LINEAR DISCRETE TIME SYSTEMS WITH BOX CONSTRAINTS

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1 Extended Abstract

We consider systems described by

$$x(t+1) = Ax(t) + Bu(t) + Cv(t) \tag{1}$$

where t is the time-index, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the control, $v \in \mathbb{R}^p$ the disturbance and matrices A, B, C are assumed known.

We define three sets

$$X = \{x : \underline{x} \leq x \leq \bar{x}\} \tag{2}$$
$$U = \{u : \underline{u} \leq u \leq \bar{u}\} \tag{3}$$
$$V = \{v : \underline{v} \leq v \leq \bar{v}\}. \tag{4}$$

A control is *feasible* if it takes values in U. A disturbance is *admissible* if it takes values in V. We assume

A1. the disturbance $v(t)$ is unknown but admissible for all t.
A2. the state $x(t)$ is observed at all t.

We recall

Definition 1 *X is invariant for (1) if for any $x(0) \in X$ there exists a sequence of feasible controls such that for $t > 0, x(t) \in X$ for all admissible disturbances.*

Definition 2 *$X_0 \subset X$ is safe if for any $x(0) \in X_0$ there exists a sequence of controls $u(t) \in U$ such that $x(t) \in X$ for all admissible disturbances.*

We tackle the following problems

P1. Given X find U such that X is invariant.
P2. Given U find X such that X is invariant.
P3. Given X find U, X_0 such that X_0 is safe.

We use repeatedly the notion of a *reach set* $\mathcal{R}(X)$. With reference to (1), $\mathcal{R}(X)$ is the set of states that can reach X in one step with a feasible control for all admissible disturbances. We recall the obvious but basic result

Theorem 3 *Invariance of X holds if and only if*

$$X \subset \mathcal{R}(X). \tag{5}$$

We give analytic form to condition (5) in a number of particular cases. Before proceeding we introduce the following notation. We denote by A_i the $i-$th row of A. Given matrix $A = [a_{ij}]$, $A^+ = [\max(a_{ij}, 0)]$, $A^- = [\min(a_{ij}, 0)]$ (and $A = A^+ + A^-$ with $A^+ \geq 0, A^- \leq 0$).

2 Case $B = I$

In this case

$$\begin{aligned} \mathcal{R}(X) &= \{x : \exists u \in U : \underline{x} \leq Ax + u + Cv \leq \bar{x}, \forall v \in V\} \\ &= \{x : \exists u \in U : \underline{x} - \underline{w} \leq Ax + u \leq \bar{x} - \bar{w}\} \end{aligned} \tag{6}$$

with

$$\underline{w}_i = \min_{\underline{v} \leq v \leq \bar{v}} C_i v = C_i^+ \underline{v} + C_i^- \bar{v} \tag{7}$$

$$\bar{w}_i = \max_{\underline{v} \leq v \leq \bar{v}} C_i v = C_i^+ \bar{v} + C_i^- \underline{v} \tag{8}$$

$$\mathcal{R}(X) \neq \emptyset \qquad \Rightarrow \qquad \bar{w} - \underline{w} \leq \bar{x} - \underline{x} \tag{9}$$

$$\begin{aligned}\mathcal{R}(X) &= \{x : \exists u \in U : \underline{x} - \underline{w} - Ax \le u \le \bar{x} - \bar{w} - Ax\} \\ &= \{x : \underline{x} - \underline{w} - Ax \le \bar{u}\} \cap \{x : \underline{u} \le \bar{x} - \bar{w} - Ax\} \\ &\cap \{x : \underline{x} - \underline{w} - Ax \le \bar{x} - \bar{w} - Ax\}\end{aligned}$$

In view of (9) the last intersection can be dropped and the first is equivalent to

$$\mathcal{R}(X) = \{x : \underline{x} - \underline{w} - \bar{u} \le Ax \le \bar{x} - \bar{w} - \underline{u}\}. \tag{10}$$

It is important to remark that the coefficients of the above linear inequalities, when put in the form

$$\begin{bmatrix} A \\ -A \end{bmatrix} x \le \begin{bmatrix} \bar{x} - \bar{w} - \underline{u} \\ -\underline{x} + \underline{w} + \bar{u} \end{bmatrix}$$

are just the entries of A and their opposites, making the computation of $\mathcal{R}(X)$ exceedingly simple: exactly $2n$ inequalities are required for an $n - th$ order system.

Next, invariance requires

$$X \subset \mathcal{R}(X) \quad \Leftrightarrow \quad \begin{cases} \max \quad A_i x \le [\bar{x} - \bar{w} - \underline{u}]_i \\ \max \, -A_i x \le [-\underline{x} + \underline{w} + \bar{u}]_i \\ \qquad \underline{x} \le x \le \bar{x} \qquad i = 1 \dots n \end{cases} \tag{11}$$

or

$$A^+ \bar{x} + A^- \underline{x} \le \bar{x} - \bar{w} - \underline{u} \tag{12}$$

$$-A^- \bar{x} - A^+ \underline{x} \le -\underline{x} + \underline{w} + \bar{u} \tag{13}$$

We assume $x \in \mathbb{R}^n$.

Collecting constraints (9,12,13)

$$\begin{bmatrix} A^+ - I & A^- \\ -A^- & -A^+ + I \\ -I & I \end{bmatrix} \begin{bmatrix} \bar{x} \\ \underline{x} \end{bmatrix} \le \begin{bmatrix} -\bar{w} - \underline{u} \\ \underline{w} + \bar{u} \\ \underline{w} - \bar{w} \end{bmatrix} \tag{14}$$

existence of an invariant box obtains if and only if (14) with $\underline{x} \le \bar{x}$ is feasible. Under (9), invariance is achieved by controls satisfying (10) or

$$\underline{k}(x) = \max(\underline{u}, \underline{x} - \underline{w} - Ax) \leq u \leq \min(\bar{u}, \bar{x} - \bar{w} - Ax) = \bar{k}(x)$$

with $\min, \max$ taken componentwise. Notice that the bounds are piecewise linear–affine functions of x. A possible control law is the *midpoint* control law, where u is chosen as the arithmetic mean of its bounds.

Notice that, for a given control law satisfying the above inequalities, control values belong to a set $\mathcal{U}(x)$ which is contained in U for all $x \in X$. We define a *control set* S as

$$S = \cup_{x \in X} \mathcal{U}(x).$$

To illustrate, a typical control set is shown below for the case

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad |v_i| \leq 1, \quad |u_i| \leq c.$$

Solving (14) it is found that no solution exists for $c < 5.05$. At $c = 5.05$ the box $|x_1| < 1, |x_2| \leq 1.025$ (dashed rectangle) is invariant. A disturbance $v(t)$ of 200 random vectors between -1 and 1 is considered. Using midpoint control, the control values are the points included (as they should) in the control set.

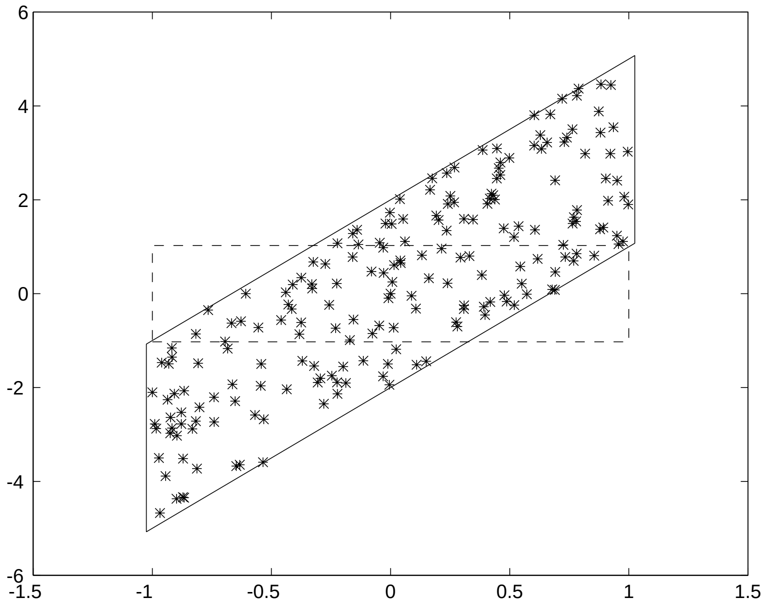

Fig. 1. Control set. Points are control values for a 200 pt sample.

3 Unbounded Box

The general case of unbounded box constraints in state space can be described as

$$\begin{cases} \bar{x}_i = \infty & \text{if } i \in \Omega \\ \underline{x}_i = -\infty & \text{if } i \in \Sigma \end{cases}$$

where Σ and Ω are (possibly overlapping) subsets of the index set $\{1 \dots n\}$. Then inequalities with row-index in Ω should be deleted from (12) and those with index in Σ should be deleted from (13). With this proviso, (14) holds (with entries suitably re–defined).

4 Tight Constraints

It is interesting to examine the case when (14) holds with =. Clearly we must assume

$$\bar{x} - \underline{x} = \bar{w} - \underline{w} \tag{15}$$

and this shows that this is the smallest[1] box in state space that is invariant wrt to $\underline{w}, \bar{w}$. It also follows that under (14) the smallest control box achieving invariance is given by

$$\begin{bmatrix} \underline{u} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} -A^+ + I & A^- \\ A^- & -A^+ + I \end{bmatrix} \begin{bmatrix} \bar{x} \\ \underline{x} \end{bmatrix} - \begin{bmatrix} \bar{w} \\ \underline{w} \end{bmatrix} \tag{16}$$

When (16) holds we shall say that tight invariance holds (or, control bounds $\underline{u}, \bar{u}$ achieve tight invariance). In the particular case $\bar{x} = \bar{w}$, $\underline{x} = \underline{x}$ we get

$$\begin{bmatrix} \underline{u} \\ \bar{u} \end{bmatrix} = - \begin{bmatrix} A^- & A^+ \\ A^+ & A^- \end{bmatrix} \begin{bmatrix} \underline{w} \\ \bar{w} \end{bmatrix} \tag{17}$$

[1] In the sense of min volume.

5 Case Square $B \neq I$

Consider

$$x(t+1) = Ax(t) + Bu(t) + Cv(t) \tag{18}$$

under constraints (2–4). We seek conditions of invariance. In principle one should calculate

$$\mathcal{R}(X) = \{x : \exists u \in U : \underline{x} - \underline{w} - Bu \leq Ax \leq \bar{x} - \bar{w} - Bu\} \tag{19}$$

$$= \{x : \exists u : \begin{bmatrix} B \\ -B \\ I \\ -I \end{bmatrix} u \leq \begin{bmatrix} \bar{x} - \bar{w} - Ax \\ -\underline{x} + \underline{w} + Ax \\ \bar{u} \\ -\underline{u} \end{bmatrix}\}. \tag{20}$$

Using the dual feasibility condition[1], it is known that $\mathcal{R}(X)$ is a convex polyhedron defined by as many linear inequalities as there are columns in

$$G = gen \begin{bmatrix} B \\ -B \\ I \\ -I \end{bmatrix}$$

where $gen(\cdot)$ is shorthand notation for the generators of the cone

$$Range\,(\cdot)^{\perp} \cap I\!R^n_+$$

(see [1] for details). The difficulty in computing G is evidenced in the table below, obtained[2] by choosing for B a random square matrix of order n

SYSTEM ORDER	SIZE(G)
$n = 1$	4×4
$n = 2$	8×12
$n = 3$	12×36
$n = 4$	mem overflow

[2] computation performed in MATLAB 4.2, on a Power Mac 6100/60

It appears that when B is a generic full-rank matrix (rather than I) the number of inequalities defining $\mathcal{R}(X)$ grows as 3^n rather than n. We circumvent the difficulty by introducing a fictitious control $c = Bu$ and then proceed as in Sec. 2. In the case of tight constraints we get

$$\begin{bmatrix} \underline{c} \\ \bar{c} \end{bmatrix} = \begin{bmatrix} -A^+ + I & A^- \\ A^- & -A^+ + I \end{bmatrix} \begin{bmatrix} \bar{x} \\ \underline{x} \end{bmatrix} - \begin{bmatrix} \bar{w} \\ \underline{w} \end{bmatrix} \tag{21}$$

It is straightforward to prove

Proposition 4 *The smallest control box achieving invariance for (18) is*

$$\text{minbox}\,(B,\, \underline{c},\, \bar{c}) = \{u : \underline{\beta} \leq u \leq \bar{\beta}\}$$

with $\quad \bar{\beta}_i = \min u_i, \quad \underline{\beta}_i = \max\, u_i \quad i = 1, 2, \ldots n$ *subject to*

$$\{u : \underline{c} \leq Bu \leq \bar{c}\} \subset \{u : \underline{\beta} \leq u \leq \bar{\beta}\}.$$

PROOF. Since constraints (14) are tight, there are no controls c outside $\{\underline{c} \leq c \leq \bar{c}\}$ achieving tight invariance of $\{\underline{x} \leq x \leq \bar{x}\}$. Assume u^* outside $\{\underline{\beta} \leq u \leq \bar{\beta}\}$ achieves tight invariance. Then control $c^* = Bu^*$ achieves tight invariance. But this is impossible as c^* falls outside $\{\underline{c} \leq c \leq \bar{c}\}$. ■

6 Case of Scalar Control

Let $B \sim n \times 1$ and define

$$(S): \quad x(t+1) = Ax(t) \;+\; Bu(t) \;+\; Cv(t) \quad t = 0, 1, 2, \cdots \tag{22}$$

$$(\tilde{S}): \quad z(t+n) = A^n z(t) + P_n \tilde{u}(t) + Q_n \tilde{v}(t) \quad t = 0, n, 2n, \cdots \tag{23}$$

with $z(0) = x(0)$ and

(A, B) *reachable*
$P_n = [A^{n-1}B \;\; \cdots \;\; AB \;\, B]$
$Q_n = [A^{n-1}C \;\; \cdots \;\; AC \;\, C]$

Notice that if

$$\tilde{u}(t) = \begin{bmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+n-1) \end{bmatrix}$$

and similarly for $\tilde{v}(t)$ then $z(t) = x(t)$ at $t = 0, n, 2n, \cdots$. It follows that if a set is invariant for $\tilde{S}$, the trajectories of S originating in this set return to it at most every $n-$th time step. We call such a property n–step invariance. Clearly, invariance is the same as 1–step invariance.

In the case of box constraints (2-4) the set of controls achieving n-step invariance of S is given by

$$\max(\underline{\beta}_\tau, \underline{x} - \underline{w}_\tau - Ax(t)) \leq u(t + \tau - 1) \leq \min(\bar{\beta}_\tau, \bar{x} - \bar{w}_\tau - Ax(t))$$

where $\tau \in \{1 \dots n-1\}$ denotes the τ–th component of $\underline{\beta}$, etc.

As n–step invariance does not rule out m–step invariance for $m < n$, it is of interest to estimate the smallest box that contains all trajectories of S originating in X. We proceed in steps, checking this condition for $t, t+1, \dots t+n$.

Let $x(t)$ lie in X. Then there exists a finite λ such that $x(t+1) \in \lambda(X - x_0) + x_0$, with x_0 chosen so as to have $0 \in X - x_0$. Choosing x_0 as the center of the box, we have

$$\frac{\bar{x} + \underline{x}}{2} - \lambda\frac{\bar{x} - \underline{x}}{2} \leq Ax + Bu + Cv \leq \frac{\bar{x} + \underline{x}}{2} + \lambda\frac{\bar{x} - \underline{x}}{2}.$$

As this must hold for all $v \in V$, we re-write the above as

$$\frac{\bar{x} + \underline{x}}{2} - \lambda\frac{\bar{x} - \underline{x}}{2} - \underline{w} \leq Ax + Bu \leq \frac{\bar{x} + \underline{x}}{2} + \lambda\frac{\bar{x} - \underline{x}}{2} - \bar{w} \tag{24}$$

with $\underline{w}\,, \bar{w}$ given by (7,8). The search is restricted to those Bu ensuring invariance of X. These are of the form

$$\underline{k}(x) \leq Bu \leq \bar{k}(x). \tag{25}$$

Finally, trajectories start in X so

$$\underline{x} \leq x \leq \bar{x}. \tag{26}$$

For given x we can compute $\hat{\lambda}(x) = \min_u \lambda$ subject to (24–26) – a LP problem.

Proposition 5 *The function $\hat{\lambda}(x)$ is convex on X.*

PROOF. The function $\hat{\lambda}(x - x_0)$ is the Minkowski functional of the convex set $X - x_0$ so it is a convex function[2] of $x - x_0$, hence of x. ■

This convexity property ensures

Corollary 6 *Let $K = 1 \dots 2^n$ and let v_k be the $k-th$ vertex of X. Then*

$$\hat{\lambda}(x) \leq \max_{k \in K} \hat{\lambda}(v_k)$$

The procedure above applies to the one-step transition from X. The computation should be repeated for the τ–step transition up to $\tau = n - 1$. This simply entails replacing A by A^τ, B by P_τ, C by Q_τ and appropriate selection of the control and disturbance bounds.

Letting $\hat{\lambda}_\tau$ be the solution of (24–26) at transition τ, we have the following

Proposition 7

$$\exists \tau : \hat{\lambda}_\tau = \hat{\lambda}_{\tau+1} = 1 \quad \Leftrightarrow \quad X \quad \textit{is invariant for (S).}$$

7 An example

Consider system (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0.2190 & 0.0470 & 0.6789 & 0.6793 & 0.9347 & 0.3835 & 0.5194 \end{bmatrix}$$

and $B = C = I$ under a disturbance vector $\mathbf{1} \leq v(t) \leq \mathbf{2}$. We consider a disturbance attenuation problem. Namely, we want to find the smallest box U centered at the origin that makes a box X invariant for (1). The results are

$$U = \{u : |u_i| \leq 2.2 \quad i = 1 \ldots 7\}$$

$$X = \left\{ \begin{bmatrix} -0.6358 \\ -0.5291 \\ -0.9209 \\ -0.9211 \\ -0.8731 \\ -0.6731 \\ -0.4731 \end{bmatrix} \leq x \leq \begin{bmatrix} 0.3642 \\ 0.4709 \\ 0.0791 \\ 0.0789 \\ 0.1269 \\ 0.3269 \\ 0.5269 \end{bmatrix} \right\}$$

Using midpoint control, the results for a simulated random disturbance $v(t) \in V$ are shown below.

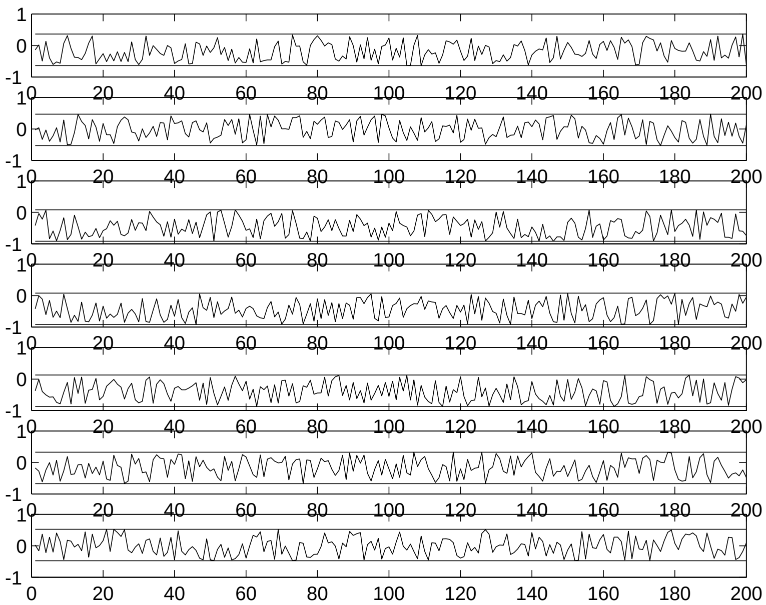

Fig. 2. State trajectories

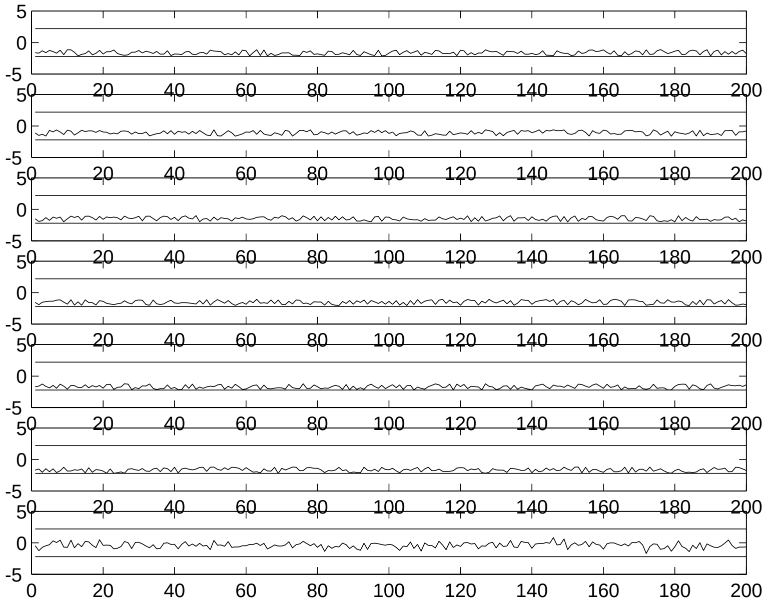

Fig. 3. Control

8 Extensions

When a box X is not invariant for a given system the procedure in Sec. 6 permits to determine U and a set of control laws able to achieve n-step invariance and, at the same time evaluate the *maximal departure* from X of a state originating in X. Departure can be minimized by solving a suitable LP problem. When this is done, one can determine a safe set X_0 by a re-scaling technique, the details of which are discussed in the full version of the paper. Also, the present study allows to evaluate safe or invariant sets under disturbance by a systematic procedure. An important problem in the control of Hybrid Systems (and in other areas) is to determine the largest invariant subset contained in a polyhedral bounding set. In principle, such an invariant can be obtained recursively, and convergence conditions are well known. However, the invariant set can be approximated from the outside or from the inside. In the first case, each approximant fails to be an invariant for the system and, in most cases serious numerical problems hinder the algorithmic success of the procedure. In the latter case, the approximants belong to an ever growing family of invariant sets, and the difficulty just mentioned largely vanishes. However, in order to start the inner approximation procedure, an invariant set must be known. Building a safe or invariant box, as discussed in this paper, might be precisely the way to overcome the difficulty. Again, these aspects are discussed at greater length in the full version of the paper.

References

[1] D'Alessandro P., (1997) *A Conical Approach to Linear Programming*, (Gordon & Breach, Amsterdam, p. 56).

[2] Kolmogorov A. N., Fomin, S. V., (1970) *Introductory Real Analysis*, (Dover Publ. NY, p. 131).