SUCCESSIVE CORRELATED DEFAULTS: PRICING TRENDS AND SIMULATION

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Abstract

We generalize the structural incomplete observation models of default proposed by Duffie & Lando (2001) and Giesecke (2001*b*) to the multi-firm case with correlated defaults. Our approach accommodates the well-documented cyclical correlation effects as well as default cascading effects, which are implied by the incomplete information of investors. We explicitly construct the pricing trend and the intensity of the first, second, etc. default. These results furnish the well-known intensity based price representations of various multi-name credit derivatives and structured credit products. Based on the pricing trend, we also formulate an algorithm for the simulation of successive, correlated, and unpredictable default arrival times.

Key words: correlated defaults; incomplete information, pricing trend, intensity, simulation.

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1 Introduction

Credit risk refers to the risk of incurring losses due to changes in a counterparty's credit quality. Credit derivatives, which have become quite successful over recent years, allow to isolate and trade that risk by providing a payoff upon a credit event arrival, such as rating downgrade, failure to pay, or bankruptcy of the reference credit. Multi-name or basket credit derivatives reference a basket of several credits. As such they provide a means to reduce or gain the credit risk profile exposure associated with a pool of debt securities. Common structures include nth-to-default swaps, in which the contingent payment is triggered by the nth default event out of a given basket of reference names. Other variations pay upon each of the first or last n credit events out of a given list of names. More complex multi-name structured credit products include Collateralized Debt Obligations (CDOs), which involve prioritized tranches whose cash flows are linked to the performance of a pool of reference debt instruments.

In this paper we propose a model and a simulation algorithm for successive correlated default events, which has direct applications in design, analysis, and valuation of single and multi-name credit derivatives, credit derivatives signed with defaultable counterparties, and structured credit products.

There are currently two approaches to the modeling of an individual default event. One approach, called the *structural approach*, takes as given the dynamics of a firm's asset value and defines default as the first time the value of the assets falls to some lower threshold. In the second approach, called the *reduced form approach*, the default event is given exogenously and its stochastic structure is directly prescribed by an intensity, which can be interpreted as a conditional default arrival rate. In general these two approaches are not consistent with each other, since in the structural approach the default is typically predictable and therefore does not admit an intensity. However, as recently explored by Duffie & Lando (2001), Giesecke (2001*b*), and by Cetin, Jarrow, Protter & Yildirim (2002) in a somewhat different setup, this dichotomy vanishes if one allows for incomplete observation of the firm's assets and/or the threshold asset level at which the firm is liquidated. This makes the structural model not only more realistic and empirically more plausible, but also paves the way for the existence of a default intensity. In fact, the above mentioned contributions establish the intensity in terms of fundamental firm variables, thereby providing a bridge between structural and reduced form approaches.

This paper generalizes the incomplete observation models of Duffie & Lando (2001) and Giesecke (2001*b*) to the multi-firm case with correlated defaults. As for the information structure, we assume that bond investors observe the default events as they arrive over time, but have no information on a firm's assets and the threshold asset level at which the firm will be liquidated. This basically leads to defaults becoming *un*predictable, which is in accordance with empirical observations. Specifically, credit spreads do not go to zero with maturity going to zero, as implied by a complete information model. As for firm interrelation, our model accommodates the effects of two distinct sources of default correlation: dependence of firms on general (macro-) economic factors, and direct inter-firm linkages such as parent-subsidiary relationships, trade credits, or similar lending contracts. While the former finds its expression in the well-documented cyclical behavior of aggregate default rates, the latter typically leads to cascading effects, which involve the direct propagation of financial distress from one firm to one or more other firms. Evidence of such effects are the jumps in a firm's credit spreads upon the default of some close competitor or major stakeholder.

Cyclical effects are introduced through assuming that firms' assets are correlated through time, as in the two-firm complete information model of Zhou (2001). Additionally to these cyclical default correlation effects, *cascading effects* are implied by the lack of complete information on fundamental firm variables, as in Giesecke (2001*a*). Investors use the publicly available default status information to constantly update the prior distribution they form on the firms' default threshold levels. This updating leads to *information based cascading*: upon the unpredictable default of a particular firm in the market, conditional default probabilities and spreads of closely linked firms jump immediately up or down, reflecting the arrival of new and important information which bears on the credit quality of the dependent firms. The size of the jump depends on the extent of the relationship, whereas the sign depends on whether the firms are positively or negatively related. The fact that such jump pattern can actually be observed in bond markets allows to calibrate the model implied cascading effects to market data.

In this multi-firm structural model with incomplete information, we explicitly characterize the *pricing trend* of successive correlated default arrivals in terms of investors' prior default threshold distribution and the distribution of the assets' running minimum (that is, the assets' historic low). The *intensity* is the density of the pricing trend; we provide sufficient conditions for the existence of the intensity of the nth default arrival, and characterize these intensities explicitly. Given this characterization, one can exploit the well-known intensity based representations of prices of contingent claims with payoffs dependent on the first, the second, etc., default arrival out of a given list of reference events. These tractable reduced form pricing relationships allow the valuation of the multi-name credit derivatives and structured credit products mentioned above. Another potential application of our results relates to the analysis and valuation of contingent claims written by counterparties which are itself subject to default, cf. the recent results of Jarrow & Yu (2001).

Based on the explicitly characterized default pricing trends, we provide an algorithm for the simulation of successive, correlated, and unpredictable default arrival times, which proves to be useful in the Monte Carlo based valuation of more complex instruments. If compared to conventional methods which involve the simulation of (correlated) jump-diffusion intensity paths, this algorithm turns out to be quite computationally efficient.

The remainder of this paper is organized as follows. In Section 2, we begin by discussing the instructive one-firm structural model with incomplete information, which forms the basis for the subsequent analysis. In Section 3, we consider the practically most relevant case of the first-to-default arrival, while in Section 4 we extend to the general case of successive correlated default arrivals. The specific modeling of default correlation is emphasized in Section 5. In Section 6, we discuss some parametric modeling examples in the first-to-default context as well as some calibration procedures. Section 7 concludes. The appendix contains the proofs.

2 Single-Entity Trend and Simulation

We begin in this section by a discussing the instructive single-entity case. Subsequent sections extent to the general multi-entity situation.

2.1 A Structural Model of Default

We consider an economy where uncertainty is modeled by a probability space $(\Omega, \mathcal{G}, P)$ equipped with a right-continuous and complete filtration $(\mathcal{G}_t)_{t\geq 0}$ describing the information flow over time. In risk measurement and management applications, P is taken to be the physical probability; for derivatives valuation purposes P is taken to be some risk-neutral probability.

Investors can invest in the bonds of a single firm (we will use the term 'bond' as a generic term to denote the debt instruments of the firm, which may not trade publicly). We take as given the dynamics of the firm's asset value V and assume that the firm defaults when the assets fall to some threshold D

for the first time. That is, the firm's default time τ is given by

$$\tau = \inf\{t \geq 0 : V_t \leq D\}, \tag{1}$$

so that τ is a random variable taking values in $(0, \infty]$. The associated default indicator process is denoted $N = (N_t)_{t\geq 0}$, where $N_t = 1_{\{t\geq\tau\}}$.

Suppose for the moment that assets follow some continuous process and that investors have complete information on assets and threshold. In this case the default occurs never unexpectedly: Investors can observe at any time the nearness of the assets to the default threshold level. Consequently, they are warned in advance when a default is imminent.[1] In practice, however, it is typically difficult if not impossible to directly observe the assets of an issuer and the threshold asset level at which the firm will be liquidated. This observation motivated the contributions of Duffie & Lando (2001), Giesecke (2001*a*), Giesecke (2001*b*), and Cetin et al. (2002), who allowed for incomplete information of public investors.[2] We will follow this route and suppose that only the firm's default is a publicly observable event; the firm's assets V and its default threshold D are unknown to investors in the firm's debt. We set accordingly

$$\mathcal{G}_t = \sigma(N_s : s \leq t).$$

In this situation investors are always uncertain about the nearness of the assets to the default threshold, so that a default appears completely unexpectedly. In fact, below we will provide a sufficient condition under which τ is a totally inaccessible stopping time in bond investors' information filtration $(\mathcal{G}_t)$.[3] This property is in accordance with empirical observations.

For unpredictable defaults we can establish tractable reduced form representations of default probabilities and default-contingent claim prices in terms of the corresponding *pricing trend*, cf. Giesecke (2001*b*). This is an increasing function A starting at zero such that the difference process $N - A^\tau$ is a martingale. Here A^τ denotes the function A stopped at default: $A^\tau_t = A(t \wedge \tau)$; this process is called the *default compensator*. If the trend is absolutely continuous

[1] In this case τ is called *predictable*, i.e. there is an increasing sequence of stopping times (T_n) such that $\tau > T_n$ and $\lim_n T_n = \tau$. Intuitively, one can foretell the default event by observing a succession of 'forerunners'. We also say that (T_n) announces τ.

[2] Hull & White (2000) and Avellaneda & Zhu (2001) take the threshold to be an unknown deterministic function of time and show how to determine this function from observed defaultable bond prices.

[3] The stopping time τ is called *totally inaccessible* or simply unpredictable, if $P[\tau = T < \infty] = 0$ for all predictable times T. Here an announcing sequence does not exist.

with respect to Lebesgue measure,

$$A(t) = \int_0^t \lambda(s)ds, \tag{2}$$

then the density λ is called the *intensity* for τ. From (2) and the martingale property of the process $N - A^\tau$ we find that

$$\lambda(t) = \lim_{h \downarrow 0} \frac{1}{h} P[\tau \le t + h \,|\, \mathcal{G}_t] \quad \text{a.s.} \quad t < \tau,$$

so that the intensity λ can be interpreted as a conditional default arrival rate.

Without loss of generality we normalize $V_0 = 0$. The following basic result characterizes the pricing trend in terms of the given prior distribution $G(x) = \int_{-\infty}^x g(y)dy$ investors form on the default threshold $D \in (-\infty, 0)$ and the distribution function $H(t, \cdot)$ of the *running minimum asset value* M_t, defined by

$$M_t = \min\{V_s \mid 0 \le s \le t\}.$$

We assume that D is independent of V.

Proposition 2.1. *Let $H(t, x)$ be continuous in t for fixed $x \le 0$. Then the pricing trend is given by*

$$A(t) = -\ln\left(1 - \int_{-\infty}^0 H(t, x)g(x)dx\right).$$

If the derivative $\dot{H}(t, x) = \frac{\partial}{\partial t} H(t, x)$ is well-defined and uniformly bounded for $x \le 0$, then the default time τ admits the unique intensity λ given by

$$\lambda(t) = \frac{\int_{-\infty}^0 \dot{H}(t, x)\, g(x)dx}{1 - \int_{-\infty}^0 H(t, x)\, g(x)dx} \quad \text{on} \quad \{t < \tau\}. \tag{3}$$

Let us recall that the default time τ is totally inaccessible if and only if the default compensator A^τ is continuous (see, for example, Dellacherie & Meyer (1982, Theorem IV.78)). The continuity of the conditional distribution function $H(t, \cdot)$ of the running minimum asset value M_t in time t is therefore a necessary and sufficient condition for the default to be unpredictable.

Example 2.2. *Assume, in line with the majority of structural approaches, that firm's assets V follow a Brownian motion with drift $\mu \in \mathbb{R}$ and volatility $\sigma > 0$:*

$$dV_t = \mu dt + \sigma dB_t,$$

where B is a standard Brownian motion. For $x \leq 0$ and $t > 0$ we then have for the distribution function of the running minimum asset value

$$H(t,x) = P[M_t \leq x] = \Phi\left(\frac{x - \mu t}{\sigma\sqrt{t}}\right) + \exp\left(\frac{2\mu x}{\sigma^2}\right)\Phi\left(\frac{x + \mu t}{\sigma\sqrt{t}}\right), \tag{4}$$

where Φ is the standard normal distribution function. It is clear that $H(t,\cdot)$ is continuous on $(0,\infty)$, so that the default is unpredictable with Brownian asset dynamics. The derivative $\dot{H}(t,x)$ is given by

$$\dot{H}(t,x) = \frac{1}{2\sigma}\left[\left(\frac{\mu}{\sqrt{t}} - \frac{x}{\sqrt{t^3}}\right)e^{\frac{2\mu x}{\sigma^2}}\phi\left(\frac{x+\mu t}{\sigma\sqrt{t}}\right) - \left(\frac{x}{\sqrt{t^3}} + \frac{\mu}{\sqrt{t}}\right)\phi\left(\frac{\mu t - x}{\sigma\sqrt{t}}\right)\right]$$

where ϕ is the standard normal density function, so that τ admits the intensity (3) with Brownian asset dynamics.

The intensity has an instructive interpretation: it constitutes the *short credit spread*, i.e. the excess interest bond investors demand over the riskless rate in compensation for assuming the risk of default of the bond issuer over an infinitesimal time period. Proposition 2.1 shows that this excess short interest is positive, which is in accordance with empirical data. This is in contrast to the usual structural model with complete information, where the default is predictable and such an intensity does *not* exist (indeed, here we have $A^\tau = N$). According to Proposition 2.1, the intensity can be calculated by uninformed bond investors in terms of their prior belief on the firm's default threshold D as well as the distribution of the assets' historic low M_t. In fact, under the conditions of Proposition 2.1 the price of a zero-recovery defaultable zero coupon bond, paying 1 at T if there is no default by T and zero otherwise, can be represented in terms of A for $t < \tau$ and $t \leq T$ as

$$\begin{aligned} E[e^{-\int_t^T r_s ds} 1_{\{\tau > T\}} \,|\, \mathcal{G}_t] &= E[e^{-\int_t^T r_s ds + A(t) - A(T)} \,|\, \mathcal{G}_t] \\ &= E[e^{-\int_t^T [r_s + \lambda(s)] ds} \,|\, \mathcal{G}_t], \end{aligned} \tag{5}$$

where r is some bounded riskless short rate process and where we take our reference probability P be some risk-neutral probability. The right side of equation (5) shows that the price of the defaultable bond can be calculated as if it were default-free, by just using the default-adjusted short rate $r + \lambda$ for discounting. Analogous price representation results can be derived for general defaultable claims; see, for example, Elliott, Jeanblanc & Yor (2000), Rutkowski (1999), or Giesecke (2001*b*) (these contributions also provide reduced form price representations in terms of A if this function is merely continuous). Default probabilities can be similarly characterized: on the set $\{t < \tau\}$ we have

$$P[\tau \leq T \,|\, \mathcal{G}_t] = 1 - e^{A(t) - A(T)} = 1 - e^{-\int_t^T \lambda(s) ds}, \quad t \leq T. \tag{6}$$

2.2 Simulating the Default

In some complex valuation problems, estimation of security prices by Monte Carlo methods may be unavoidable. One conventional method of simulating the default in a structural model would consist of simulating a sample path of V and an independent random variate d from G, and then recording the paths' first passage time to d. Besides the fact that path simulation for (jump-) diffusion asset processes V might be burdensome for a large number of entities over long time horizons, the discretization of V may lead to a biased Monte Carlo price estimate. We now devise a default time simulation algorithm which does not require the simulation of paths.

Taking the continuous increasing function A as given through Proposition 2.1, our basic idea is to construct an inaccessible stopping time δ such that $A(\cdot \wedge \delta)$ is its compensator. The resulting algorithm is in fact equivalent to the standard inverse transform approach (cf. for example Devroye (1986)).

Basic Algorithm:

(1) Simulate an independent standard uniform random variable U.

(2) Set $\delta = \inf\{t \geq 0 : A(t) \geq -\ln U\}$.

From Step (2) we obtain immediately

$$P[\delta > t] = P[U < e^{-A(t)}] = e^{-A(t)} = P[\tau > t], \tag{7}$$

where the last equality follows from (6). This means that δ is equivalent in distribution to τ, as desired.

In the intensity based reduced form credit risk literature, Step (2) of the basic algorithm is in fact often used to *define* the default time δ associated with some typically exogenously given intensity process h via

$$\delta = \inf\{t \geq 0 : \int_0^t h_s ds \geq -\ln U\}, \tag{8}$$

see, for example, Lando (1998), or Schönbucher & Schubert (2001). Proposition 2.1, where the intensity is calculated in terms of fundamental firm variables, provides the economic rationale for this procedure in terms of an underlying structural model of default. Definition (8) also provides immediately the algorithm for default time simulation via simulation of intensity paths. However, for the very common (jump) diffusion models for h, this algorithm may be relatively computationally intensive. The basic algorithm we suggest here avoids the simulation of intensity paths by directly considering the "integrated intensity" A, cf. (2).

3 First-to-Default Pricing Trend

We now extend our basic one-name setup to n correlated firms, where we focus on the default event which arrives first out of a given list of events. This is instructive and constitutes a practically most relevant case. We extend our analysis to general successive defaults in Section 4.

3.1 A Multi-Firm Structural Model

Given the dynamics of its asset value V^i satisfying $V_0^i = 0$, we assume that firm $i \in \{1, 2, \ldots, n\}$ defaults when its assets fall to the threshold $D_i \in (-\infty, 0)$ for the first time:

$$\tau_i = \inf\{t \geq 0 : V_t^i \leq D_i\}, \tag{9}$$

with associated indicator process N^i. In analogy to the single-entity model, investors observe the defaults of firms as they are arriving, but they have no information on the firms' assets and default thresholds. We thus set

$$\mathcal{G}_t = \sigma(N_s^i : s \leq t, i = 1, \ldots, n).$$

In lack of threshold information, investors form a continuous prior distribution G on the threshold vector $D = (D_1, \ldots, D_n)$ on $\mathbb{R}_-^n$, which we take as given. D is assumed to be independent of assets $V^1, \ldots, V^n$.

The ordered sequence of default times (τ_i) is denoted by (T_i). The following result characterizes the *first-to-default pricing trend* in terms of G and the density $h(t, \cdot)$ of the running minimum asset vector $M_t = (M_t^1, \ldots, M_t^n)$, where $M_t^i = \min_{s \leq t} V_s^i$ (assuming that h exists).

Proposition 3.1. *If the density $h(t, x)$ is continuous in t for $x \in \mathbb{R}_-^n$, then the first-to-default pricing trend is given by*

$$A_1(t) = -\ln \int_{\mathbb{R}_-^n} G(x)\, h(t, x)\, dx. \tag{10}$$

If additionally the derivative $\dot{h}(t, x) = \frac{\partial}{\partial t} h(t, x)$ is well-defined and uniformly bounded for $x \in \mathbb{R}_-^n$, then the first-to-default time T_1 admits the unique inten-

sity λ_1 *given by*

$$\lambda_1(t) = -\frac{\int_{\mathbb{R}^n_-} G(x)\,\dot{h}(t,x)\,dx}{\int_{\mathbb{R}^n_-} G(x)\,h(t,x)\,dx} \quad on \quad \{t < T_1\}. \tag{11}$$

Again, the continuity of the running minimum asset density $h(t,\cdot)$ is a necessary and sufficient condition for the first-to-default time to be totally inaccessible in bond investors' information filtration $(\mathcal{G}_t)$.

Given the intensity, reduced form representations of general first-to-default-contingent claims can be derived analogously to the single-entity case (5). In particular, conditional first-to-default probabilities can be written as

$$P[T_1 \leq T \,|\, \mathcal{G}_t] = 1 - e^{A_1(t) - A_1(T)} = 1 - e^{-\int_t^T \lambda_1(s)ds}, \quad t \leq T, \quad t < T_1.$$

Let us compare our first-to-default intensity formula (11) with a related result of Duffie (1998), who constructs an intensity of the first default as the sum of the exogenously given single-entity default intensities $\lambda(i,t)$:

$$\lambda_1(t) = \sum_{i=1}^{n} \lambda(i,t), \quad t \geq 0, \tag{12}$$

under the assumption that $P[\tau_i = \tau_j] = 0$ for all $i \neq j$. The surprising feature of this result is that it does not require knowledge of the joint distribution of the default times, once the single-entity intensities $\lambda(i,t)$ are taken as given. The key to this is the choice of the reference filtration $(\mathcal{G}_t)$. In Duffie (1998), the $\lambda(i,t)$ are defined with respect to the large filtration generated by the default indicators N^i for all i. That means that single-entity intensities directly reflect the defaults of dependent firms as they arrive over time. Given some strong kind of interdependence (perhaps some parent-subsidiary or similar contractual relationship), this would correspond to jumps in the single-entity intensity upon defaults of dependent entities. However, the construction of such intensities appears to be very difficult, and requires of course knowledge of the complete default dependence structure. A first step in this direction has been recently made by Jarrow & Yu (2001). If the intensities are only correlated trough time within a diffusion setup, then one would ignore the information on defaults of correlated firms which becomes available over time (defaults of dependent firms do not feed back). That means that commonly applied diffusion intensity models do not seem to be an appropriate basis for Duffie's (1998) result (12); it appears that this result is in fact only of limited applicability. A similar conclusion, though based on different arguments, is drawn by Rutkowski (1999).

We circumvent these potential difficulties by constructing the first-to-default intensity directly without reference to the single-entity intensity (3). In our model, the latter is defined with respect to the (small) filtration generated by N^i for a single fixed firm i only. Clearly, this intensity need not be an intensity with respect to the large filtration $(\mathcal{G}_t)$, and therefore Duffie's (1998) result (12) and our result (11) are not consistent in general.

3.2 Simulating the First-to-Default

We now wish to simulate an unpredictable first-to-default stopping time δ_1. Our algorithm proceeds in two steps: Given the continuous increasing function A_1 from Proposition 3.1, we first construct a stopping time δ_1 having compensator $A_1(\cdot \wedge \delta_1)$. This proceeds as described in Section 2.2, and results in δ_1 being equivalent in distribution to T_1. The second step is similar to the intensity based algorithm of Duffie (1998), and consists of simulating the identity of the first defaulter given δ_1. For this step we define the identity density q_1 by

$$\begin{aligned} q_1(i,t)dt &= P[\delta_1 = \tau_i, \delta_1 \in (t, t+dt)] \\ &= P[\tau_i \in (t, t+dt), \tau_j > t \ (j \neq i)]. \end{aligned}$$

Assuming that the partial derivative $G_{z_i}(z_1, \ldots, z_n)$ of G with respect to its ith argument is well-defined and that each V^i is continuous, using the fact that $\{\tau_i = t\} = \{D_i = M^i_t\}$ we can express q_1 in terms of the given G_{z_i} and h as follows:

$$q_1(i,t) = \int_{\mathbb{R}^n_-} G_{z_i}(x)\, h(t,x)\, dx, \quad i = 1, \ldots, n, \tag{13}$$

which differs from Duffie (1998) in the respect that we construct this density directly without reference to the single-name intensities.

First-to-Default Algorithm:

(1) Simulate an independent standard uniform random variable U.

(2) Set $\delta_1 = \inf\{t \geq 0 : A_1(t) \geq -\ln U\}$.

(3) Simulate the identity of the first defaulter, i.e. simulate a random variable I_i valued in $\{1, \ldots, n\}$, with the conditional probability given δ_1 that $I_1 = i$ (that is, $T_1 = \tau_i$) equal to

$$P[I_1 = i \,|\, \sigma(\delta_1)] = \frac{q_1(i, \delta_1)}{q_1(1, \delta_1) + \ldots + q_1(n, \delta_1)}. \tag{14}$$

This algorithm appears to be computationally more efficient than approaches relying on the simulation of single-entity intensity paths to generate unpredictable correlated default times via (8), as in Schönbucher & Schubert (2001), for example. In the first-to-default context, an approach based on (8) might be even more burdensome than in the single-entity context, as one might have to simulate intensities of up to n entities in order to obtain a single first-to-default time. The same type of criticism applies for simulation of first-to-default times via the simulation of n individual firms' first passage times (generate asset paths and default thresholds for each firm, and record the minimum of the individual first passage times).

If the vector D can be easily simulated from G, then a further reduction in computational expenses may be achieved as follows. Denote by $\bar{H}(t,x) = P[M_t > x]$ for $x \in \mathbb{R}^n_-$ the joint survival function of M_t. Assuming that $h(t,x)$ is continuous in t, we get from Proposition 3.1 that

$$A_1(t) = -\ln \bar{H}(t, D).$$

Likewise, we have for the first-to-default identity probability (13) that

$$q_1(i,t) = -\bar{H}_{z_i}(t, D),$$

given that the derivative $\bar{H}_{z_i}(z_1, \ldots, z_n) = \frac{\partial}{\partial z_i} \bar{H}(z_1, \ldots, z_n)$ is well-defined. Depending on the structure of $\bar{H}$ and the ability to simulate D, by exploiting these simpler expressions for A_1 and q_1 the efficiency of first-to-default simulation may be improved for higher dimensions.

4 Successive Correlated Defaults

4.1 Pricing Trends

In this section we extend our first-to-default results to general successive correlated default arrivals. The setting is as in Section 3: the ordered sequence of default times (τ_i) is denoted by (T_i). We denote by G the prior distribution function of D and by $h(t_1, \ldots, t_n; \cdot)$ the density of $(M^1_{t_1}, \ldots, M^n_{t_n})$ (assuming that h exists). Let G^m and h_m denote the m-dimensional marginal of G and h, respectively, with respect to their first m arguments.

Proposition 4.1. *Fix some $k \geq 2$ and assume that the first default times have identities $I_i = i$ for $1 \leq i \leq k-1$. Assume that the partial derivative $G_{z_1 \cdots z_{k-1}}$ of G with respect to its first $k-1$ arguments is well-defined. If*

$h(\tau_1, \ldots, \tau_{k-1}, t, \ldots, t; x)$ is continuous in t for $x \in \mathbb{R}^n_-$, then the kth-to-default pricing trend is given by

$$A_k(t) = -\ln \frac{\int_{\mathbb{R}^n_-} G_{z_1 \cdots z_{k-1}}(x) h(\tau_1, \ldots, \tau_{k-1}, t, \ldots, t; x) dx}{\int_{\mathbb{R}^{k-1}_-} G^{k-1}_{z_1 \cdots z_{k-1}}(x) h_{k-1}(\tau_1, \ldots, \tau_{k-1}; x) dx} \quad on \quad \{t > T_{k-1}\}.$$

If the derivative $\dot{h}(\tau_1, \ldots, \tau_{k-1}, t, \ldots, t; x) = \frac{\partial}{\partial t} h(\tau_1, \ldots, \tau_{k-1}, t, \ldots, t; x)$ is well-defined and uniformly bounded for $x \in \mathbb{R}^n_-$, then T_k admits the unique intensity λ_k given by

$$\lambda_k(t) = -\frac{\int_{\mathbb{R}^n_-} G_{z_1 \cdots z_{k-1}}(x) \dot{h}(\tau_1, \ldots, \tau_{k-1}, t, \ldots, t; x) dx}{\int_{\mathbb{R}^n_-} G_{z_1 \cdots z_{k-1}}(x) h(\tau_1, \ldots, \tau_{k-1}, t, \ldots, t; x) dx} \quad on \quad \{T_{k-1} < t < T_k\}.$$

Let us remark that the assumption $I_i = i$ for $1 \leq i \leq k-1$ is chosen for notational convenience only; for arbitrary identities of the first $k-1$ defaulters the calculations are analogous.

In analogy to the single-entity case, using this result we can exploit convenient and tractable reduced form representations of prices of claims contingent on the kth default. Moreover, we can establish an algorithm for the simulation of successive correlated default times, to which we turn next.

4.2 Simulating Successive Defaults

Simulation of $m \leq n$ successive correlated and unpredictable default times $\delta_1, \ldots, \delta_m$ is possible by iterating the first-to-occur algorithm. Given the continuous increasing function A_1 provided by Proposition 3.1 with $A_1(t) = 0$, we start by constructing an unpredictable first-to-default stopping time δ_1. Conditional on δ_1, subsequently the identity of the first defaulter is simulated. Now we iterate this procedure: given the continuous increasing $\mathcal{G}_{T_1}$-measurable function A_2 provided by Proposition 4.1 with $A_2(t) = 0$ for $0 \leq t \leq \delta_1$, we next simulate the second-to-default time δ_2 with identity, and so on.

For the successive-event algorithm, we fix some $k \geq 2$ and let R_k denote the set of surviving entities after the kth event arrival. Letting $Z_k = (\delta_i, I_i)_{i \leq k}$ be the first k simulated default times and their identities, we define the identity density

$$\begin{aligned} q_k(i,t)dt &= P[\delta_k = \tau_i,\, \delta_k \in (t, t+dt) \,|\, \sigma(Z_{k-1})] \\ &= P[\tau_i \in (t, t+dt), \tau_j > t \; (j \in R_{k-1} - \{i\}) \,|\, \sigma(Z_{k-1})]. \end{aligned}$$

In order to provide an example for the calculation of q_k, let us assume that the first $k-1$ simulated default times have simulated identities $I_i = i$ for

$1 \leq i \leq k-1$ and that $i = k \in R_{k-1}$ (the calculations for arbitrary identities are analogous). Then we can write

$$q_k(k,t)dt = P[\tau_k \in (t, t+dt), \tau_{k+1} > t, \ldots, \tau_n > t \,|\, \sigma(Z_{k-1})].$$

Assuming furthermore that the partial derivative $G_{z_1 \cdots z_k}$ of G with respect to its first k arguments is well-defined and that each V^i is continuous, by applying Bayes' rule for $t > \delta_{k-1}$ we get

$$q_k(k,t) = \frac{\int_{\mathbb{R}_-^n} G_{z_1 \cdots z_k}(x) h(\delta_1, \ldots, \delta_{k-1}, t, \ldots, t; x) dx}{\int_{\mathbb{R}_-^{k-1}} G^{k-1}_{z_1 \cdots z_{k-1}}(x) h_{k-1}(\delta_1, \ldots, \delta_{k-1}; x) dx},$$

where, as in the previous section, G^m and h_m denote the m-dimensional marginal of G and h, respectively, with respect to their first m arguments.

Successive Event Algorithm:

(1) Initialize $R_0 = \{1, \ldots, n\}$ and $k = 1$.

(2) Simulate an independent standard uniform random variable U_k.

(3) Set $\delta_k = \inf\{t \geq 0 : A_k(t) \geq -\ln U_k\}$.

(4) Simulate the identity of the kth defaulter, i.e. simulate a random variable I_k valued in R_{k-1}, with the conditional probability given the previous defaulters and their identities $(\delta_i, I_i)_{i \leq k-1}$ as well as δ_k that $\tau_i = T_k$ equal to

$$P[I_k = i \,|\, \sigma(\delta_k, Z_{k-1})] = \frac{q_k(i, \delta_k)}{\sum_{i \in R_{k-1}} q_k(i, \delta_k)}.$$

(5) Set $R_k = R_{k-1} - I_k$.

(6) If $k = m$ then stop, else set $k = k+1$ and go back to Step (2).

As described in the context of first-to-default simulation, the efficiency of the successive-event algorithm might be further improved if one is able to generate random samples of G easily (we shall comment on that in Section 6 below).

5 Imposing Default Correlation

In the last section we have established the pricing trends of successive correlated defaults in a multi-firm structural model with incomplete information. This structural model allows for two natural and intuitive ways of incorporating correlation between individual firms. These "correlation mechanisms" can be imposed simultaneously or alternatively, depending on what correlation pattern is to be modeled, which data as a basis for estimation is available, or simply on computational tractability.

Asset value correlation corresponds to cyclical default correlation effects induced by the dependence of firms on common (macro-) economic factors. If, as in Zhou (2001), one assumes that assets V follow an n-dimensional Brownian motion, then this would formally be represented by the correlation matrix $(\rho_{ij})_{n\times n}$, where ρ_{ij} denotes the linear correlation between V^i and V^j. There are well-known methods for estimating (ρ_{ij}) from equity market data, see Kealhofer (1998) and Crouhy, Galai & Mark (2000) for more explicit comments.

Default threshold dependence corresponds to default cascading effects induced by direct inter-firm linkages such as parent-subsidiary relationships or mutual substantial capital holdings. The idea is as follows (Giesecke (2001*a*)). It is reasonable to suppose that the default threshold D_i of an individual firm i is chosen by the firm's management or shareholders so as to maximize the value of their stake in the firm. This liquidation strategy is therefore not disclosed to firm outsiders such as investors in the firm's debt; equity investors will use that information in their own interest and transfer value from the bond investors' stake in the firm to their own. If firms are closely linked through a parent-subsidiary or other contractual relation, then a firm's equity investors cannot choose their liquidation strategy independently from the strategy of the equity investors of linked firms. Put another way, the liquidation strategies of closely linked firms are based on common factors, for example profit sharing agreements in the case of a parent-subsidiary relation. Public debt investors anticipate these common factors in their corresponding prior distribution G which they form on the default thresholds $D_1, \ldots, D_n$ of the firms.

Investors use the default status information of the firms in the market to constantly update their prior distribution G. The updating mechanism leads to jumps in bond prices and credit spreads whenever unpredictable defaults arrive (these are observed by investors only in the moment they occur). This can be interpreted as *information based cascading*: upon a default of a particular firm in the market, default probabilities and spreads of closely linked firms jump immediately up or down, reflecting the arrival of new and important information which bears on the credit quality of the connected firms. The size of the jump depends on the extent of the relationship, whereas the sign depends on whether the firms are positively or negatively related. An analysis of these cascading effects can be found in Giesecke (2001*a*).

6 Default Threshold Dependence

The effects of asset correlation on joint default probabilities in a Brownian motion based first-passage framework with complete information have already been examined in Zhou (2001). In this section we study the modeling and estimation of default threshold dependence, which implies the cascading effects in our setup of incomplete information.

6.1 Threshold Copula

Taking investors' threshold prior in form of the joint distribution function G as given, we can separate the threshold dependence structure from the marginal behavior of the individual thresholds D_i by means of the *copula* of G. In fact, for any G we can find a copula function $C : [0,1]^n \to [0,1]$ such that

$$G(x_1, \ldots, x_n) = C(G_1(x_1), \ldots, G_n(x_n)), \quad x_i \leq 0, \tag{15}$$

where G_i is the distribution function of the default threshold D_i of firm i (since the G_i are continuous, the copula is unique). The copula is thus nothing more than a joint distribution function with all marginals being standard uniform (for more details we refer to Nelsen (1999)).

The motivation for introducing copulas is three-fold. First, the copula describes the complete non-linear dependence between the random variables $D_1, \ldots, D_n$, irrespective of their joint distribution type. Linear correlation, in contrast, is only the natural measure of dependence for joint elliptical random vectors, see Embrechts, McNeil & Straumann (2001) in that respect. Second, copulas will allow us to examine the effects of dependence between thresholds on joint default probabilities separately from effects induced by the distribution G_i of the individual thresholds. Third, the problem of estimating G is divided into two sub-problems: estimation of the marginals G_i, which correspond to idiosyncratic factors, and estimation of the copula C, which represents common threshold determinants which are due to the direct links between firms. We will discuss the calibration of the threshold copula in Section 6.3 below.

A further advantage of introducing copulas is related to the proposed simulation algorithms. We have mentioned in Section 3.2 that the efficiency of simulating successive default arrival times might be improved by first simulating random variates from the threshold distribution G. In the copula framework, this is easily achieved: given a realization $(W_1, \ldots, W_n)$ from C, the vector

$$(G_1^{-1}(W_1), \ldots, G_n^{-1}(W_n))$$

has joint distribution G. For the generation of realizations from a copula several efficient algorithms are available, see Embrechts et al. (2001) and Devroye (1986). Lindskog (2000) provides specialized algorithms for families C_θ belonging to the class of Archimedean copulas (the families discussed below belong to this class, for example).

6.2 Parametric Copulas for First-to-Default Baskets

Having separated the default threshold dependence structure from their marginal behavior, let us now examine the effects of threshold dependence for several parametric copula families.

One of the most popular multi-name credit derivative structures is a first-to-default swap, which pays upon the first default in a given basket of firms. In response to this popularity, we shall focus our attention here on a first-to-default basket with $n = 5$ names. As for the asset dynamics, we suppose that issuers' assets follow a standard Brownian motion. The density of M_t is then given by $h(t, \cdot) = h_1(t, \cdot) \cdots h_5(t, \cdot)$, where

$$h_i(t, x) = \frac{1}{\sigma_i \sqrt{t}} \phi \left(\frac{\mu_i t - x}{\sigma_i \sqrt{t}} \right) + e^{\frac{2\mu_i x}{\sigma_i^2}} \left[\frac{2\mu_i}{\sigma_i^2} \Phi \left(\frac{x + \mu_i t}{\sigma_i \sqrt{t}} \right) + \frac{1}{\sigma_i \sqrt{t}} \phi \left(\frac{x + \mu_i t}{\sigma_i \sqrt{t}} \right) \right]$$

is the density of M_t^i, which is straightforwardly derived from (4). ϕ is the standard normal density function.

We start by modeling the threshold dependence structure by the Clayton copula family. As is not uncommon for first-to-default baskets, we suppose that the correlation structure in the basket is symmetric. We can therefore choose the one-parameter version of the Clayton family, which is given by

$$C_\theta^C(u_1, \ldots, u_5) = (u_1^{-\theta} + \cdots + u_5^{-\theta} - 4)^{-\frac{1}{\theta}}, \quad u_i \in [0, 1], \quad \theta > 0. \qquad (16)$$

The parameter θ controls the degree of threshold dependence: $\theta \to \infty$ reflects perfect positive dependence, and $\theta \to 0$ corresponds to independence. The degree of monotonic threshold dependence can be expressed in terms of Kendall's pairwise rank correlation $\rho^K \in [-1, 1]$. We have $\rho^K = -1$ iff the thresholds are perfectly negatively related, $\rho^K = 1$ iff they are perfectly positively related, and $\rho^K = 0$ in case of independence. For the Clayton family we have

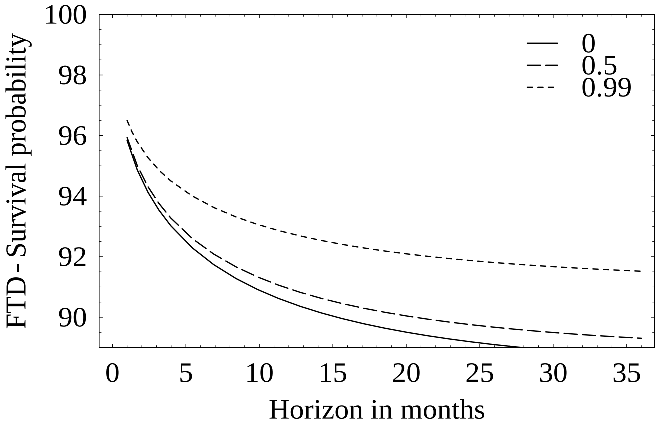

Figure 1: Term structure of first-to-default survival probabilities, varying rank threshold correlation.

$\rho^K = \theta/(\theta+2)$ (with $\theta > 0$ ρ^K is positive as well and (16) expresses positive dependence). Assuming that the default threshold of firm i has distribution function $G_i(x) = e^x$, from (15) the joint threshold distribution is

$$G^C_\theta(x_1, \ldots, x_5) = (e^{-\theta x_1} + \cdots + e^{-\theta x_5} - 4)^{-\frac{1}{\theta}}, \quad x_i \leq 0. \tag{17}$$

In Figure 1, we plot the term structure of risk-neutral first-to-default survival probabilities

$$L_1(t) = P[T_1 > t] = \int_{\mathbb{R}^5_-} G^C_\theta(x)\, h(t,x)\, dx$$

for varying degrees of rank threshold correlation ρ^K. We set $\mu_i = 6\%$ (the riskless rate) and $\sigma_i = 20\%$, the average volatility of a conservative low-risk S&P 500 firm. In terms of asset volatility, the firms in the basket are of high quality, so that individual default probabilities are low. Positive threshold dependence ($\rho^K > 0$) corresponds to positive dependence between firm defaults; for an formal statement of the relationship between threshold copula C and the copula of $(\tau_1, \ldots, \tau_5)$ as a measure of default dependence we refer to Giesecke (2001*a*). For a given horizon T, $L_1(T)$ is increasing in the degree of default dependence. This effect is made more explicit in Figure 2, which shows $L_1(T)$ for $T = 12$ months as a function of rank threshold correlation ρ^K for varying asset volatilities $\sigma = \sigma_i$.

This observation can be explained as follows. Loosely, positive monotonic threshold dependence as measured by $\rho^K > 0$ means that the default thresholds of different firms are likely to cluster around a common level dictated by

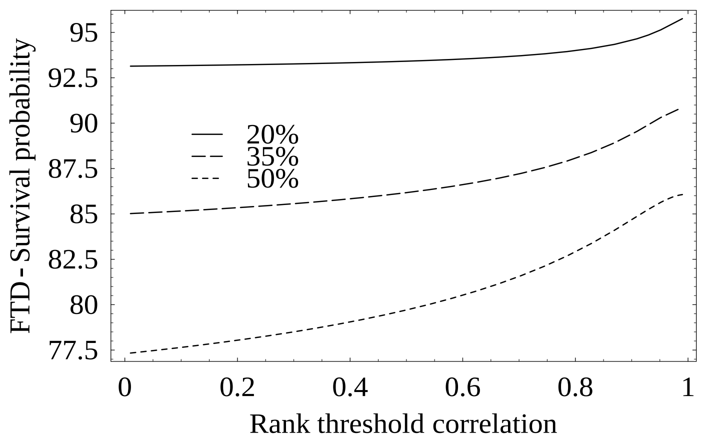

Figure 2: First-to-default survival probability as a function of rank threshold correlation, varying asset volatility.

the marginals G_i (which are equal for all firms). In the extreme case $\rho^K = 1$, we have $D_i = D_j$ almost surely (see Embrechts et al. (2001)). Even with uncorrelated assets, it follows that the likelihood of several firms' assets hitting a similar threshold level before a given horizon is higher than with independent thresholds, where such a clustering is not present. Positive asset correlation would increase that likelihood further. Thus, holding individual default probabilities fixed, the higher the positive threshold dependence, the higher are joint default probabilities, and the higher are survival probabilities of the first-to-default. This can be most easily seen in case $n = 2$, where

$$L_1(t) = 1 - P[\tau_1 \leq t] - P[\tau_2 \leq t] + P[\tau_1 \leq t, \tau_2 \leq t]. \tag{18}$$

The relation between survival probabilities and default correlation observed here is consistent with the pricing of a first-to-default contract, which pays off upon the first default in the basket. As with increasing default dependence the survival probability of T_1 increases, the payoff probability decreases, and the contract price decreases. This relationship corresponds to the idea that the stronger the positive dependence between the firms in the basket, the less value the first-to-default contract's hedging capability has for the holder of the basket. Indeed, fixing firms' individual default probabilities, positive default dependence increases the likelihood of several firms defaulting before a given horizon, but the contract covers the first default only. The price of the first-to-default contract therefore reaches its maximum if the firms are perfectly negatively dependent, i.e. in the case $\rho^K = -1$ for the thresholds, and its min-

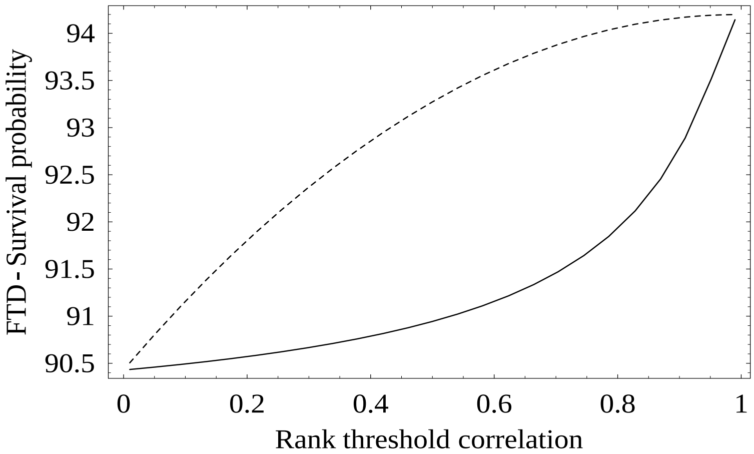

Figure 3: First-to-default survival probability as a function of rank threshold correlation for different copula families (solid line: Clayton, dashed line: Gumbel).

imum if firms are perfectly positively correlated, corresponding to $\rho^K = 1$ for the thresholds.

So far we have examined the sensitivity of the survival probability with respect to threshold dependence and asset volatility for the Clayton family. The survival probability is however also sensitive to the choice of the family itself. To study this, let us introduce the Gumbel family with parameter $\theta \geq 1$:

$$C_\theta^G(u_1, \ldots, u_5) = \exp\big(- [(-\ln u_1)^\theta + \ldots + (-\ln u_5)^\theta]^{\frac{1}{\theta}}\big), \quad u_i \in [0,1]. \quad (19)$$

The value $\theta = 1$ corresponds to independence, while $\theta \to \infty$ reflects perfect positive dependence. For the Gumbel family the pairwise rank correlation is $\rho^K = 1 - 1/\theta$. With $G_i(x) = e^x$, we have for the joint threshold distribution

$$G_\theta^G(x_1, \ldots, x_5) = \exp\big(- [(-x_1)^\theta + \ldots + (-x_5)^\theta]^{\frac{1}{\theta}}\big), \quad x_i \leq 0. \quad (20)$$

The choice of the copula family has indeed significant effects on the resulting arrival probabilities. Figure 3 displays the (risk-neutral) 12 month first-to-default survival probability $L_1(12)$ as a function of rank threshold correlation ρ^K for both Clayton and Gumbel threshold copulas. The asset volatility σ_i is set to 20% for all names. The differences in the survival probability for the two families are due to their tail dependence properties. The Gumbel copula exhibits upper tail dependence, which refers to the pronounced tendency of a copula to generate high threshold values in all marginals simultaneously (for a formal definition we refer to Nelsen (1999)). All else being equal, this implies in turn an increased likelihood of joint defaults, which leads to higher survival probabilities $L_1(12)$ of T_1, cf. (18). The Clayton copula exhibits lower tail dependence, which leads to opposite effects. Consequently, for a given horizon the first-to-default survival probability with the Gumbel threshold copula is at least as high as with the Clayton copula.

If we have evidence for such particular default correlation pattern in our basket, we can model them by choosing tail-dependent threshold copula families. In view of the uncertainty surrounding the choice of a copula family, if such evidence is not available it seems reasonable to confine to families which display asymptotic independence in both tails. A simple closed-form family that satisfies this property is the Frank family, which is in the one-parameter version for $\theta > 0$ defined by

$$C^F_\theta(u_1, \ldots, u_5) = -\frac{1}{\theta} \ln \left(1 + \frac{(e^{-\theta u_1} - 1) \cdots (e^{-\theta u_5} - 1)}{(e^{-\theta} - 1)^4} \right), \quad u_i \in [0, 1].$$

For more families suitable for modeling the threshold copula, and methods to construct copula families, we refer to Nelsen (1999).

6.3 Calibrating the Threshold Copula

In this section we discuss the parametric estimation of the threshold dependence structure C. This is much more challenging than the estimation of asset correlation, which is discussed in Kealhofer (1998) and Crouhy et al. (2000) in some detail. Throughout, we fix some copula family C_θ with parameter vector $\theta \in \mathbb{R}^m$, which we wish to estimate. We refer to Nelsen (1999) and Lindskog (2000) for a wealth of families with different properties, and methods to construct them. Durrleman, Nikeghbali & Roncalli (2001) discuss a strategy to choose an appropriate copula family.

A first estimation strategy is based on the *Diversity Score* provided by the rating agency Moody's, cf. Jouanin, Rapuch, Riboulet & Roncalli (2001). The Diversity Score is an attempt to rate the quality (the degree of diversification) of the collateral asset pool of a Collateralized Debt Obligation. The basic idea underlying the diversity score is to replace the original collateral portfolio of n dependent issuers with a portfolio of $d < n$ independent issuers with identical default probability $p(t)$ and identical notional of (n/d) times the original notional. The size of the comparison portfolio d is chosen such that the first two moments of the distribution of the number of survivors in these two portfolios,

$$\sum_{i=1}^{n} 1_{\{\tau_i>t\}} \quad \text{and} \quad \frac{n}{d}\sum_{i=1}^{d} 1_{\{\sigma_i>t\}}, \tag{21}$$

are equal for some fixed horizon t, say one year. The survival indicators $1_{\{\sigma_i>t\}}$ of the comparison portfolio are i.i.d. Bernoulli with success probability $p(t)$. The size of the comparison portfolio d is called the diversity score.

The mean matching procedure yields the equality

$$\sum_{i=1}^{n} q_i(t) = np(t) \tag{22}$$

where $q_i(t) = P[\tau_i > t] = 1 - \int_{-\infty}^{0} H_i(t,x)g_i(x)dx$ is the survival probability of firm i, $H_i(t,\cdot)$ is the distribution function of the running minimum asset value M_t^i, and g_i is the density of the default threshold D_i, which we assume to exist. This survival probability corresponds to the structural model of Section 2.1. Variance matching yields

$$\sum_{i,j=1}^{n} (q_{ij}(t,t) - q_i(t)q_j(t)) = \frac{n^2}{d^2}(p(t)(1-p(t))), \tag{23}$$

where $q_{ij}(t,s) = P[\tau_i > t, \tau_j > s]$ is the joint survival probability of firms i and j for $j \neq i$:

$$q_{ij}(t,s) = \int_{-\infty}^{0}\int_{-\infty}^{0} C_\theta^{ij}(G_i(x), G_j(y))h_{ij}(t,s;x,y)dxdy, \tag{24}$$

and $q_{ij} = q_i$ for $i = j$. Here, $C_\theta^{ij}(u_i,u_j) = C_\theta(1,\ldots,1,u_i,1,\ldots,1,u_j,1,\ldots,1)$ is the bivariate copula of the default threshold vector (D_i,D_j), G_k is the distribution function of the threshold D_k, and $h_{ij}(t,s;\cdot,\cdot)$ is the density of the running minimum asset vector (M_t^i, M_s^j), which we assume to exist. This bivariate survival probability corresponds to the structural model of Section 3.1.

Suppose we have estimated a model for individual assets dynamics, have fixed distributions G_i for individual default thresholds, and are given the diversity score d for an underlying portfolio of n instruments from Moody's. Together with an appropriate assumption or estimate of asset correlation, equations (22) and (23) allow to calibrate C_θ under the assumption that the threshold dependence structure is bivariate and symmetric.

Another calibration procedure is based on the capacity of our model to forecast default cascading or contagion effects: upon a default of some firm, credit spreads of closely associated firms jump up or down. This is a reflection of the fact that suddenly appearing information is used by bond investors to immediately re-assess the credit quality of associated firms. These cascading effects are modeled through the dependence between the thresholds at which individual firms default, cf. Giesecke (2001*a*).

In view of this, a reasonable strategy seems to be using the market-observed price jumps in traded instruments to calibrate the threshold copula C_θ. This analysis may also include quotes of suitable peer issuers. Instead of using market prices, one can also use *anticipated* spread/price jumps. This allows to calibrate the implied cascading effects to subjective beliefs of traders in the underlying or associated credit derivatives on the underlying, allowing to leverage traders' unique and valuable experience. In fact, it is often more intuitive to estimate the jump in spreads given certain default scenarios in the market than to come up with an asset correlation estimate. This is in particular the case for names where the methods described in Kealhofer (1998) are not applicable due to lack of appropriate equity data.

The procedure based on observed or anticipated default-induced price jumps involves more effort than that based on Moody's Diversity Score; it hence seems more appropriate for smaller pools of issuers, such as those underlying basket credit derivative transactions. In order to outline this procedure, suppose we are given the jump

$$\Delta_i^j(t,T), \quad t \leq T, \quad j \neq i$$

in the time-t-conditional probability of survival of firm i by time T, if firm j were to default at time t. Given appropriate assumptions on recovery rates and riskless interest rates, $\Delta_i^j(t,T)$ can be backed out from quotes of traded bonds maturing at T or credit swap spreads of firm i around the default of issuers comparable to j. Another way is to calculate the jump in the survival probability $\Delta_i^j(t,T)$ from the anticipated jump in bond prices or swap spreads in the appropriate default scenario.

Let us consider the instructive case with $n = 2$ issuers. Using Bayes' rule we have for conditional default probabilities $q_1(t,T) = P[\tau_1 > T \mid \mathcal{G}_t]$

$$q_1(t,T) = \frac{q_{12}(T,t)}{q_{12}(t,t)} \quad \text{on} \quad \{T_1 > t\},$$

where $T_1 = \tau_1 \wedge \tau_2$ and $q_{12}(t,T)$ is the time-t conditional joint survival probability defined in (24), and, assuming sufficient regularity of h_{12},

$$q_1(t,T) = \frac{\frac{\partial}{\partial s} q_{12}(T,s)|_{s=t}}{\frac{\partial}{\partial s} q_{12}(t,s)|_{s=t}} \quad \text{on} \quad \{T_1 = \tau_2 = t\}.$$

For the jump in conditional default probabilities we therefore obtain

$$\Delta_1^2(t,T) = \frac{\frac{\partial}{\partial s} q_{12}(T,s)|_{s=t}}{\frac{\partial}{\partial s} q_{12}(t,s)|_{s=t}} - \frac{q_{12}(T,t)}{q_{12}(t,t)}. \tag{25}$$

Suppose we have estimated a model for individual assets dynamics, have fixed distributions G_i for individual default thresholds, and are given the jump $\Delta_1^2(t,T)$. Together with an appropriate assumption or estimate of asset correlation, equation (25) allows to calibrate C_θ under the assumption that the threshold dependence structure is symmetric.

7 Conclusion

This paper generalizes the structural incomplete observation models of Duffie & Lando (2001) and Giesecke (2001*b*) to the multi-firm case with correlated defaults. Our approach accommodates the well-documented cyclical correlation effects as well as default cascading effects. The latter are implied in our setup by the incomplete information of investors.

We explicitly construct the pricing trend and the arrival intensity for the first, second, etc. default in terms of fundamental firm variables, and we provide sufficient conditions for the existence of that intensity. Based on the pricing trend, we formulate a computationally efficient algorithm for the simulation of successive, correlated, and unpredictable defaults.

The proposed model and the associated algorithm have direct applications in design, analysis, and valuation of single and multi-name credit derivatives, credit derivatives signed with defaultable counterparties, and structured credit products.

A Proofs

Proof of Proposition 2.1. Let $L(t) = P[\tau > t]$ denote the survival function of τ. Noting the independence of D and V and that $\{\tau > t\} = \{M_t > D\}$, we have

$$L(t) = P[M_t > D] = 1 - \int_{-\infty}^{0} H(t,x) g(x) dx, \tag{26}$$

It is a classic result, due to Dellacherie (1970), that the compensator of N is given through $A^\tau = A(\cdot \wedge \tau)$ with

$$A(t) = -\int_0^t \frac{dL(s)}{L(s-)}. \tag{27}$$

If $H(t,x)$ is continuous in t for $x \le 0$, then the survival function L is continuous, and we obtain from (27) and (26) that

$$A(t) = -\ln L(t) = -\ln\left(1 - \int_{-\infty}^{0} H(t,x) g(x) dx\right), \tag{28}$$

which proves the first statement.

With our assumptions on H, Aven's (1985) conditions are satisfied so that there exists a λ given by (3) such that

$$A(t) = \int_0^t \lambda(s) ds,$$

and this yields the second statement. Since λ is a deterministic function of time, it is predictable. This implies uniqueness, cf. Brémaud (1980). □

Proof of Proposition 3.1. Let $L_1(t) = P[T_1 > t]$ denote the survival function of T_1. Using the fact that $\{\tau_i > t\} = \{M_t^i > D_i\}$ and that D is independent of V, we get

$$\begin{aligned} L_1(t) &= P[\min_i(\tau_1, \ldots, \tau_n) > t] \\ &= P[M_t^1 > D_1, \ldots, M_t^n > D_n] \\ &= \int_{\mathbb{R}_-^n} G(x)\, h(t,x)\, dx. \end{aligned} \tag{29}$$

It follows from the results in Chou & Meyer (1975), that if L_1 is continuous, then the process $A^{T_1} = A_1(\cdot \wedge T_1)$ given through

$$A_1(t) = -\ln L_1(t) \tag{30}$$

is the compensator of T_1. This proves the first statement.

With our assumptions on h, Aven's (1985) conditions are satisfied so that there exists a λ_1 such that

$$A_1(t) = \int_0^t \lambda_1(s) ds,$$

which yields the second statement. The uniqueness of the intensity follows from the fact that λ_1 is a deterministic function of time. □

Proof of Proposition 4.1. For the kth arrival time T_k, we define the $\mathcal{G}_{T_{k-1}}$-measurable survival function L_k by

$$L_k(t) = P[T_k > t \,|\, \sigma(Z_{k-1})], \quad k \ge 2, \tag{31}$$

where $Z_k = (T_i, I_i)_{i \leq k}$ and $I_k \in \{1, \ldots, n\}$ is the identity of the kth defaulter. That is, $L_k(t)$ denotes the conditional probability that the kth default is after time t, given the arrival times and identities of the first $k-1$ defaults. Note that $L_k(t) = 1$ for all $t \leq T_{k-1}$. If the first $k-1$ default times have identities $I_i = i$ for $1 \leq i \leq k-1$, then with the definition of a default event in (9) we get

$$\begin{aligned} L_k(t) &= P[\tau_k > t, \ldots, \tau_n > t \mid \sigma(\tau_1, \ldots, \tau_{k-1})] \\ &= P[D_k < M_t^k, \ldots, D_n < M_t^n \mid D_1 = M_{\tau_1}^1, \ldots, D_1 = M_{\tau_{k-1}}^{k-1}]. \end{aligned}$$

With an application of Bayes' rule we obtain

$$L_k(t) = \frac{\int_{\mathbb{R}_-^n} G_{z_1 \cdots z_{k-1}}(x) h(\tau_1, \ldots, \tau_{k-1}, t, \ldots, t; x) dx}{\int_{\mathbb{R}_-^{k-1}} G_{z_1 \cdots z_{k-1}}^{k-1}(x) h_{k-1}(\tau_1, \ldots, \tau_{k-1}; x) dx}$$

for $t > \tau_{k-1}$.

Now we put $N_t = \sum_i 1_{\{t \geq T_i\}}$ and consider the compensator of the point process N. From the results in Chou & Meyer (1975), if L_k is continuous, then the process A^N given by

$$A_t^N = \begin{cases} -\ln L_1(T_1) - \ldots - \ln L_k(t) & : \quad T_{k-1} \leq t < T_k \\ -\ln L_1(T_1) - \ldots - \ln L_n(T_n) & : \quad T_n \leq t \end{cases} \tag{32}$$

is the compensator of the process N. This implies that the compensator of the kth default indicator process N_k is given by $A^{T_k} = A_k(\cdot \wedge T_k)$ with

$$A_k(t) = -\ln L_k(t), \quad \text{on} \quad \{t > T_{k-1}\}, \tag{33}$$

which proves the first statement.

With our assumptions on h, Aven's (1985) conditions are satisfied so that for $t > T_{k-1}$ we get

$$A_k(t) = \int_{T_{k-1}}^t \lambda_k(s) ds$$

with

$$\lambda_k(t) = A_k'(t) = -\frac{L_k'(t)}{L_k(t)} = -\frac{\int_{\mathbb{R}_-^n} G_{z_1 \cdots z_{k-1}}(x) \dot{h}(\tau_1, \ldots, \tau_{k-1}, t, \ldots, t; x) dx}{\int_{\mathbb{R}_-^n} G_{z_1 \cdots z_{k-1}}(x) h(\tau_1, \ldots, \tau_{k-1}, t, \ldots, t; x) dx},$$

which yields the second statement. Uniqueness of λ_k follows from the predictability of the mapping $t \to \lambda_k(t)$. □

References

Avellaneda, Marco & Jingyi Zhu (2001), Modeling the distance-to-default process of a firm. Working Paper, Courant Institute.

Aven, Terje (1985), 'A theorem for determining the compensator of a counting process', *Scandinavian Journal of Statistics* **12**, 69–72.

Brémaud, Pierre (1980), *Point Processes and Queues – Martingale Dynamics*, Springer-Verlag, New York.

Cetin, Umut, Robert Jarrow, Philip Protter & Yildiray Yildirim (2002), Modeling credit risk with partial information. Working Paper, Cornell University.

Chou, C. & P.A. Meyer (1975), Sur la représentation des martingales comme intégrales stochastiques dans la processus ponctuels, *in* 'Séminaire de Probabilités IX, Lecture Note in Mathematics', Springer-Verlag Berlin, pp. 60–70.

Crouhy, Michel, Dan Galai & Robert Mark (2000), 'A comparative analysis of current credit risk models', *Journal of Banking and Finance* **24**, 59–117.

Dellacherie, C. (1970), Un exemple de la théorie générale des processus, *in* 'Séminaire de Probabilités IV, Lecture Note in Mathematics 124', Springer-Verlag Berlin, pp. 60–70.

Dellacherie, Claude & Paul-André Meyer (1982), *Probabilities and Potential*, North Holland, Amsterdam.

Devroye, L. (1986), *Non-Uniform Random Variate Generation*, Springer-Verlag, New York.

Duffie, Darrell (1998), First-to-default valuation. Working Paper, GSB, Stanford University.

Duffie, Darrell & David Lando (2001), 'Term structures of credit spreads with incomplete accounting information', *Econometrica* **69**(3), 633–664.

Durrleman, Valdo, Ashkan Nikeghbali & Thierry Roncalli (2001), Which copula is the right one? Working Paper, GRO, Crédit Lyonnais.

Elliott, Robert, Monique Jeanblanc & Marc Yor (2000), 'On models of default risk', *Mathematical Finance* **10**(2), 179–195.

Embrechts, Paul, Alexander J. McNeil & Daniel Straumann (2001), Correlation and dependence in risk management, *in* M.Dempster, ed., 'Risk management: value at risk and beyond', Cambridge University Press, Cambridge.

Giesecke, Kay (2001*a*), Correlated default with incomplete information. Working Paper, Cornell University. To appear in Journal of Banking and Finance.

Giesecke, Kay (2001*b*), Default and information. Working Paper, Cornell University.

Hull, John & Alan White (2000), 'Valuing credit default swaps i: No counterparty default risk', *Journal of Derivatives* **8**, 29–40.

Jarrow, Robert A. & Fan Yu (2001), 'Counterparty risk and the pricing of defaultable securities', *Journal of Finance* **56**(5), 555–576.

Jouanin, J.-F., G. Rapuch, G. Riboulet & T. Roncalli (2001), Modeling dependence for credit derivatives with copulas. Working Paper, GRO, Crédit Lyonnais.

Kealhofer, Stephen (1998), Portfolio management of default risk. Working Paper, KMV Corp.

Lando, David (1998), 'On cox processes and credit risky securities', *Review of Derivatives Research* **2**, 99–120.

Lindskog, Filip (2000), Modeling dependence with copulas. Dissertation, Department of Mathematics, ETH Zürich.

Nelsen, Roger (1999), *An Introduction to Copulas*, Springer-Verlag, New York.

Rutkowski, Marek (1999), On models of default risk. Working Paper, Department of Mathematics, University of Warsaw.

Schönbucher, Philipp & Dirk Schubert (2001), Copula-dependent default risk in intensity models. Working Paper, Universität Bonn.

Zhou, Chunsheng (2001), 'An analysis of default correlation and multiple defaults', *Review of Financial Studies* **14**(2), 555–576.