accurate dynamics whenever the first-order dynamics are stable.

The Matlab code that was built along with this paper is available at http://eco-072399b.princeton.edu/yftp/gensys2/, where the current version of this paper will also be found.

II. The General Form of the Model

We suppose a model that takes the form

$$\underset{n\times 1}{K} \left(\underset{n\times 1}{w_t}, \underset{n\times 1}{w_{t-1}}, \underset{m\times 1}{\sigma\varepsilon_t} \right) + \underset{p\times 1}{\Pi\sigma\eta_t} = 0 \,, \tag{1}$$

where $E_t\eta_{t+1} = 0$ and $E_t\varepsilon_{t+1} = 0$.[2] The equations hold for $t = 0, \ldots, \infty$, as does the $E_t\varepsilon_{t+1} = 0$ condition. The disturbances ε_t are exogenously given, while η_t is

determined as a function of ε when the model is solved, if the solution exists and is unique. Note that because there is no assumption at all about η_0, it is a free vector that is likely to make certain linear combinations of the equations tautological at the initial date.

The scale factor σ is introduced to allow us to shrink the distribution of ε_t toward zero as we seek a domain of validity for our local approximation. The distribution of ε_t itself is assumed to be constant across time t and invariant to changes in σ, so that in particular it has a fixed covariance matrix Ω.

The equation system could be written equivalently as

$$Q_1 K(w_t, w_{t-1}, \sigma\varepsilon_t) = 0 \tag{2}$$

$$E_t[Q_2 K(w_{t+1}, w_t, \sigma\varepsilon_{t+1})] = 0 \,, \tag{3}$$

where Q_1 is any matrix such that $Q_1\Pi = 0$ and $[Q_1', Q_2']$ is a full rank square matrix. The "forward-shift" of the expectational block reflects the absence of any restriction on η_0.

We assume that the solution will imply that w_t remains always on a stable manifold, defined by $H(w_t, \sigma) = 0$ and satisfying

$$\{ \underset{n_u \times 1}{H} (w_t, \sigma) = 0, H(w_{t+1}, \sigma) = 0 \text{ a.s.} \quad \text{and} \quad Q_1 K(w_{t+1}, w_t, \sigma\varepsilon_{t+1}) = 0 \text{ a.s.}\}$$

$$\Rightarrow E_t[Q_2 K(w_{t+1}, w_t, \sigma\varepsilon_{t+1})] = 0 \,. \tag{4}$$

We consider expansion of the system about a deterministic steady state $\bar{w}$, i.e. a point satisfying $K(\bar{w}, \bar{w}, 0) = 0$. We do not need to assume the steady state is unique, so the situation arising in unit root models, where there is a continuum of steady states, is not ruled out.

We also assume that the nonlinear system (1) is formulated in such a way that its first-order expansion characterizes the first-order behavior of the deterministic solution. That is, we assume that solving the first-order expansion of (1) about $\bar{w}$,

$$K_1 dw_t = -K_2 dw_{t-1} - K_3 \sigma\varepsilon_t + \Pi\eta_t \,, \tag{5}$$

as a linear system results in a unique stable saddle path in the neighborhood of the deterministic steady state. If so, this saddle path characterizes the first-order behavior of the system. We assume further that $H_1(\bar{w}, 0)$ is of full row rank, so that the first-order character of the saddle path is determined by the first-order expansion of H.[3]

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[2]This form is more general than it might seem. See Sims (2002) for examples showing how models with explicit expectations operators, including lagged expectations, can be cast into this form.

The system (1) has the second-order Taylor expansion about $\bar{w}$

$$K_{1ij}dw_{jt} = -K_{2ij}dw_{j,t-1} - K_{3ij}\sigma\varepsilon_{jt} + \Pi_{ij}\eta_{jt}$$
$$- \tfrac{1}{2}(K_{11ijk}dw_{jt}dw_{kt} + 2K_{12ijk}dw_{jt}dw_{k,t-1} + 2K_{13ijk}dw_{jt}\sigma\varepsilon_{kt}$$
$$+ K_{22ijk}dw_{j,t-1}dw_{k,t-1} + 2K_{23ijk}dw_{j,t-1}\sigma\varepsilon_{kt} + K_{33ijk}\sigma^2\varepsilon_{jt}\varepsilon_{kt}), \quad (6)$$

where we have resorted to tensor notation. That is, we are using the notation that

$$A_{ijk}B_{mnjq} = C_{ikmnq} \quad \Leftrightarrow \quad c_{ikmnq} = \sum_j a_{ijk}b_{mnjq}. \quad (7)$$

where a, b, c in this expression refer to individual elements of multidimensional arrays, while A, B, C refer to the arrays themselves. As special case, for example, ordinary matrix multiplication is $AB = A_{ij}B_{jk}$ and the usual matrix expression $A'BA$ becomes $A_{ji}B_{jk}A_{km}$. Note that we are distinguishing the array K_{mij} of first derivatives from the array K_{mnijk} of second derivatives only by the number of indexing subscripts the two arrays have.

III. REGULARITY CONDITIONS

Because we are taking first and second derivatives and because we are expanding about the steady state $\bar{w}$, it is clear that we require existence of first and second derivatives of K at $\bar{w}$. We have also directly assumed that the first order behavior of K near $\bar{w}$ determines $H(\cdot, 0)$. In order to make our local expansion in dw, $\sigma\varepsilon$, and ε work, we will need that $H(w, \sigma)$ is continuous and twice-differentiable in both its arguments.

It may seem that these are all standard assumptions on the degree of differentiability of the system near $\bar{w}$. Consider what emerges, though, when we split the system into expectational and non-expectational components as in (2)-(3). If we replace (3) with its second-order expansion and take some expectations explicitly, we arrive at

$$E_t\big[Q_2(K_{1ij}dw_{j,t+1} + K_{2ij}dw_{jt} + \tfrac{1}{2}(K_{11ijk}dw_{j,t+1}dw_{k,t+1} + 2K_{12ijk}dw_{j,t+1}dw_{k,t}$$
$$+ K_{22ijk}dw_{j,t}dw_{k,t} + K_{33ijk}\Omega_{jk}\sigma^2))\big] = 0, \quad (8)$$

and find ourselves needing to assert that ε_t has finite second moments, which is not a local property. That is, if ε_t does not have second moments, shrinking σ will not make $\sigma\varepsilon_t$ have finite second moments. The same point applies to (3) in its original nonlinear form. If it is to be differentiable in w_t and σ, we will in general

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[3]This assumption on H is not restrictive. So long as there is a continuous, differentiable saddle manifold, we will be able to choose an H satisfying this condition.

need to impose restrictions on the distribution of ε_t. Jin and Judd (2002) have an example of a model in which some apparently natural choices of a distribution for ε_t imply that $E_t[Q_2 K(w_{t+1}, w_t, \sigma \varepsilon_{t+1})]$ is discontinuous in σ at $\sigma = 0$, even though K has plenty of derivatives at the steady state.

IV. SOLUTION METHOD

The solution we are looking for can be written in the form

$$w_t = F^*(w_{t-1}, \sigma \varepsilon_t, \sigma) . \tag{9}$$

Because we know the saddle manifold characterized by H exists and that $H_1(\bar{w}, \sigma)$ has full row rank n_u, we can use H to express n_u linear combinations of w's in terms of the remaining $n_s = n - n_u$. Let the n_s linear combinations of w's chosen as "explanatory" variables in this relation be

$$y_t = \underset{n_s \times n}{\Phi} w_t . \tag{10}$$

Then the solution (9) can be expressed equivalently, in a neighborhood of $\bar{w}$, as

$$y_t = \Phi \, F^*(w_{t-1}, \sigma \varepsilon_t, \sigma) = F(y_{t-1}, x_{t-1}, \sigma \varepsilon_t, \sigma) \tag{11}$$

$$\underset{n_u \times 1}{x_t} = h(y_t, \sigma) , \tag{12}$$

where (12) is just the solved version of the $H = 0$ equation that characterizes the stable manifold. Here of course x, like y, is a linear combination of w's.

The appearance of x_{t-1} in (11) may seem redundant, since along the solution path we will have $x_t = h(y_t, \sigma)$, but at the initial date the lagged w vector might not satisfy this restriction. This is likely in a growth model with multiple types of capital, for example, where there may be optimal proportions of capital of different types, but no physical requirement that the initial endowments are in these proportions.[4]

The solution method for linear rational expectations systems described in Sims (2002) begins by applying linear transformations to the list of variables and to the equation system to produce an upper triangular block recursive system. In the transformed system, the unstable roots of the system are all associated with the lower right block, η_t does not appear in the upper set of equations in the system,[5] and the upper part of the equation system is normalized to have the identity as the coefficient matrix on current values of the upper part of the transformed variable

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[4]See section V below for further discussion of this point.

[5]It may not be possible in fact to eliminate η_t from the upper part of the system. When it is not, the solution is not unique. The programs signal the non-uniqueness and deliver one solution, in which the η's are set to zero in the upper block of this system.

matrix. In other words, by applying to the equation system the same sequence of linear operations as applied in the earlier paper to a linear system[6], we can transform (6) to

$$
\begin{aligned}
dy_{it} =& G_{1ij}dx_{jt} + G_{2ij}dv_{j,t-1} + G_{3ij}\varepsilon_{jt} + \tfrac{1}{2}\Big(G_{11ijk}dv_{jt}dv_{kt} + 2G_{12ijk}dv_{jt}dv_{k,t-1} \\
&+ 2G_{13ijk}dv_{jt}\varepsilon_{kt} + G_{22ijk}dv_{j,t-1}dv_{k,t-1} + 2G_{23ijk}dv_{j,t-1}\varepsilon_{kt} + G_{33ijk}\varepsilon_{jt}\varepsilon_{kt}\Big)
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
J_{1ij}dx_{jt} =& J_{2ij}dx_{j,t-1} + J_{3ij}\varepsilon_{jt} + \Pi^*\eta_t + \tfrac{1}{2}\Big(J_{11ijk}dv_{jt}dv_{kt} + 2J_{12ijk}dv_{jt}dv_{k,t-1} \\
&+ 2J_{13ijk}dv_{jt}\varepsilon_{kt} + J_{22ijk}dv_{j,t-1}dv_{k,t-1} + 2J_{23ijk}dv_{j,t-1}\varepsilon_{kt} + J_{33ijk}\varepsilon_{jt}\varepsilon_{kt}\Big),
\end{aligned}
\tag{14}
$$

where $v_t = [y_t; x_t]$, i.e. the y and x vectors stacked up.

Now the y and x introduced above may seem to have no connection to the y and x in terms of which we wrote the solution (11)-(12). But that solution has second-order expansion

$$
\begin{aligned}
dy_{it} =& F_{1ij}dv_{j,t-1} + F_{2ij}\varepsilon_{jt} + F_{3i}\sigma^2 \\
&+ \tfrac{1}{2}\Big(F_{11ijk}dv_{j,t-1}dv_{k,t-1} + 2F_{12ijk}dv_{j,t-1}\varepsilon_{kt} + F_{33ijk}\varepsilon_{jt}\varepsilon_{kt}\Big)
\end{aligned}
\tag{15}
$$

$$
dx_{it} = \tfrac{1}{2}M_{11ijk}dy_{jt}dy_{kt} + M_2\sigma^2.
\tag{16}
$$

Of course if x were chosen as an arbitrary linear combination of w's, there would in general be a first-order term in dy_t on the right-hand side of (16). However, we can always move such terms to the left-hand side and then redefine x to include them. We will now proceed to show that the dy and dx in (15)-(16) are indeed those in (13)-(14), and that indeed we can construct the coefficient matrices in the former from knowledge of the coefficient matrices in the latter.

The terms in σ in (15)-(16) deserve discussion. As can be seen from (8), the appearance of expectations operators in our system makes it depend on the distribution of ε, not just on realized values of ε. But there is only one term in (8) that is first-order in dw_{t+1}. All the other terms are second-order, or depend on dw_t or σ^2, not σ. Therefore if there were a component of $Q_1K_1dw_{t+1}$ that depended on σ (rather than σ^2), that term could not be zero as the equation requires. Hence we can be sure that there is no term linear in σ in the second order expansion of (2)-(3), and thus none in (15)-(16). This then also rules out any term of the form $\sigma \cdot \sigma\varepsilon_{t+1}$ also, since such a term could enter only through the cross products in $dw_{t+1}\sigma\varepsilon_{t+1}$ or through the $dw_{t+1}dw_{t+1}$ terms, and without a first-order term in σ in dw_{t+1}, these cross products can generate no $\sigma \cdot \sigma\varepsilon_{t+1}$ terms.

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[6]and implemented in the Matlab function `gensys.m`

Observe that dx_t in (13)-(14) must be zero to first order (except for $t = -1$), because otherwise there would be an explosive component in the first order part of the solution, contradicting the stability assumption. Therefore, F_1 is exactly G_2 from (13). Clearly also $F_2 = G_3$. Therefore we have a complete first-order solution for dy and dx in hand:

$$dy_t \doteq F_1 dv_{t-1} + F_2 \varepsilon_t \tag{17}$$

$$dx_t \doteq 0 . \tag{18}$$

We find the second order terms in the following steps. First shift (14) forward in time by one (so that the left-hand side is dx_{t+1}) and substitute the right-hand side of (16), shifted forward in time by 1, for the dx_{t+1} on the left. Then substitute the right-hand-side of (17), shifted forward by 1, for all occurrences of dy_{t+1} in the resulting system. Finally apply the E_t operator to the result. In doing this, we are dropping all the second order terms in the solution for dy and dx when these terms themselves occur in second order terms. This makes sense because cross products involving terms higher than first order are third order or higher, and thus do not contribute to the second order expansion. Note that this means that, since dx is zero to first order, in (13)-(14) all the second-order terms in dv can be written in terms of dy alone. We will abuse notation by using the same G and J labels for the smaller second-order coefficient matrices that apply to dy alone that we use in (13)-(14) for the second order terms involving the full v vector. In this way we arrive at

$$J_{1ij}\left(\tfrac{1}{2}\left(M_{11jk\ell}F_{1kr}dy_{rt}F_{1\ell s}dy_{st} + M_{11jk\ell}F_{2kr}F_{2\ell s}\Omega_{rs}\sigma^2\right) + M_{2j}\sigma^2\right)$$
$$= J_{2ij}\left(\tfrac{1}{2}M_{11jk\ell}dy_{kt}dy_{\ell t} + M_{2j}\sigma^2\right) + \tfrac{1}{2}\Big(J_{11ijk}\left(F_{1jr}F_{1ks}dy_{rt}dy_{st}\right.$$
$$\left. + F_{2jr}F_{2ks}\Omega_{rs}\sigma^2\right) + 2J_{12ijk}F_{1jr}dy_{rt}dy_{kt} + 2J_{13ijk}F_{2jr}\Omega_{rk}\sigma^2$$
$$+ J_{22ijk}dy_{jt}dy_{kt} + J_{33ijk}\Omega_{jk}\sigma^2\Big) , \quad (19)$$

Where we have set $\mathrm{Var}(\varepsilon_t) = \sigma^2\Omega$.

For this equation to hold for all dy and σ^2 values, we must match coefficients on common terms. Therefore, looking at the $dy_t \cdot dy_t$ terms, we conclude that

$$J_{1ij}M_{11jkt}F_{1kr}F_{1\ell s} = J_{2ij}M_{11jrs} + J_{11ijk}F_{1jr}F_{1ks} + 2J_{12ijs}F_{1jr} + J_{22ijk} . \tag{20}$$

This is a linear equation, and every element of it is known except for $M_{11...}$. The transformations that produced the block-recursive system with ordered roots guarantee that $J_{2..}$, an ordinary 2×2 matrix, has all its eigenvalues above the critical stability value. It is therefore invertible, and we can multiply (20) through on the

left by J_2^{-1}, to get a system in the form

$$AM^*F_1 \otimes F_1 = M^* + B . \qquad (21)$$

In this equation, M^* is the ordinary $n_s \times n_s^2$ matrix obtained by stacking up the second and third dimensions of $M_{11...}$, $A = J_2\backslash J_1$,[7] and B is everything else in the equation that doesn't depend on M^*. If the dividing line we have specified between stable and unstable roots is $1 + \delta$, then our construction of the block-recursive system has guaranteed that $J_2\backslash J_1$ has all its eigenvalues $\leq 1/(1 + \delta)$, while at the same time it is a condition on the solution that all the eigenvalues of F_1 be $< 1 + \delta$. To guarantee that a second-order solution exists, we require that the product of the largest eigenvalue of $F_1 \otimes F_1$, which is the square of the largest eigenvalue of F_1, be less than the inverse of the largest eigenvalue of $A = J_2\backslash J_1$. If $\delta = 0$ this condition is automatically satisfied. Otherwise, there is an extra condition that was not required for finding a solution to the linear system: the smallest unstable root must exceed the square of the largest stable root.

Assuming this condition holds, (21) has the form of a discrete Lyapunov or Sylvester equation that is guaranteed to have a solution. Because of the special structure of $F_1 \otimes F_1$, it would be very inefficient to solve this system with standard packages (like Matlab's `lyap.m`), but it is easy to exploit the special structure with a doubling algorithm to obtain an efficient solution for M^*.

With $M_{11...}$ in hand, it is easy to see from (19) that we can obtain a solution for M_2 by matching coefficients on σ^2. The only slightly demanding calculation is a required inversion of $J_2 - J_1$. But since $J_2\backslash J_1$ has all its eigenvalues less than one, this $J_2 - J_1$ is guaranteed to be nonsingular.

The next step is to use (16) to substitute for the first-order term in dx_t on the right of (13) and (17)-(18) to substitute for all occurrences of dy_t and dx_t in second-order terms on the right in the resulting equation. This produces an equation with dy_t on the left, and first and second-order terms in dy_{t-1} and ε_t and terms in σ^2 on the right. With M_{11} and M_2 in hand, it turns out that it is only a matter of bookkeeping to read off the values of F_{12}, F_{22}, and F_3 by matching them to the collected coefficients in this equation.[8]

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[7]We are using the nonstandard notation $A\backslash B$ to stand for $A^{-1}B$.

[8]This bookkeeping is not trivial to program, but it is probably best for those who need to program it to consult the program, rather than take up space here with the bookkeeping.

V. ANALYZING THE STATE REPRESENTATION

The `gensys.m` program produces as output, among other things, a first-order expansion of (9), as

$$dw_t \doteq F_1^* dw_{t-1} + F_2^* \sigma \varepsilon_t \,. \tag{22}$$

To find a conventional state-space representation of such a system, we can form a singular value decomposition

$$[F_1^* F_2^*] = \begin{bmatrix} U & V \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_1' & R_2' \\ S_1' & S_2' \end{bmatrix} \,, \tag{23}$$

where $[U \ V]$ and $[R \ S]$ are orthonormal matrices and D is diagonal. Any state vector z_t that has the property that w_t is determined by z_t in this system will have to be of the form $z_t = \theta U' w_t$. The only way w_{t-1} affects current w_t is via $R_1' w_{t-1}$. While R_1' can be the same row rank as U, it can also be less, so that a smaller "state" vector summarizes the past than is needed to characterize the current situation. Also, the rank of F_1^* can be below its number of non-zero singular values. In this case it may be possible to find a z_t that , after the system has run a few periods, summarizes the past and/or characterizes the current situation yet is of lower dimension than the rank of D.

The program `gstate.m` takes as input F_1^* and F_2^*, together with an optional candidate matrix ϕ of coefficients that might form a state vector as $z_t = \phi w_t$. The program checks whether ϕ lies in U's or R_1''s row space and returns U and R_1' for further analysis.

Once a state/co-state representation of the form $v_t = \Phi w_t$ has been settled on, where Φ is non-singular and the $v = [y; x]$ vector is partitioned into state and costate, it is straightforward to convert a first or second-order approximate solution from one co-ordinate system into the another.

VI. THE LOCAL ACCURACY OF THE APPROXIMATION

Once we have a second-order accurate approximation to the dynamics, in the form (15)-(16), we can make a claim to local accuracy of the following form:

$$dv_{t+1} = \hat{F}(dv_t, \sigma \varepsilon_{t+1}, \sigma) + o_p(\|dv_t, \sigma\|^2) \,, \tag{24}$$

where o_p means "order in probability" and $\hat{F}$ is the second-order approximation to the dynamics. That is, the error in the approximation is claimed to converge in probability to zero, at a more rapid rate than $\|dv_t, \sigma\|^2$, when $\|dv_t, \sigma\|^2$ goes to zero. This rate is the weakest kind of claim that can be made for a Taylor expansion. If we are willing to claim that third derivatives exist at the deterministic steady state, then we can replace the error term with $O_p(\|dv_t, \sigma\|^3)$. This claim does not

depend on strict boundedness of the support of the distribution of ε_t, because we are only claiming our local accuracy with a certain (high) probability. Whatever the distribution of ε_t, $\sigma\varepsilon_t$ converges in probability to zero as $\sigma \to 0$, allowing us to make this claim. Of course this is all dependent on the underlying assumption that the original nonlinear model has dynamics differentiable of sufficiently high order in σ in the neighborhood of deterministic steady state, and on the existence and continuity of the expectations that occur in the statement of the model.

This one-step-ahead "local accuracy in probability" claim obviously can be extended to a corresponding claim to accuracy n-steps-ahead for any finite n. We have made no appeal to stationarity of the system in making these claims. Of course the size of the n for which accuracy remains good at a given level of σ will in general be smaller for systems that are not stationary. But the qualitative nature of the accuracy claim is no different for non-stationary systems.

This type of finite-time-span, accuracy-in-probability claim is exactly what is appropriate for purposes of fitting a model to data — which always cover a finite time span — or for purposes of simulating the model from given initial conditions over a finite span of time. It is also exactly appropriate for the correct calculation of expected welfare, when welfare is constructed as a discounted sum of period utilities. The discounting means that accuracy of the approximation is unimportant after some time horizon in the future.

Finite-time-span, accuracy-in-probability claims will not justify estimating unconditional expectations of any functions of variables in the model via simulation. To make the effects of initial conditions die away, such simulations must cover long spans of time. If the second-order approximation is non-stationary, expectations calculated from simulations of it will of course not converge. If the true nonlinear model is non-stationary, then the true unconditional expectations will in general not exist, even though it is possible that the local second-order approximation is stationary, so again in this case it will not be possible to estimate unconditional expectations from simulated paths.

When both the true nonlinear model and the second order approximate model are stationary and ergodic, and the true unconditional expectation in question is a twice-differentiable function of σ in the neighborhood of $\sigma = 0$, then it is possible to estimate the expectation from long simulations of the approximate model, with the estimates accurate locally in σ in the usual sense. This is true even though it may be (e.g. because of unbounded support of ε_t) that with probability one the path of the model repeatedly enters regions where the local approximation is inaccurate. This is possible because as $\sigma \to 0$ the fraction of time spent in these regions goes to zero, for both the true and the approximate model.

However it will often be preferable to estimate an expectation by using the second-order approximation analytically, expanding the function whose expectation is being taken as a Taylor series and applying the methods of the next section.

VII. FORECASTING AND SIMULATION

Forecasts s steps ahead, $E_t[dw_{t+s}]$ and $\text{Var}_t[dw_{t+s}]$ are the building blocks for the calculation of impulse response functions as well as welfare.

We build the forecasts from the second-order accurate dynamic model given by (15)-(16), modified here to reflect our assumption that the initial conditions satisfy the equations of the model and that therefore $dx_{t-1} = 0$ to first order. We abuse notation by using the same F's here, for the pieces of the original F matrices corresponding to dy's, as we did for the original F matrices in (15)-(16) that corresponded to the full $dv = [dy, dx]$ vector.

$$dy_t \doteq F_{1j}dy_{j,t-1} + F_{2j}\sigma\varepsilon_{j,t} + F_3\sigma^2$$
$$+ \tfrac{1}{2}F_{11jk}dy_{j,t-1}dy_{k,t-1} + F_{12jk}\sigma dy_{j,t-1}\varepsilon_{k,t} + \tfrac{1}{2}F_{22jk}\sigma^2\varepsilon_{j,t}\varepsilon_{k,t} \tag{25}$$

$$dx_t \doteq \tfrac{1}{2}M_{11jk}dy_{j,t}dy_{k,t} + M_2\sigma^2 \tag{26}$$

We would then like to calculate, to second order accuracy, $E_t[dy_{t+s}]$ and $\text{Var}_t[y_{t+s}]$.

To begin with, note that, since the conditional mean of dy_{t+s} is of second order, the variance terms $\hat{\Sigma}_s \equiv \text{Var}_t(y_{t+s})$ are correct to second order accuracy when computed from the first-order terms in the expansion (25) alone and that, to second-order accuracy, $\text{Var}_t(x_{t+s}) = 0$ since dx_t itself is of second order.

For $s = 1$, it is easy to see from (25-26) that we have

$$d\hat{y}_{t+1} = E_t[dy_{t+1}] \doteq F_{1j}dy_{j,t} + F_3\sigma^2$$
$$+ \tfrac{1}{2}F_{11jk}dy_{jt}dy_{kt} + \frac{\sigma^2}{2}F_{22jk}\Omega_{jk} \tag{27}$$

$$d\hat{x}_{t+1} = E_t[dx_{t+1}] \doteq \tfrac{1}{2}M_{11jk}\left(d\hat{y}_{j,t+1}d\hat{y}_{k,t+1} + \hat{\Sigma}_{1jk}\right) + M_2\sigma^2. \tag{28}$$

The expression in (28) for determining $E_t[dx_{t+1}]$ from the conditional mean and variance of dy_{t+1} works equally well for determining $E_t[dx_{t+s}]$ from the conditional mean and variance of dy_{t+s} for $s > 1$. The straightforward approach to determining $d\hat{y}_{t+s}$ and $d\hat{x}_{t+s}$ is to apply (27) recursively, computing $d\hat{y}_{t+s}$ from $d\hat{y}_{t+s-1}$ and $\hat{\Sigma}_{k-1}$, etc. This procedure is in fact second-order accurate, but it introduces higher order terms into the expansion. For example, since $d\hat{y}_{t+1}$ contains quadratic terms in dy_t, and (27) makes $d\hat{y}_{t+2}$ quadratic in $d\hat{y}_{t+1}$, $d\hat{y}_{t+2}$, in a simple recursive computation, becomes quartic in dy_t. These extra high-order terms do not in general increase accuracy of the approximation, as they do not correspond to higher order

coefficients in a Taylor series expansion of the true dynamic system, and in practice often lead to explosive time paths for $d\hat{y}_{t+s}$.

To see what goes wrong, consider the simple univariate model

$$y_t = \rho y_{t-1} + \alpha y_{t-1}^2 + \varepsilon_t,$$

where $|\rho| < 1$ and $\alpha > 0$. Though this model is locally stable about its unique deterministic steady state of $\bar{y} = 0$, it has a second steady-state, at $(1 - \rho)/\alpha$. If x exceeds the other steady state, it will tend to diverge. This is likely to be a generic problem with quadratic expansions — they will have extra steady states not present in the original model, and some of these steady states are likely to mark transitions to unstable behavior.

Since the unique local dynamics are stable in a neighborhood of the steady state, it will be desirable to choose amongst the second order accurate expansions one that implies stability. Deriving sufficient conditions on the support of ε_t to guarantee non explosiveness under the iterative scheme (27)-(28) is in general a non-trivial task and therefore it is useful to have available an algorithm which generates non-explosive forecasts and simulations without imposing explicit conditions on the support of ε_t. The mere fact that the generated forecasts are stable of course does not imply superior accuracy in general, especially when shocks are not bounded. However, stationarity will in general imply that, for a given neighborhood $\mathcal{U}$ of the steady state and a given time horizon T, we can restrict σ in such a way as to make the probability of leaving $\mathcal{U}$ in time T arbitrarily small.

Obtaining a stable solution based on (27) can be achieved by pruning out the extraneous high-order terms in each iteration by computing the projections of the second order terms based on a *first-order* expansion, $d\bar{y}$, as follows:

$$d\hat{y}_{t+s} \doteq F_{1j}d\hat{y}_{j,t+s-1} + F_3\sigma^2$$

$$+ \tfrac{1}{2}F_{11jk}\left(d\bar{y}_{j,t+s-1}d\bar{y}_{k,t+s-1} + \hat{\Sigma}_{k-1,jk}\right) + \tfrac{\sigma^2}{2}F_{22jk}\Omega_{jk} \tag{29}$$

$$d\hat{x}_{t+s} \doteq \tfrac{1}{2}M_{11jk}\left(d\bar{y}_{j,t+s}d\bar{y}_{k,t+s} + \hat{\Sigma}_{s,jk}\right) + M_2\sigma^2 \tag{30}$$

$$d\bar{y}_{t+s} \doteq F_{1j}d\bar{y}_{j,t+s-1} \tag{31}$$

$$\hat{\Sigma}_{ij,s} = \sigma^2 F_{2ik}\Omega_{k\ell}F_{2j\ell} + F_{1ik}\hat{\Sigma}_{k\ell,s-1}F_{1j\ell}. \tag{32}$$

Using these equations recursively results in a $d\hat{y}_{t+s}$ series which, by construction, is quadratic in dy_t for all s. Furthermore, when the eigenvalues of F_1 are less than one in absolute value, the first order accurate solution $d\bar{y}_{t+s}$ is stable and hence so is the squared process $\left(d\bar{y}_{j,t+s}d\bar{y}_{k,t+s}\right)$. It follows that $d\hat{y}_t$ must be stable

as well.[9] Note that the F_{12} component of the second order expansion — the coefficients of the interactions between dy_{t-1} and ε_t — do not enter this recursion at all.

The same issues arise if the aim is to generate simulated time paths, rather than simply conditional expectations and variances of future variables. For this purpose, we can introduce the notation $dy_{t+s}^{(1)}$ and $dy_{t+s}^{(2)}$ for first and second order accurate simulated time paths, respectively. A recursive, non-explosive, "pruned" simulation scheme is then given by

$$
dy_{t+s}^{(2)} \doteq F_{1 \cdot j} dy_{j,t+s-1}^{(2)} + F_{2 \cdot j} \sigma \varepsilon_{j,t+s} + F_3 \sigma^2
$$
$$
+ \tfrac{1}{2} F_{11 \cdot jk} dy_{j,t+s-1}^{(1)} dy_{k,t+s-1}^{(1)} + \sigma F_{12 \cdot jk} y_{j,t+s-1}^{(1)} \varepsilon_{k,t+s} + \tfrac{\sigma^2}{2} F_{22 \cdot jk} \varepsilon_{j,t+s} \varepsilon_{k,t+s} \tag{33}
$$

$$
dx_{t+s}^{(2)} \doteq \tfrac{1}{2} M_{11 \cdot jk} dy_{j,t+s}^{(1)} dy_{k,t+s}^{(1)} + M_2 \sigma^2 \tag{34}
$$

$$
dy_{t+s}^{(1)} \doteq F_{1 \cdot j} dy_{j,t+s-1}^{(1)} + F_{2 \cdot j} \sigma \varepsilon_{j,t+s} , \tag{35}
$$

where the F_{12} terms that could be ignored in forming conditional expectations have necessarily returned for generation of accurate simulations. By preventing buildup of spurious higher-order terms, we make stability of the simulation over a long time path more likely, while at the same time preserving second-order accuracy of the mapping from initial variable values y_t, x_t, shocks $\varepsilon_{t+1}, \ldots, \varepsilon_{t+s}$, and σ to the simulated values $y_{t+1}^{(2)}, \ldots, y_{t+s}^{(2)}$.

It can help in understanding these recursions to append the vector $dy^{(1)} \otimes dy^{(1)}$ to $dy^{(2)}$ and use matrix notation:

$$
\begin{bmatrix} dy_{t+1}^{(2)} \\ (dy_{t+1}^{(1)} \otimes dy_{t+1}^{(1)}) \end{bmatrix} = \Theta_1 \begin{bmatrix} dy_t^{(2)} \\ (dy_t^{(1)} \otimes dy_t^{(1)}) \end{bmatrix} + \Theta_2 \, \sigma^2 + \xi_{t+1} \tag{36}
$$

with

$$
\Theta_1 = \begin{bmatrix} F_1 & \tfrac{1}{2}(F_{11}^*) \\ 0 & (F_1 \otimes F_1) \end{bmatrix} \tag{37}
$$

$$
\Theta_2 = \begin{bmatrix} F_3 + \tfrac{1}{2} F_{22 \cdot jk} \Omega_{jk} \\ (F_2 \otimes F_2) \operatorname{vec}(\Omega) \end{bmatrix} \tag{38}
$$

$$
\xi_t = \begin{bmatrix} \sigma \varepsilon_t + \sigma F_{23 \cdot jk} \varepsilon_{k,t} dy_{j,t-1}^{(1)} + \tfrac{\sigma^2}{2} F_{33 \cdot jk} (\varepsilon_{jt} \varepsilon_{kt} - \Omega_{jk}) \\ \sigma^2 (\varepsilon_t \otimes \varepsilon_t - (F_2 \otimes F_2) \operatorname{vec}(\Omega)) \end{bmatrix} . \tag{39}
$$

---

[9]The same matrix eigenvalue conditions are at issue here as in section IV's discussion of existence of the solution to (21)

The F_{11}^* in the definition of Θ_1 (37) is a matrix with number of rows equal to the length of y and with the second and third dimensions of the array vectorized into a row vector — so it is an $n_s \times n_s^2$ matrix. Note that Θ_1 is upper block triangular and is stable exactly when the eigenvalues of F_1 are less than one in absolute value. Note also that, to second order accuracy,

$$\text{Var}(\xi_t) = \begin{bmatrix} \sigma^2 \Omega & 0 \\ 0 & 0 \end{bmatrix}.$$

Calculations of conditional and unconditional first and second moments can therefore be carried out using (36) as if it were an ordinary first order AR. This can be an aid to understanding, or to computation in small models, though for larger systems it is likely to be important for computational efficiency to take account of the special structure of the Θ_j matrices in (36).

VIII. WELFARE

One can easily produce cases where the second-order approximation is necessary to get an accurate evaluation of certain aspects of the model. Utility-based welfare calculation is one case. For example, calculating welfare effects of various monetary and fiscal policies or welfare effects of changes in economic environment such as financial market structure should include second-order or even higher-order terms in order to get an accurate measure. Kim and Kim (forthcoming) present an example of how inaccurate the linearized solution can be in calculating welfare using a two-country model. Using the linearized solution, welfare of autarky can appear to be higher than that of the complete markets, solely because of the inaccuracy of the linearization method. Another application in which second-order approximation is important is examination of asset price behavior in DSGE's. Linearized solutions will imply equal expected returns on all assets. Second order solutions will generate correct risk premia, though generally to analyze time variation in risk premia will require higher than second-order accuracy.

Equation (36) makes it relatively straightforward to see how to carry out a second-order accurate welfare calculation. Welfare is defined as a discounted sum of expected utility. Let the period utility function be given by $u : \mathbb{R}^{n_s} \to \mathbb{R}$.[10] Then the

---

[10]Of course often in growth models utility is a function of consumption, which is not a conventional state variable. To use the formulation we develop here, then, consumption (an x variable) has to be replaced by the corresponding component of $h(y, \sigma)$. Also, because we work entirely in terms of y, we are not covering the case where the initial distribution of w does not lie on the saddle path. The methods we describe here can be expanded to cover this case and to allow x to enter u, at the cost of some increase in the burden of notation.

utility conditional on an initial distribution of y_0 with mean and variance (μ, Σ) is

$$
U(\mu, \Sigma) = E_0 \left[\sum_{t=0}^{\infty} \beta^t u(y_t) \right] \approx
$$
$$
\frac{u(\bar{y})}{1 - \beta} + E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\nabla u(\bar{y}) dy_t^{(2)} + \tfrac{1}{2} \operatorname{vec}(\nabla^2 u(\bar{y}))' (dy_t^{(1)} \otimes dy_t^{(1)}) \right) \right] \Rightarrow \tag{40}
$$

$$
U(\mu, \Sigma) = \frac{u(\bar{y})}{1 - \beta} + \left[\nabla u(\bar{y}) \quad \tfrac{1}{2} \operatorname{vec}(\nabla^2 u(\bar{y}))' \right]
$$
$$
\cdot [I - \beta \Theta_1]^{-1} \left(\begin{bmatrix} \mu \\ \operatorname{vec}(\Sigma + \mu\mu') \end{bmatrix} + \beta(1 - \beta)^{-1} \Theta_2 \sigma^2 \right) \tag{41}
$$

If we are interested only in unconditional expected u, we can arrive at the correct formula by multiplying (41) through by $1 - \beta$ and taking the limit as $\beta \to 1$, giving us

$$
E[u(y_t)] = u(\bar{y}) + \left[\nabla u(\bar{y}) \quad \tfrac{1}{2} \operatorname{vec}(\nabla^2 u(\bar{y}))' \right] (I - \Theta_1)^{-1} \Theta_2 \sigma^2 . \tag{42}
$$

Note that in (41) we make no use, explicitly or implicitly, of F_{12}. Also note that though the matrix $I - \beta \Theta_1$ appears in the formula inverted, the utility calculation only requires

$$
\left[\nabla u(\bar{y}) \quad \tfrac{1}{2} \operatorname{vec}(\nabla^2 u(\bar{y}))' \right] \cdot (I - \beta \Theta_1)^{-1} ,
$$

whose computation is only an equation-solving problem, not a full inversion; furthermore, this part of the computation does not need to be repeated as μ and Σ are varied. Finally, note that (42) uses only $(I - \Theta_1)^{-1} \Theta_2$, regardless of the form of u. This is again an equation-solving problem. So if we are interested only in unconditional expectations, even in unconditional expectations of many different functions u, the computation of a full second-order correction may be much simpler than calculation of the full second-order expansion of the dynamics.

It is these simplifications, applied to particular models, that are the insights provided by the papers that have put forward "bias-correction" methods for making second-order accurate expected welfare computations in DSGE models (Kim and Kim, forthcoming; Sutherland, 2002).

VIII.1. **Conditional vs. Unconditional welfare.** Many papers have analyzed the welfare effects of government policies based on various utility based welfare criteria. When one compares welfare of one environment with another, there are two ways to do that. One is to compare the unconditional welfare of the two states (average value of expected utility) and another is to calculate the level of welfare conditional on an initial state (discounted sum of expected utility conditioned on the information at the initial state).[11] Conventionally, many researchers have used

unconditional welfare for evaluating policies. Examples of using unconditional welfare include Clarida, Galí, and Gertler (1999), Rotemberg and Woodford (1997, 1999), Sutherland (2002) and Kollmann (2002). Unconditional welfare compares the welfare level of the economy at the steady state. We know that it takes time for one steady state to reach another steady state and unconditional welfare neglects the welfare effects during the transitional period. It is therefore generally not in fact optimal, in problems with discounting, to use policies that maximize the unconditional expectation of one-period welfare. This is not a new point — it is the same point as the non-optimality of driving the rate of return to zero in a growth model — and it has been recognized in the DSGE literature in , e.g, Kim and Kim (2003), and Woodford (2002).

Because unconditional welfare can be computed easily, using the "bias correction" shortcut, it is important to note that using unconditional welfare can give nonsensical results. Kim and Kim (forthcoming) construct a two-country DSGE model and compute risk-sharing gains from autarky to the complete-markets economy using a second-order approximation method. Welfare is defined as conditional welfare and the results show that there are positive welfare gains from autarky to the complete-markets economy. But the unconditional welfare measure can for certain parameter values produce the paradoxical result that autarky generates a higher level of welfare than the complete markets.

The use of conditional welfare does not imply that results necessarily are tied to some particular initial state. One can condition on a distribution of values for the initial state. The critical point is that when comparing two policies or equilibria one should use the same distribution for the initial state for each. When there is no time-inconsistency problem the optimal policy will have the property that no matter what initial distribution is specified for the state, it will produce a higher conditional expectation of welfare than any other policy. However, when comparing a collection of policies that are not optimal, one may find that rankings of policies vary with the assumed distribution of the initial state.

When there is a time-inconsistency problem, the optimal policy generally depends on the initial conditions, even if we restrict attention to policy rules that are a fixed mapping from state to actions. Using a conditional expectation as the welfare measure does not avoid this problem. One attempt to get around this issue is the suggestion in, e.g., Giannoni and Woodford (2002) that policy should follow the rule that would prevail under commitment in the limit as the initial conditions recede into the past. This "timeless perspective" policy can be implemented

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[11]Woodford (2002) discusses the differences between unconditional and conditional welfare in calculating welfare effects of monetary policies.

by treating the Lagrange multipliers on private sector Euler equations as "states", and then maximizing conditional expected discounted utility.

IX. Conclusion

Use of perturbation methods to improve analysis of DSGE models is still in its early stages. Programs that automate computations for models higher than second order are just beginning to emerge. Methods of dealing with the kinds of singularities that show up in economic models — for example the indeterminacy of asset allocations in standard portfolio problems when variances are zero — are still not widely understood. And we have only begun to get a feel for where these methods are useful and what their limitations are. Real progress is being made, however, in an atmosphere that is both competitive enough to be stimulating and cooperative enough that researchers located around the world are benefiting from each others' insights.

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