Cartel Pricing Dynamics in the Presence of an Antitrust Authority*

Joseph E. Harrington, Jr.
Department of Economics
Johns Hopkins University
Baltimore, MD 21218
410-516-7615, -7600 (Fax)
joe.harrington@jhu.edu
www.econ.jhu.edu/People/Harrington

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Abstract

Price-fixing is characterized when firms are concerned about creating suspicions that a cartel has formed. Antitrust laws have a complex effect on pricing as they interact with the conditions determining the internal stability of the cartel. Dynamics are driven by two forces - the sensitivity of detection to price movements causes a cartel to gradually raise price while the sensitivity of penalties to the price level induces the cartel to lower price over time in order to maintain the stability of the cartel. While antitrust laws can lower collusive prices, they can also raise them by making it easier for firms to collude.

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1 Introduction

As evidenced by recent cases in lysine, graphite electrodes, vitamins, and auction houses, price-fixing remains a perennial problem. Though there is a voluminous theoretical literature on collusive pricing, an important dimension to price-fixing cartels has received little attention. In light of its illegality, a critical goal faced by a cartel is to avoid the appearance that there is a cartel. Firms want to raise prices but not suspicions that they are coordinating their behavior. If high prices or rapidly increasing prices or, more generally, anomalous price movements may make customers and the antitrust authorities suspicious that a cartel is operating, one would expect this to have implications for how the cartel prices. The objective of this paper is to explore these implications especially with respect to pricing dynamics. Also of interest is understanding the impact of antitrust policy.

In an earlier paper (Harrington, 2002a), the joint profit maximizing price path was characterized under the constraint of possible detection and antitrust penalties. Previous static models assumed that detection is more likely when the price is higher (Block, Nold, and Sidak, 1981). Examining the dynamic extension of those models, the counterfactual result was derived that, after initially raising price, the cartel price path is decreasing over time. When instead the probability of detection is specified to be increasing in the extent of price *changes* rather than the price level, the cartel gradually raises price and price converges. Comparative statics on the steady-state price reveal that it is decreasing in the damage multiple and the probability of detection but is independent of the level of fixed fines. Furthermore, if fines are the only penalty, the cartel's steady-state price is the same as in the absence of antitrust laws, though fines do affect the path to the steady-state. Another intriguing result is that a more stringent standard for calculating damages increases the steady-state price.

That analysis presumed the incentive compatibility constraints ensuring the internal stability of the cartel were not binding. In the current paper, these constraints are explicitly introduced and allowed to bind. The optimal cartel price path is characterized and three considerations come into play - a desire to set high prices to realize high profit, a desire to gradually raise price so as to make detection less likely, and a need to adjust price so as to maintain the internal stability of the cartel. After laying out the model in Section 2 and defining an optimal collusive price path in Section 3, properties of that price path are characterized in Sections 4 and 5. At play are two distinct sources of dynamics as the price path influences both the probability of detection and the penalties in the event of detection. I consider each of these dynamics in turn in Section 4. When penalties are fixed but the probability of detection is endogenous, the cartel price path is shown to maintain the property that it is increasing over time even when incentive compatibility constraints bind. When instead the probability of detection is fixed but penalties are endogenous, the cartel price path is decreasing over time (after price is raised in the first period of collusion). Numerical work is in progress to allow for both of these dynamics to operate. In Section 5, I explore the impact of antitrust laws by comparing the cartel price path with that which would occur if cartels were legal (though not enforceable by the courts). While antitrust laws can have their desired effect of lowering price, they can also raise price because the prospect of detection and possible penalties can enhance cartel stability and thereby allow firms to set higher prices.

2 Model

Consider an industry with n symmetric firms. $\overline{\pi}(P_i, P_{-i})$ denotes firm i's profit when its price is P_i and all other firms charge a common price of P_{-i}. Define $\pi(P) \equiv \overline{\pi}(P, P)$ to be a firm's profit and $D(P)$ a firm's demand when every firm charges P. The space of feasible prices is Ω which is assumed to be a non-empty, compact, convex subset of $\Re_+$. An additional restriction will be placed on Ω later.

A1 Either: i) $\overline{\pi}(P_i, P_{-i})$ is continuous in P_i and P_{-i}, quasi-concave in P_i, $\exists \psi(P_{-i}) \in \arg\max \overline{\pi}(P_i, P_{-i})$, and $\exists$ unique $\widehat{P}$ such that $P \gtreqless \psi(P)$ as $P \lesseqgtr \widehat{P}$; or ii)

$$\overline{\pi}(P_i, P_{-i}) = \begin{cases} (P_i - c)\, nD(P_i) & \text{if } P_i < P_{-i} \\ (P_i - c)\, D(P_i) & \text{if } P_i = P_{-i} \\ 0 & \text{if } P_i > P_{-i} \end{cases}$$

and $D(c) > 0$.

Part (i) of A1 results in the stage game encompassing many differentiated products models while (ii) makes it inclusive of the Bertrand price game (homogeneous goods and constant marginal cost). Allowing for the latter is important for some existence results though our characterization results are much more general. $\widehat{P}$ will denote a symmetric Nash equilibrium price under either (i) or (ii) where, in the latter case, $\widehat{P} = c$. Let $\widehat{\pi} \equiv \pi\left(\widehat{P}\right)$ be the associated profit. As a convention, $\overline{\pi}(\psi(P), P) = (P - c)\, nD(P)$ under (ii). A2 defines the cartel profit function and the joint profit-maximizing price.

A2 $\pi(P)$ is differentiable and quasi-concave in P, if $\pi(P) > 0$ then it is strictly quasi-concave in P, and $\exists P^m > \widehat{P}$ such that $\pi(P^m) > \pi(P) \forall P \neq P^m$.

Firms engage in this price game for an infinite number of periods. The setting is one of perfect monitoring so that firms' prices over the preceding $t-1$ periods are common knowledge in period t. Assume a firm's payoff is the expected discounted sum of its income stream where the common discount factor is $\delta \in (0,1)$.

If firms form a cartel, they meet to determine price. Assume these meetings, and any associated documentation, provides the "smoking gun" if an investigation is pursued.[1] The cartel is detected with some probability and incurs penalties in that event. Assume, for simplicity, that detection results in the discontinuance of collusion forever. Detection in period t then generates a terminal payoff of $[\widehat{\pi}/(1-\delta)] - X^t - F$ where X^t is a firm's damages and F is any (fixed) fines (which may include the monetary equivalent of prison sentences).[2] If not detected, collusion continues on to the next period.

Damages are assumed to evolve in the following manner:

$$X^t = \beta X^{t-1} + \gamma x\left(P^t\right) \text{ where } \beta \in [0,1) \text{ and } \gamma \geq 0.$$

As time progresses, damages incurred in previous periods become increasingly difficult to document and $1-\beta$ measures the rate of the deterioration of the evidence. $x(P^t)$ is the level of damages incurred by each firm in the current period where γ is the damage multiple applied. While U.S. antitrust law specifies treble damages, γ could be less than three because a case is settled out-of-court. Single damages are not unusual for an out-of-court settlement.

A3 $x : \Omega \rightarrow \Re_+$ is bounded and continuous and is non-decreasing over $\left[\widehat{P}, P^m\right]$.

Current U.S. antitrust practice is $x(P^t) = \left(P^t - \widehat{P}\right) D(P^t)$ where $\widehat{P}$ is referred to as the "but for price" and is the price that would have occurred but for collusion. By the boundedness of $x(\cdot)$, it follows that damages are bounded by $\overline{X} \equiv \overline{x}/(1-\beta)$ where $\overline{x} \geq x(P)\, \forall P \in \Omega$. We then have that $X^{t-1} \in \left[0, \overline{X}\right]$.

Detection of a cartel can occur from many sources; some of which are related to price - such as customer complaints - and some of which are unrelated to price - such as internal whistleblowers. Hay and Kelley (1974) find that detection was attributed to a complaint by a customer or a local, state, or federal agency in 13 of 49 price-fixing

[1] Though it is assumed that suspicions lead to an investigation and conviction with probability one, all results go through if the probability of these events is simply positive.

[2] In that our focus will be on symmetric cartel solutions, firms will always have the same level of damages. In this model, damages refers to any penalty that is sensitive to the price charged and the length of time the cartel has been in place while fines refer to penalties that are fixed with respect to the endogenous variables.

cases. In the recent graphite electrodes case, it was reported that the investigation began with a complaint from a steel manufacturer which is a purchaser of graphite electrodes (Levenstein and Suslow, 2001). High prices or anomalous price movements may cause customers to become suspicious and pursue legal action or share their suspicions with the antitrust authorities.[3] Though it isn't important for my model, I do imagine that buyers (in many price-fixing cases, they are industrial buyers) are the ones who may become suspicious about collusion.[4]

To capture these ideas in a tractable manner, I specify an exogenous probability of detection function which depends on the current and previous periods' price vectors. $\phi\left(\underline{P}^t, \underline{P}^{t-1}\right)$ is the probability of detection when the cartel is active where $\underline{P}^t \equiv \left(P_1^t, \ldots, P_n^t\right)$ is the vector of firms' prices.[5] When firms charge a common price, the vector will be replaced with that common scalar. This specification can capture how high prices and big price changes can create suspicions among buyers that firms may not be competing.[6] The impact of the properties of this detection technology on the joint profit-maximizing price path was explored in Harrington (2002a). There I found that cartel pricing dynamics are empirically plausible when detection is driven by price changes rather than price levels. As a result, in this paper I will largely focus on when detection is sensitive to price changes. While additional structure will be imposed later, one common assumption to the ensuing analysis is that the probability of detection is minimized when prices don't change and is weakly higher with respect to price increases. In A4, note that $\phi(P, P)$ is allowed to vary with P in which case price levels can matter. For example, if $\phi(P, P)$ is increasing in P then, when price is stable, detection is more likely when price is higher.

[3] The Nasdaq case is one in which truly anomalous pricing resulted in suspicions about collusion. It was scholars rather than market participants who observed that dealers avoided odd-eighth quotes and ultimately explained it as a form of collusive behavior (Christie and Schultz, 1994). Though the market-makers did not admit guilt, they did pay an out-of-court settlement of around $1 billion.

[4] "As a general rule, the [Antitrust] Division follows leads generated by disgruntled employees, unhappy customers, or witnesses from ongoing investigations. As such, it is very much a reactive agency with respect to the search for criminal antitrust violations. ... Customers, especially federal, state, and local procurement agencies, play a role in identifying suspicious pricing, bid, or shipment patterns." [McAnney, 1991, pp. 529, 530]

[5] In much of the analysis, it is not necessary to specify the exact form of detection when the cartel collapses. At this point, it is sufficient to suppose that detection can occur during the post-cartel phase but it need not be as likely as when the cartel was active.

[6] While customers are implicitly assumed to be forgetful in that their likelihood of becoming suspicious depends only on recent prices, the inclusion of a more comprehensive price history would significantly complicate the analysis, by greatly expanding the state space, without any apparent gain in insight.

A4 $\phi : \Omega^{2n} \rightarrow [0,1]$ is continuous and: i) $\phi(P^o, P^o) \leq \phi(P', P^o)$ and $\phi(P^o, P^o) \leq \phi(P^o, P')$, $\forall P', P^o \in \Omega$; ii) if $\underline{P}'' \geq \underline{P}' \geq \underline{P}^o$ (component-wide) then $\phi\left(\underline{P}'', \underline{P}^o\right) \geq \phi\left(\underline{P}', \underline{P}^o\right)$.

As further motivation for the emphasis on price movements, it is worth noting that while it may be difficult for individuals external to a firm, such as buyers, to have reasonably-informed beliefs about the level of cost and demand, it may be quite reasonable for them to receive noisy signals of changes in cost or demand and be able to make assessments about what is a reasonable change in price. Rather than explicitly model these cost and demand shocks, I have instead postulated that bigger price changes are more likely to trigger suspicions. In the context of a stationary environment, greater price fluctuations should seem more puzzling to buyers. The virtue to this approach is that the cartel's price path is deterministic, which would not be the case with the presence of cost or demand shocks, and this will make it possible to derive analytical results. Future research will explicitly consider encompassing such shocks though will probably require using numerical analysis.

This modelling of detection warrants further discussion since it does not model those agents who might engage in detection. The first point to make concerns tractability. With two distinct sources of structural dynamics - detection and antitrust penalties - in addition to the usual (repeated game-style) behavioral dynamics, this model is rich enough to provide new insight into cartel pricing dynamics even with a simple modelling of the detection process and its complexity already pushes the boundaries of formal analysis. Tractability issues aside, there is another motivation for our approach. The objective of this paper is to develop insight and testable hypotheses about cartel pricing. A good model of the detection process is then one that is a plausible description of how cartel members *perceive* the detection process. To my knowledge, there is little evidence from past cases that cartels hold a sophisticated view of buyers (which is implied if one were to model buyers as strategic agents and derive an equilibrium). It strikes me as quite reasonable that firms might simply postulate that higher prices or bigger price changes result in a greater likelihood of creating suspicions without having derived that relationship from first principles about buyers. Thus, even if this modelling of the detection process is wrong, the resulting statements about cartel pricing may be right if that model is a reasonable representation of firms' perceptions.[7]

[7] Nor do I believe it is inconsistent to model firms as choosing prices optimally - as that is a statement about what one thinks is best for one's self - and, at the same time, suppose that firms do not derive

In period 1, firms have the choice of forming a cartel, and risking detection and penalties, or earning non-collusive profit of $\widehat{\pi}$. If they choose the former, they can, at any time, choose to discontinue colluding. However, a finitely-lived cartel will cause collusion to unravel so that, in equilibrium, firms either collude forever or not at all (subject to the cartel being exogenously terminated because of detection). Firms are then not allowed to form and dissolve a cartel more than once. While the possibility of temporarily shutting down the cartel is not unreasonable (firms may want to "lay low" for a bit of time), the analysis is complicated enough without allowing for such. Exploration of that strategic option is left for future research.

Related Work Though no previous work on the topic allows for the rich set of dynamics of this model, there have been papers which take account of detection considerations in the context of cartel pricing. I have already mentioned the static analysis of Block, Nold, and Sidak (1981) which is extended by Spiller (1986), Salant (1987), and Baker (1988) to allow buyers to adjust their purchases under the anticipation that they may be able to collect multiple damages if sellers are shown to have been colluding. Also within a static setting, Besanko and Spulber (1989, 1990), LaCasse (1995), Polo (1997), and Souam (2001) explore a context in which firms have private information, which influences whether or not they collude, and either the government or buyers must decide whether to pursue costly legal action. Three papers consider a dynamic setting. Cyrenne (1999) modifies Green and Porter (1984) by assuming that a price war, and the ensuing raising of price after the war, results in detection for sure and with it a fixed fine. Spagnolo (2000) and Motta and Polo (2001) consider the effects of leniency programs on the incentives to collude in a repeated game of perfect monitoring.[8] Though considering collusive behavior in a dynamic setting with antitrust laws, these papers exclude the sources of dynamics that are the foci of the current analysis; specifically, that the probability of detection and penalties are sensitive to firms' pricing behavior. It is that sensitivity that will generate predictions about cartel pricing dynamics.

In relating this model to the broader literature on collusive pricing, it is relatively unique in allowing for endogenous state variables that firms continually influence. There is some work which allows for exogenously-determined state variables in that demand

buyers' optimal detection behavior - as that is a statement about what one thinks is best for others. An agent may know what is best for themselves without having a clue as to what is best for someone else.

[8] Rey (2001) offers a nice review of some of this work along with other theoretical analyses pertinent to optimal antitrust policy.

evolves over time; see Kandori (1991) and Bagwell and Staiger (1997).[9] Then there is a line of models that allow firms to make a restricted set of decisions which determine a state variable. Davidson and Deneckere (1990) and Benoît and Krishna (1987, 1991) have firms make a one-time capacity decision, where capacity is infinitely-durable, prior to then colluding in price.[10] Chang (1992) considers a setting in which firms initially locate their products in product characteristic space and then play an infinitely repeated price game. He also allows firms to re-locate at a fixed cost though, in equilibrium, there is no re-location.

Most closely related to the style of analysis here is recent computational work by Fershtman and Pakes (2000), de Roos (2000), and Byzakov (2002). These papers allow firms to invest where investment stochastically influences some firm-specific trait like product quality. Those traits represent state variables in the system. While investment is made non-cooperatively, firms may collude in price. Thus, there is collusion with endogenous and continuously changing state variables; features present in my model. By deploying numerical analysis, these papers allow for a richer set of dynamics but, for the same reason, they are highly restrictive in the modelling of collusion. They define a single collusive outcome for the static game using, for example, the Nash Bargaining Solution. The current collusive outcome is then calculated independently of the dynamic equilibrium. If it is incentive compatible for firms to behave according to the specified collusive solution then collusion occurs and otherwise firms act according to the non-collusive solution. In a sense, firms are given only two options - either maximally collude (according to the specified collusive solution) or not collude at all. The possibility that firms could collude to a lesser degree, when maximal collusion is not incentive compatible, is not permitted. By deploying analytical methods, I am able to model that option so that if state variables make collusion more difficult, firms can scale down the degree of collusion rather than give up colluding altogether. Furthermore, when the cartel chooses price today, they take into account how it may affect the future degree of collusion. The collusive solution is then an integral part of the dynamic equilibrium and this has a substantive impact on price levels and pricing dynamics.

[9] This work builds on Rotemberg and Saloner (1986). Also see Haltiwanger and Harrington (1991) which has demand evolve though in a deterministic fashion.

[10] Benoît and Krishna (1987) also consider the restricted case in which firms can add a certain amount to capacity after their initial capacity investment. Related is the work on semi-collusion starting with Fershtman and Muller (1986). However, that work assumes firms collude in price or quantity and does not model it as an infinitely-repeated game.

3 Optimal Symmetric Subgame Perfect Equilibrium

The cartel's problem is to choose an infinite price path so as to maximize the expected sum of discounted income subject to the price path being incentive compatible (IC). In determining the set of IC price paths, the assumption is made that deviation from the collusive path results in the cartel being dissolved and firms behaving according to a Markov Perfect Equilibrium (MPE).

Suppose a firm deviates and collusion dissolves. Since cartel meetings are no longer taking place, the damage variable simply depreciates at the exogenous rate of $1-\beta$: $X^t = \beta X^{t-1}$.[11] This is still a dynamic problem, however, in that price movements can create suspicions and, while firms are no longer colluding, an investigation could reveal evidence of past collusion. The state variables at t are the vector of lagged prices, $\underline{P}^{t-1}$, and (common) damages, X^{t-1}. A MPE is then defined by a stationary policy function which maps $\Omega^n \times \Re_+$ into Ω. Let $V_i^{mpe}\left(\underline{P}^{t-1}, X^{t-1}\right)$ denote firm i's payoff at a MPE. When there is a symmetric state and a symmetric MPE, the payoff is denoted $V^{mpe}\left(P^{t-1}, X^{t-1}\right)$.

For the characterization of cartel pricing, it is not necessary to characterize the MPE as it is generally sufficient for the ensuing results that the MPE payoff satisfy the following condition:

$$\widehat{\pi}/\left(1-\delta\right) \geq V_i^{mpe}\left(\underline{P}^{t-1}, X^{t-1}\right) \geq \widehat{\pi}/\left(1-\delta\right) - \beta X^{t-1} - F, \forall \left(\underline{P}^{t-1}, X^{t-1}\right) \in \Omega^n \times \left[0, \overline{X}\right]; \tag{1}$$

that is, a MPE results in a payoff weakly lower than the static Nash equilibrium payoff but weakly higher than the static Nash equilibrium payoff less the cost of incurring the penalties for sure. The issue then is under what conditions does a MPE exist that satisfies (1). Note that it holds if the post-cartel price path is sufficiently close to pricing at $\widehat{P}$ and the likelihood of incurring penalties during the post-cartel phase is sufficiently great. For example, (1) holds when infinite repetition of the static Nash equilibrium is a MPE. Appendix B provides two sets of sufficient conditions for that to be the case: i) homogeneous products and constant marginal cost; and ii) the probability of detection (when the cartel is inactive) is independent of an individual firm's price when that price is different from a common price charged by other firms. In the ensuing analysis, (1) is assumed in some cases and in others occurs for free; being implied by other assumptions. Though this

[11] Implicit in this specification is that damages stop accumulating once the cartel is dismantled. This is a useful approximation though need not be descriptively accurate. If the post-cartel price exceeds $\widehat{P}$, it is because of past collusion so one could argue that additional damages should be assessed. Whether, in practice, they are assessed is another matter.

property need not always hold (for an example where it doesn't, see Harrington 2002b), it is useful to limit our attention to when it does so as to be able to provide a coherent set of results. Let me emphasize that I could have done away with (1) by simply focusing on the Bertrand price game. The route I have taken is more general as, by assuming the MPE payoff satisfies (1), it includes the Bertrand price game as a special case.

It is natural to assume that, at the start of the cartel, damages are zero and firms are charging the non-collusive price: $\left(P^0, X^0\right) = \left(\widehat{P}, 0\right)$. While many of the ensuing results are robust to these initial conditions, they will be assumed throughout the paper so as to simplify some of the proofs. Before providing the conditions defining the cartel solution, the assumption is made that damages are assessed only in those periods for which the cartel has been functioning properly and, more specifically, are not assessed when a firm deviates from the cartel price. Thus, when a firm considers cheating on the agreement, it assumes the act of deviation negates damages for that period. In practice, it is not clear when damages are no longer assessed and this assumption is probably as good as any other. Furthermore, it has a nice property which is useful for both analytical and numerical work. If damages were assigned in the period that a firm deviated then, entering the post-deviation phase, firms would have different levels of damages and this would complicate the characterization of a MPE path. Under this assumption, firms have a common damage variable in the post-deviation phase which means that the state space is $\Omega^n \times \Re_+$ rather than $\Omega^n \times \Re_+^n$.[12]

As the focus is on symmetric collusive solutions, it is sufficient to define the state variables as $\left(P^{t-1}, X^{t-1}\right) \in \Omega \times \left[0, \overline{X}\right]$. The firms' problem is either to not form a cartel - and price at $\widehat{P}$ in every period with each firm receiving a payoff of $\widehat{\pi}/\left(1-\delta\right)$ - or form a cartel and choose a price path so as to:

$$\max_{\{P^t\}_{t=1}^{\infty} \in \Gamma} \sum_{t=1}^{\infty} \delta^{t-1} \Pi_{j=1}^{t-1} \left[1 - \phi\left(P^j, P^{j-1}\right)\right] \pi\left(P^t\right) \tag{2}$$

$$+ \sum_{t=1}^{\infty} \delta^t \phi\left(P^t, P^{t-1}\right) \Pi_{j=1}^{t-1} \left[1 - \phi\left(P^j, P^{j-1}\right)\right] \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \sum_{j=1}^{t} \beta^{t-j} \gamma x\left(P^j\right) - F\right]$$

[12] Let me add that all results have been derived when it is instead assumed that damages are assessed in the period of deviation based on the price that the cartel set (which also serves to preserve common damages). I conjecture, but have not shown, that results are also true when damages are assessed for the deviating firm based on its price and quantity. In any case, I feel it is a second-order effect as to how damages are determined in the period of deviation and I have no reason to believe that results are sensitive to this assumption.

where

$$\begin{aligned}
\Gamma &\equiv \{\{P^t\}_{t=1}^{\infty} \in \Omega^\infty : \sum_{\tau=t}^{\infty} \delta^{t-\tau} \Pi_{j=t}^{\tau-1} \left[1-\phi\left(P^j, P^{j-1}\right)\right] \pi\left(P^\tau\right) \\
&\quad + \sum_{\tau=t}^{\infty} \delta^{\tau-t+1} \phi\left(P^\tau, P^{\tau-1}\right) \Pi_{j=t}^{\tau-1} \left[1-\phi\left(P^j, P^{j-1}\right)\right] \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \sum_{j=1}^{\tau} \beta^{\tau-j} \gamma x\left(P^j\right) - F\right] \\
&\geq \max_{P_i} \overline{\pi}\left(P_i, P^t\right) + \delta\phi\left(\left(P^t, \ldots, P_i, \ldots, P^t\right), P^{t-1}\right) \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \sum_{j=1}^{t-1} \beta^{t-j} \gamma x\left(P^j\right) - F\right] \\
&\quad + \delta\left[1-\phi\left(\left(P^t, \ldots, P_i, \ldots, P^t\right), P^{t-1}\right)\right] V_i^{mpe}\left(\left(P^t, \ldots, P_i, \ldots, P^t\right), \sum_{j=1}^{t-1} \beta^{t-j} \gamma x\left(P^j\right)\right), \\
\forall t &\geq 1\}
\end{aligned}$$

In (2), $\Pi_{j=1}^{t-1}\left[1-\phi\left(P^j, P^{j-1}\right)\right]$ is the probability that the cartel has not been detected as of period t. Γ is the set of price paths that satisfy the incentive compatibility constraints (ICCs). A solution to (2) is referred to as an Optimal Symmetric Subgame Perfect Equilibrium (OSSPE) price path.

As I do not have a general proof of existence for a pure-strategy MPE, it is necessary to assume A5 so as to provide a general proof of the existence of an OSSPE price path.[13] Recall that if the stage game is the Bertrand price game then infinite repetition of the static Nash equilibrium is a MPE. This satisfies both the existence and continuity specified in A5.

A5 $\forall\left(\underline{P}^{t-1}, X^{t-1}\right) \in \Omega^n \times \left[0, \overline{X}\right]$, $\exists$ a Markov Perfect Equilibrium and, furthermore, $\exists$ a continuous function $V_i^{mpe} : \Omega^n \times \left[0, \overline{X}\right] \rightarrow \Re$ such that $V_i^{mpe}\left(\underline{P}^{t-1}, X^{t-1}\right)$ is the payoff associated with a Markov Perfect Equilibrium.

Define the firms' choice set as $\{No\ Cartel\} \cup \Gamma$ where it is understood that choosing an element from Γ implies forming a cartel while choosing *No Cartel* implies all firms price at $\widehat{P}$ in all periods. An OSSPE price path is a selection from $\{No\ Cartel\} \cup \Gamma$ that maximizes each firm's payoff. All proofs are in Appendix A.

Theorem 1 *If A1-A5 hold then an OSSPE price path exists.*

$V\left(P^{t-1}, X^{t-1}\right)$ will denote the payoff that is associated with an OSSPE path. When something is stated to be a property of an OSSPE path, it is meant to refer to an OSSPE path that involves cartel formation.

[13] To my knowledge, there is no general existence theorem for Markov Perfect Equilibrium, even in mixed strategies, when the state space is uncountable; see Fudenberg and Tirole (1991).

In order to simplify the proofs, the assumption is made from hereon that $\Omega = [0, P^m]$ so that the cartel does not set price above the simple monopoly price. While I don't believe this assumption is essential for any result, I cannot dismiss the possibility that an OSSPE path would have price exceed the simple monopoly price in some periods. I will later elaborate on this point and will note in the proofs where this assumption is used. However, I also conjecture that the most relevant part of the parameter space is where an OSSPE path lies below P^m. If all of this creates doubt for the reader, a sufficient condition for this assumption to be made without loss of generality is for demand to be perfectly (or sufficiently) inelastic up to some maximal price, P^m, and zero thereafter. Prices in excess of P^m then generate zero demand and can be shown never to be part of an OSSPE path.

4 Dynamic Properties of the Collusive Price Path

When ICCs are not binding, an OSSPE price path is increasing over time (Harrington, 2002a). When those constraints are binding, the analysis is sufficiently more complex that it prevents any such general result. Both the probability of detection and the associated antitrust penalties are evolving over time and how they impact the price path depends on whether ICCs are tightening or loosening. If those constraints are loosening then one would expect the price path to be increasing as if it is IC to initially raise price then it is increasingly easy to do so. If instead those constraints are tightening then one might imagine that price may initially rise but then decline as the constraints bind; forcing the cartel to set a lower price in order to stabilize it. Of course, there is nothing assuring us that the ICCs are monotonically changing; they could become looser then tighter than looser and so forth. Indeed, the combination of two dynamics - the sensitivity of detection to price changes and the dependence of penalties to past prices - means the model is sufficiently rich that general analytical results are not to be had. My approach will then be a blend of analytical and numerical results. First, I isolate the implications of each one of these two dynamics. In Section 4.1, penalties are fixed (so there are fines but no damages) but the probability of detection remains endogenous. As when ICCs do not bind, I find that an OSSPE price path is increasing over time. The strategy of the proof is to show that if it is IC to raise price to some level then it is IC to keep it at that level. Thus, the cartel never needs to lower price so as to maintain cartel stability. Then, in Section 4.2, the probability of detection is fixed but the penalty is endogenous. On an

OSSPE path, it is established that the penalty rises over time which lowers the payoff both from colluding and from deviating. However, the rate of decline in the collusive payoff is shown to be faster than that for the deviator's payoff which causes ICCs to tighten. To ensure cartel stability, the cartel then lowers price over time after initially raising it in the first period. The second step in my research plan is to use numerical analysis to explore what happens when both of these dynamics are operative. That work is in progress and will be reported in a future version of this paper.

4.1 Pricing Dynamics with Endogenous Detection

Assume there are only fines: $\gamma = 0$ and $F > 0$. The lone state variable for the cartel is lagged price and, in the event of a deviation, the vector of lagged prices. Though penalties are fixed, the probability of detection is sensitive to how the cartel prices as specified in A4. However, further structure is required to establish our main result.

B1 If $P' \geq P$ and $P' > P^o$ then $\phi(P', P) - \phi((P', \ldots, P^o, \ldots, P'), P)$ is non-increasing in P.

To interpret B1, suppose that the lagged cartel price is P and the cartel is to raise price to P'. If an individual firm considers deviating to a price of P^o, $\phi(P', P) - \phi((P', \ldots, P^o, \ldots, P'), P)$ is the associated difference in the probability of detection between colluding and deviating. B1 says that this change is weakly larger when the increase in the collusive price is larger (that is, P is smaller and farther away from P'). Some examples of ϕ satisfying B1 are provided in Appendix C. One such example is when ϕ is additively separable in the individual price changes and, when the price change is negative, the probability of detection is non-increasing in the price change and, when the price change is positive, it is a weakly convex non-decreasing function of the price change. Roughly, B1 requires convexity with respect to price increases.

The presumed property for a MPE is imposed by B2.

B2 $\widehat{\pi}/(1-\delta) \geq V_i^{mpe}(\underline{P}) \geq (\widehat{\pi}/(1-\delta)) - F, \forall \underline{P} \in \Omega^n$.

Theorem 2 shows that when penalties are fixed and only detection is sensitive to the price path, the cartel price path is non-decreasing over time.

Theorem 2 *Assume A1-A2, A4-A5, B1-B2, and* $\gamma = 0$. *If* $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ *is an OSSPE path then* $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ *is non-decreasing over time.*

When the cartel is unconstrained by concerns about stability (that is, the ICCs are not binding), the optimal price path is non-decreasing over time (Harrington, 2002a). Since bigger price movements are more likely to trigger suspicions about a cartel having been formed, the cartel gradually raises price so as to balance profit and the probability of detection. Thus, if, when ICCs bind, the price path is decreasing, it is because incentive compatibility requires it. The issue then is under what circumstances does the cartel find itself charging a price that it can't sustain. In the proof of Theorem 2, it is established that if it is IC to raise price to some level then it is IC to keep price at that level. Therefore, it is never necessary to reduce price in order to maintain the stability of the cartel which implies that the price path is non-decreasing over time.

There are two key assumptions which are critical in showing that if it is IC to raise price to some level then it is IC to keep it there. First, the property of the probability of detection function as described in B1. Consider the ICC associated with the cartel currently pricing at P''. If a firm deviates and prices at $P^o < P''$, it changes its current profit by an amount $\overline{\pi}(P^o, P'') - \pi(P'')$ and alters the future payoff, in the event the cartel is not detected, by an amount $V_i^{mpe}(P'', \ldots, P^o, \ldots, P'') - V(P'')$. Those components to the ICC are the same regardless of whether the cartel is raising price to P'' or keeping it there. What differs, however, is how cheating influences the current likelihood of detection. Suppose the cartel is raising price from P' to P''. If a firm goes along with that price increase, detection occurs with probability $\phi(P'', P')$ while if a firm deviates by pricing at P^o then the probability of detection is $\phi((P'', \ldots, P^o, \ldots, P''), P')$. Thus, cheating changes the probability of detection by $\phi(P'', P') - \phi((P'', \ldots, P^o, \ldots, P''), P')$. If instead the cartel is maintaining price at P'' then cheating alters the probability of detection by $\phi(P'', P'') - \phi((P'', \ldots, P^o, \ldots, P''), P'')$. B1 ensures us that

$$\phi(P'', P') - \phi((P'', \ldots, P^o, \ldots, P''), P') \geq \phi(P'', P'') - \phi((P'', \ldots, P^o, \ldots, P''), P'')$$

so that cheating has a more favorable effect on the likelihood of detection when the cartel is raising price than when it is keeping price constant. Given the other components of the ICCs are identical, if it is IC to raise price to P'' then it is IC to keep price at P''. The second key assumption used in proving Theorem 2 is that penalties are fixed. If penalties were endogenous then, depending on how they are changing and how they influence ICCs, it is possible that the cartel could find it IC to raise price to some level but find it is not IC to keep it there because the penalty has changed. In conclusion, when the dynamics are solely due to how the price path influences the likelihood of detection, concerns about cartel stability do not alter the qualitative properties of the cartel price path.

4.2 Pricing Dynamics with Endogenous Penalties

While the previous section allowed detection to be endogenous and penalties to be fixed, let us now assume the contrary. Suppose detection is independent of prices - being exclusively driven by such factors as internal whistleblowers - and $\gamma > 0$ so that penalties are sensitive to the prices set.

C1 $\exists \phi^o \in (0,1)$ such that $\phi\left(\underline{P}', \underline{P}^o\right) = \phi^o \; \forall \underline{P}', \underline{P}^o \in \Omega^n$.

It will be necessary to specify the likelihood of detection after the cartel has collapsed. Let $\rho(\tau)$ denote the probability of detection τ periods after the last cartel meeting (which was in the period during which a firm cheated). As specified in C2, detection is weakly less likely when the cartel is inactive and may decline as the time since collusion grows.

C2 $\rho(0) = \phi^o$ and $\rho(\tau)$ is non-increasing in τ.

As the probability of detection is fixed, the problem simplifies considerably. First, as argued in Appendix B, the unique MPE is the infinite repetition of the static Nash equilibrium. Second, the optimal deviation price is that which maximizes current profit, $\psi(P^t)$. Since the probability of detection is fixed and the price at which a firm deviates doesn't influence penalties (where recall it is assumed that damages are not assessed when the cartel is not functioning), a deviating firm's price only affects current profit. Using these properties, the cartel's problem can be stated as:

$$\max_{\{P^t\}_{t=1}^{\infty} \in \Omega^{\infty}} \sum_{t=1}^{\infty} \delta^{t-1} (1-\phi^o)^{t-1} \left[\pi\left(P^t\right) - \Delta\gamma x\left(P^t\right)\right] + \left[\delta\phi^o / (1-\delta(1-\phi^o))\right]\left[(\widehat{\pi}/(1-\delta)) - F\right] \tag{3}$$

subject to:

$$\begin{aligned} &\sum_{\tau=t}^{\infty} \delta^{\tau-t} (1-\phi^o)^{\tau-t} \left[\pi(P^\tau) - \Delta\gamma x(P^\tau)\right] \\ &\quad + \left[\delta\phi^o / (1-\delta(1-\phi^o))\right]\left[(\widehat{\pi}/(1-\delta)) - F\right] - \Delta\beta X^{t-1} \quad \geq \\ &\overline{\pi}\left(\psi\left(P^t\right); P^t\right) + \delta\left(\widehat{\pi}/(1-\delta)\right) - \theta\beta X^{t-1} - \kappa F, \; \forall t \geq 1, \end{aligned} \tag{4}$$

where $\Delta \equiv \delta\phi^o / [1 - \delta\beta(1-\phi^o)]$, $\theta \equiv \sum_{\tau=0}^{\infty} \delta(\delta\beta)^\tau \Pi_{h=0}^{\tau-1}(1-\rho(h))\rho(\tau)$, and $\kappa \equiv \sum_{\tau=0}^{\infty} \delta^{\tau+1} \Pi_{h=0}^{\tau-1}(1-\rho(h))\rho(\tau)$. $\pi(P^t) - \Delta\gamma x(P^t)$ represents the net income from collusion in period t. A firm receives profit of $\pi(P^t)$ by colluding but incurs a liability in the form of $\Delta\gamma x(P^t)$ which is the expected present value of damages.[14] This expression

[14] More specifically, the expected present value of damages is $\gamma x\left(P^t\right) \sum_{\tau=0}^{\infty} \delta\left[\delta\beta(1-\phi^o)\right]^\tau \phi^o$ where $(1-\phi^o)^\tau \phi^o$ is the probability of detection in τ periods and $\beta^\tau \gamma x\left(P^t\right)$ is the value of damages at that time.

is multiplied by $(1-\phi^o)^{t-1}$ which is the probability that the cartel has not yet been detected. Turning to the payoff to deviating in (4), θ and κ measure the marginal effect of damages and fines, respectively, on the punishment payoff. It follows from C1-C2 that $\theta \in [0,\Delta]$ and $\kappa \in [0, \delta\phi^o/(1-\delta(1-\phi^o))]$. A key implication is that if, starting from period t, some price path is IC given $X^{t-1} = X'$ then it is also IC if $X^{t-1} < X'$ as the collusive payoff is decreasing with respect to damages at a (weakly) faster rate then the deviation payoff.

The next assumption says that the difference between the maximal current profit and the collusive profit is increasing in the collusive price. It'll imply that if a price path is IC then so is a price path which is identical except that the period t price is lower, holding X^t fixed.

C3 $\overline{\pi}(\psi(P), P) - \pi(P)$ is increasing in P $\forall P \geq \widehat{P}$.

In proving the results of this section, it will be useful to pose the cartel's problem as choosing a level of damages rather than price. As this approach requires that $x(\cdot)$ be one-to-one, C4 strengthens A3 by assuming the damage function is strictly monotonic over the relevant domain.

C4 $x(\cdot)$ is differentiable and non-decreasing, $x\left(\widehat{P}\right) = 0$, and x is strictly increasing over $\left[\widehat{P}, P^m\right)$.

Defining $\xi(x)$ as the price that generates current damage penalties of d, it is implicitly defined by: $d = \gamma x(\xi(d))$. ξ is well-defined $\forall d \in \left[\gamma x\left(\widehat{P}\right), \gamma x(P^m)\right]$.

C5 $\pi(\xi(d))$ is concave in d $\forall d \in \left[\gamma x\left(\widehat{P}\right), \gamma x(P^m)\right]$.

It is shown in Appendix D that C5 holds when demand is weakly concave, marginal cost is constant, damages take the standard form, and the but for price weakly exceeds the competitive price. Note that C4 is also implied by these conditions.

The next result shows that damages are non-decreasing over time so that the penalty from detection is weakly increasing on an OSSPE path.

Lemma 3 *Assume A1-A2 and C1-C5. If* $\left\{\overline{X}^t\right\}_{t=1}^{\infty}$ *is consistent with an OSSPE then* $\left\{\overline{X}^t\right\}_{t=1}^{\infty}$ *is non-decreasing.*

C6 imposes quasi-concavity of net income - profit less the expected present value of damages. Sufficient conditions for C6 are strict concavity of the profit and damage functions.

C6 $\exists P^{+} \in \left(\widehat{P}, P^{m}\right]$ such that $\pi'(P) - \Delta\gamma x'(P) \gtreqless 0$ as $P \lesseqgtr P^{+}$ $\forall P \in \left[\widehat{P}, P^{m}\right]$.

The next result shows that though the cartel raises price in the first period, it will (weakly) decrease price thereafter when the probability of detection is fixed and penalties are sensitive to the price path.[15]

Theorem 4 *Assume A1-A2 and C1-C6. If* $\left\{\overline{P}^{t}\right\}_{t=1}^{\infty}$ *is an OSSPE price path then* $\overline{P}^{1} > P^{0}$ *and it is non-increasing* $\forall t \geq 1$.

As the probability of detection is independent of lagged prices, all dynamics come from the evolution of damages. Since detection is more likely when the cartel is active, the collusive payoff is then more sensitive than the deviation payoff to damages. Given that damages grow over an OSSPE path (Lemma 3), the collusive payoff is then declining faster than the deviation payoff over time. This tightens ICCs and, in order to ensure they are satisfied, the cartel may need to lower price (by C3). Though the price is falling over time, its decline is sufficiently mild so that damages continue to rise.

The idea of a cartel raising price for one period to then monotonically lower it is, I believe, counterfactual. However, if one appends this analysis with that of the preceding subsection (thereby allowing the probability of detection to be increasing in the extent of the price change), I conjecture the following price paths will emerge. Initially, the cartel gradually raises price so as to balance earning higher profit with avoiding suspicions about collusion. When ICCs do not bind, the price path continues to increase and eventually asymptotes some steady-state price (Harrington, 2002a). When these constraints do bind, however, price may either continue to rise or instead peak in finite time after which it declines and asymptotes some lower price. In that damages are growing, the equilibrium constraints are tightening which forces the cartel to reduce price for the purposes of internal stability rather than considerations related to detection (indeed, reducing price could raise the likelihood of detection relative to keeping price fixed). Numerical analysis is in progress to assess the validity of this conjecture.

5 The Effect of Antitrust Laws on Price Levels

Having identified some properties of cartel pricing dynamics, the next step is to explore the impact of antitrust laws on the level of cartel prices. Of course, the primary goal of

[15] It is worth noting that, when the ICCs do not bind, the cartel raises price in the first period and keeps it fixed thereafter when the probability of detection is fixed.

antitrust laws is not to induce a cartel to price lower but rather to deter cartel formation altogether. In practice, the considerable length of time before cartels are detected (if they are detected at all) combined with the weak penalties (recall that single damages are not atypical) suggests that few cartels are discouraged from forming. However, even if a cartel is formed, one hopes that antitrust laws will induce the cartel to price lower to reduce the risk of detection and penalties in the event of detection. Furthermore, if the cartel price path is shifted down then clearly these laws reduce the profitability of forming a cartel - the cartel is induced to price lower and there is the possibility of penalties - and thus makes it less likely a cartel will form. If, however, antitrust laws induce the cartel to price higher than it is much more problematic as to whether these laws are even desirable.

To address the impact of antitrust laws on the cartel price path, the first task is to define the benchmark collusive price in the absence of antitrust laws. If detection considerations are removed then the model becomes a classical repeated game. In that the unique MPE for that game is infinite repetition of the static Nash equilibrium and given that we use MPE for the punishment in the game with antitrust laws, it is appropriate to make the benchmark price to be the highest price supportable by a grim trigger strategy, which is denoted $\widetilde{P}$.

A6 $\widetilde{P}$ exists and is unique where if

$$\pi(P)/(1-\delta) \geq \overline{\pi}(\psi(P), P) + \delta(\widehat{\pi}/(1-\delta)) \;\; \forall P \in \left[\widehat{P}, P^m\right]$$

then $\widetilde{P} = P^m$ and otherwise $\widetilde{P} \in \left[\widehat{P}, P^m\right)$ and is defined by

$$\pi(P)/(1-\delta) \gtreqqless \overline{\pi}(\psi(P), P) + \delta(\widehat{\pi}/(1-\delta)) \;\; \text{as } P \lesseqqgtr \widetilde{P}, \forall P \in \left[\widehat{P}, P^m\right].$$

The first result considers the model of Section 4.2 where the probability of detection is fixed and independent of the price path. The intuitive result is derived that the introduction of antitrust laws induces a cartel to price lower in all periods. I then consider the other extreme by assuming that detection depends only on price and, more specifically, price movements. While antitrust laws are shown to initially lower prices, they eventually cause the cartel to price *higher* than would have occurred in the absence of such laws. The risk of detection and the prospect of penalties serve to loosen ICCs and thereby allow the cartel to support higher prices.

5.1 Detection is Independent of Price Movements

If detection is independent of prices then antitrust laws induce the cartel to price lower in all periods.

Theorem 5 *Assume A1-A3, A6, and C1-C2. If* $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ *is an OSSPE price path then* $\sup\left\{\overline{P}^t \,|t=1,2,\ldots\right\} \leq \widetilde{P}$ *and if* $\widetilde{P} < P^m$ *then* $\sup\left\{\overline{P}^t \,|t=1,2,\ldots\right\} < \widetilde{P}$.

This result is quite general and is due to two effects. First, introducing detection results in the cartel being finitely-lived almost surely. This reduces the collusive payoff and tightens ICCs. While we have assumed that detection results in the permanent cessation of collusion, a temporary cessation is sufficient for this force to be operative. Lest the reader thinks that this straightforward and uninteresting effect is exclusively driving our conclusions, a finitely-lived cartel in expectation is neither necessary nor sufficient for antitrust laws to cause the collusive price to be lower.[16] The second and more critical effect is due to detection depending only on whether firms are colluding, and not on the prices they set or the way in which prices change. In that case, a firm that cheats reduces the likelihood of detection by causing the cartel to dissolve. While antitrust penalties reduce both the collusive payoff and the payoff to a deviator (as detection may still occur during the period of deviation or afterwards), it has a bigger impact on the collusive payoff since detection is more likely when the cartel is active. As a result, antitrust penalties depress the collusive payoff more than the payoff to deviating. This tightens ICCs and forces the cartel to set lower prices.

5.2 Detection Depends on Price Movements

Now consider the other extreme - detection depends only on price movements. This is captured by assuming the baseline probability of detection, which is that associated with the price vector not changing, is zero.

D1 $\phi : \Omega^{2n} \rightarrow \Re_+$ is continuously differentiable.

D2 $\phi(P,P) = 0 \ \forall P \in \Omega$.

[16] Theorem 6 shows that the collusive price is higher in the long-run even though with positive probability the cartel is finitely-lived. Thus, a finitely-lived cartel is not sufficient for antitrust laws to have their intended effect. An example is provided in Harrington (2002b) for which the collusive price is lower in all periods even though the cartel is infinitely-lived with probability one. Hence, it is not a necessary condition either.

D3 If $\underline{P}' \geq \underline{P}^o$ and $\underline{P}' \geq \underline{P}''$ (component-wide) then

$$\phi\left(\underline{P}', \underline{P}^o\right) + \left[1 - \phi\left(\underline{P}', \underline{P}^o\right)\right] \phi\left(\underline{P}'', \underline{P}'\right) \geq \phi\left(\underline{P}'', \underline{P}^o\right).$$

D2 says that if prices don't change then the cartel is not detected for sure. I believe results are robust to minor variations in this assumption and this will be discussed after they are presented. In interpreting D3, the lhs is the probability of detection over two periods where prices are initially raised from $\underline{P}^o$ to $\underline{P}'$ and then lowered from $\underline{P}'$ to $\underline{P}''$. D3 specifies that this probability exceeds the probability of directly changing the price vector from $\underline{P}^o$ to $\underline{P}''$. To see why this condition is compelling - if one accepts bigger price movements are more likely to trigger suspicions - consider it at the level of an individual firm. Compare firm i: i) raising price from P_i^o to P_i' and then lowering it from P_i' to P_i''; with ii) changing price from P_i^o to P_i''. Suppose $P_i' > P_i'' > P_i^o$. Since raising price from P_i^o to P_i'' should be less likely to trigger suspicions than raising price from P_i^o to P_i', the probability of detection for (i) ought to be at least as great as that for (ii). If instead $P_i' > P_i^o > P_i''$ then lowering price from P_i' to P_i'' should be worse than lowering price from P_i^o to P_i'' so that, once again, the probability of detection for (i) ought to be at least as great as that for (ii).

A5 will be assumed so that a MPE exists. The following additional property is imposed which holds, for example, when the Bertrand price game is the stage game.

D4 $V_i^{mpe}(\underline{P}, X)$ is non-increasing in X and if $\underline{P} \neq \left(\widehat{P}, \ldots, \widehat{P}\right)$ and $X > 0$ then

$$\widehat{\pi}/(1-\delta) > V_i^{mpe}(\underline{P}, X) \geq (\widehat{\pi}/(1-\delta)) - \beta X - F.$$

While D1-D3 do not imply the probability of detection is ever positive, such is implicit in D4. Define $\overline{\Lambda}(P)$ to be the maximal payoff from deviating when the cartel is in a steady-state of charging a price of P. This means that P was charged last period and this period and damages are at their steady-state level of $\gamma x(P)/(1-\beta)$.

$$\begin{aligned}\overline{\Lambda}(P) \quad &\equiv \quad \max_{P_i \in \Omega} \pi(P_i, P) + \delta\phi((P, \ldots, P_i, \ldots, P), P)\left[(\widehat{\pi}/(1-\delta)) - \beta(\gamma x(P)/(1-\beta)) - F\right] \\ &\quad +\delta\left[1 - \phi((P, \ldots, P_i, \ldots, P), P)\right] V_i^{mpe}((P, \ldots, P_i, \ldots, P), \beta\gamma x(P)/(1-\beta)).\end{aligned}$$

Since damages of $\gamma x(P)/(1-\beta)$ are carried over from the previous period, damages in the period of deviation are $\beta(\gamma x(P)/(1-\beta))$.

Note that A1-A5 imply that $\overline{\Lambda}(P)$ is defined. In D5, P^* is defined to be the highest steady-state price path that is IC. By D2, the steady-state collusive payoff is $\pi(P)/(1-\delta)$.

D5 P^* exists and is unique where, if

$$\pi(P)/(1-\delta) \geq \overline{\Lambda}(P)\,\forall P \in \left[\widehat{P}, P^m\right]$$

then $P^* = P^m$ and, otherwise, $P^* \in \left[\widehat{P}, P^m\right)$ and is defined by

$$\pi(P)/(1-\delta) \gtreqqless \overline{\Lambda}(P) \text{ as } P \lesseqqgtr P^*, \forall P \in \left[\widehat{P}, P^m\right].$$

Furthermore, it is straightforward to show that $P^* \geq \widetilde{P}$ and $P^* > \widetilde{P}$ when $P^m > \widetilde{P}$. It follows from D4 that

$$\overline{\pi}(P_i, P) + \delta(\widehat{\pi}/(1-\delta)) > \overline{\Lambda}(P).$$

It is then true that $\pi\left(\widetilde{P}\right)/(1-\delta) \geq \overline{\Lambda}\left(\widetilde{P}\right)$ which implies $P^* \geq \widetilde{P}$. If $\widetilde{P} < P^m$ then

$$\pi\left(\widetilde{P}\right)/(1-\delta) = \overline{\pi}(P_i, P) + \delta(\widehat{\pi}/(1-\delta)) > \overline{\Lambda}\left(\widetilde{P}\right)$$

and therefore $P^* > \widetilde{P}$.

The cartel's problem is:

$$\max_{\{P^t\}_{t=1}^{\infty} \in \Omega^{\infty}} \sum_{t=1}^{\infty} \delta^{t-1} \Pi_{j=1}^{t-1}\left[1-\phi\left(P^j, P^{j-1}\right)\right]\pi\left(P^t\right) \tag{5}$$
$$+\sum_{t=1}^{\infty} \delta^t \phi\left(P^t, P^{t-1}\right) \Pi_{j=1}^{t-1}\left[1-\phi\left(P^j, P^{j-1}\right)\right]\left[(\widehat{\pi}/(1-\delta)) - \beta^t X^0 - \sum_{j=1}^{t} \beta^{t-j}\gamma x\left(P^j\right) - F\right]$$

subject to

$$\Psi\left(\{P^\tau\}_{\tau=t-1}^{\infty}, X^{t-1}\right) \geq \Lambda\left(\{P^\tau\}_{\tau=t-1}^{\infty}, X^{t-1}\right),\ \forall t \geq 1; \tag{6}$$

where

$$\begin{aligned}
\Psi\left(\{P^\tau\}_{\tau=t-1}^{\infty}, X^{t-1}\right) \equiv & \sum_{\tau=t}^{\infty} \delta^{\tau-t} \Pi_{j=t}^{\tau-1}\left[1-\phi\left(P^j, P^{j-1}\right)\right]\pi\left(P^\tau\right) \\
& +\sum_{\tau=t}^{\infty} \delta^{\tau-t+1} \Pi_{j=t}^{\tau-1}\left[1-\phi\left(P^j, P^{j-1}\right)\right]\phi\left(P^\tau, P^{\tau-1}\right) \times \\
& \left[(\widehat{\pi}/(1-\delta)) - \beta^{\tau-t+1} X^{t-1} - \sum_{j=t}^{\tau} \beta^{\tau-j}\gamma x\left(P^j\right) - F\right],
\end{aligned}$$

$$\begin{aligned}
\Lambda\left(\{P^\tau\}_{\tau=t-1}^{\infty}, X^{t-1}\right) \equiv & \max_{P_i} \overline{\pi}\left(P_i, P^t\right) \\
& +\delta\phi\left(\left(P^t, \ldots, P_i, \ldots, P^t\right), P^{t-1}\right)\left[(\widehat{\pi}/(1-\delta)) - \beta X^{t-1} - F\right] \\
& +\delta\left[1-\phi\left(\left(P^t, \ldots, P_i, \ldots, P^t\right), P^{t-1}\right)\right] \times \\
& V_i^{mpe}(\left(P^t, \ldots, P_i, \ldots, P^t\right), \beta X^{t-1}).
\end{aligned}$$

Theorem 6 states that the price path is bounded below P^* and converges to it. If ICCs are binding in the absence of antitrust laws, so that $\widetilde{P} < P^m$, then the introduction of antitrust laws and an authority to enforce them causes the cartel to eventually price higher.

Theorem 6 *Assume A1-A6 and D1-D5. If* $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ *is an OSSPE price path then* $\overline{P}^t \leq P^* \forall t$ *and* $\lim_{t\to\infty} \overline{P}^t = P^*$.

Given the prospects of detection, the cartel will tend to gradually raise price so as to reduce the likelihood of triggering suspicions that a cartel has formed. This could cause the cartel price path to initially lie below $\widetilde{P}$, which is the cartel price in the absence of antitrust laws. Theorem 6 establishes that eventually the cartel will price in excess of $\widetilde{P}$ *because* detection may occur and antitrust laws result in the levying of penalties. For example, suppose the MPE is infinite repetition of the static Nash equilibrium. The post-deviation period is then characterized by firms lowering their prices from some collusive level to $\widehat{P}$. This "price war" has associated with it some probability of triggering suspicions that firms may not be competing; leading to an investigation and the levying of costly antitrust penalties. These expected penalties represent an additional cost associated with deviation which serves to lower the payoff to deviating. Of course, detection can also occur with collusion which lowers the collusive payoff. However, since $\phi(P, P) = 0$ and the cartel price path eventually settles down, the probability of detection if firms continue colluding is approaching zero and, therefore, the collusive payoff is approaching that value which occurs without antitrust laws. In the long-run, antitrust laws then cause a loosening of ICCs which allows the cartel to support prices in excess of $\widetilde{P}$. Finally, if damages are rising over time then this effect becomes stronger as expected penalties from cheating on the cartel grow, which serves to support yet higher cartel prices. [17]

As just argued, the assumption that $\phi(P, P) = 0$ means that antitrust penalties have no impact on the collusive payoff in the long-run because the probability of detection is converging to zero. However, they do have an impact on the payoff from deviating since deviation results in price discretely falling which means a positive probability of detection. If instead $\phi(P, P) > 0$ then the presence of an antitrust authority depresses both the collusive payoff and the payoff from deviating so its effect on ICCs in the long-

[17] Let me now comment on why I cannot *a priori* dismiss the possibility that an OSSPE path could entail prices in excess of P^m. By pricing above P^m, the cartel may make deviation less profitable as it could cause the MPE price path to involve bigger price decreases and thus be more likely to induce detection.

run is ambiguous. Still, by continuity, Theorem 6 would seem to hold as long as a deviation-induced price war is more likely to generate detection than the stable prices associated with continued collusion. The more general idea is that once parties engage in a conspiracy, detection is often more likely if they discontinue it - resulting in an abrupt change in behavior that might trigger suspicions - than if they continue with the charade. This perverse effect of antitrust policy on cartel pricing may be quite general.[18]

In conclusion, it is important to note that Theorem 6 has an antecedent in Cyrenne (1999).[19] He modifies the imperfect monitoring model of Porter (1983) and Green and Porter (1984) by assuming that the transition into a punishment phase entails an additional cost which is interpreted as an antitrust fine. For the case of a quantity game, the optimal collusive price is independent of the fine though the length of the punishment is decreasing in the size of the fine. For the case of a price game (in which firms' prices are private information), the optimal collusive price is increasing in the fine. In both cases, a higher fine increases the average price set by the cartel. While this result is similar to the one here, there are two significant weaknesses to the analysis in Cyrenne (1999). First, the modelling of the detection process is nonsensical. As part of the standard Green-Porter mechanism, the cartel specifies a trigger price (for the quantity game) such that reversion to the static Nash equilibrium occurs when price falls below it. It is the process of price falling below the trigger price that brings forth cartel detection. No other element of the price series influences detection. If P' is the trigger price then the probability of detection equals 1 if firms are colluding in t and $P^t < P'$ and is zero otherwise. This has odd properties. For example, a small change in price can trigger detection - if price goes from being above P' to below P' - while a large change in price (up or down) can avoid detection as long as price remains above P'. Though Cyrenne (1999) motivates this specification by the notion that large price movements induce detection, his specification does not appear to capture that idea very well. Second, he characterized the steady-state without describing how the cartel gets to it. The transitional path is of primary consideration in the case of a cartel trying to avoid detection as that is where the potentially revealing pricing dynamics lie. A proper analysis then requires modelling the entire cartel price path.

[18] For very different reasons, Fershtman and Pakes (2000) and Athey and Bagwell (2001) also identify some perverse effects of antitrust policy on cartel pricing.

[19] I discovered this paper after developing the intuition for Theorem 6 but prior to proving it.

6 Concluding Remarks

This paper has enriched the classical repeated game model of collusion by taking account of how the manner in which the cartel prices can lead to its eventual detection and, in that event, the levying of penalties. Due to the complex way in which detection and penalties influence the conditions for the internal stability of the cartel, there is an array of implications. First, the introduction of antitrust laws can lower the prices set by the cartel but can also allow them to charge higher prices by loosening the incentive compatibility constraints associated with collusion. Second, while the optimal cartel price path is increasing when incentive compatibility constraints are not binding, when they bite the properties of the path depend on whether those constraints are loosening or tightening over time. When penalties are exogenously set, collusion becomes easier over time and this results in the price path being increasing. When penalties are endogenous but the probability of detection is fixed, collusion becomes more difficult over time as penalties accumulate. As a result, the cartel price path is decreasing over time, after initially being raised right after cartel formation. Our conjecture is that when both penalties and detection are sensitive to cartel behavior that there are two possible paths: i) the cartel price path is monotonically increasing; and ii) the cartel gradually raises price but, after some point, lowers price so as to maintain the stability of the cartel. Numerical analysis is in progress that will assess the accuracy of that conjecture.

7 Appendices

7.1 Appendix A

Proof of Theorem 1: Suppose Γ is empty. As the choice set is the singleton $\{No\ Cartel\}$, the OSSPE price path is $\widehat{P}$ forever. For the remainder of the proof, suppose Γ is non-empty. Consider the payoff function in (2). Since $\pi(\cdot)$ and $x(\cdot)$ are bounded functions and $\delta, \beta \in (0,1)$, the payoff function is defined for all price paths. The payoff function is continuous in $\{P^t\}_{t=1}^{\infty}$ by the continuity of $\pi(\cdot)$, $x(\cdot)$, and $\phi(\cdot)$. To show that Γ is a compact set, first note that it is a subset of Ω^{∞} which, by the compactness of Ω and Tychonoff's Product Theorem, is itself compact. The lhs expression of the ICC is continuous in $\{P^t\}_{t=1}^{\infty}$. Under (i) of A1, the rhs expression is continuous (using A5). Under (ii), the rhs takes the form:

$$\max_{P_i \le P^t} n\pi(P_i) + \delta\phi\left(\left(P^t, \ldots, P_i, \ldots, P^t\right), P^{t-1}\right)\left[\left(\widehat{\pi}/(1-\delta)\right) - \sum_{j=1}^{t-1}\beta^{t-j}\gamma x\left(P^j\right) - F\right]$$

$$+\delta\left[1-\phi\left(\left(P^t, \ldots, P_i, \ldots, P^t\right), P^{t-1}\right)\right] V_i^{mpe}((P^t, \ldots, P_i, \ldots, P^t), \sum_{j=1}^{t-1}\beta^{t-j}\gamma x\left(P^j\right))$$

which is also continuous in P^t. It follows that Γ is a closed set. Since Γ is a closed subset of a compact set, Γ is compact. There is then a solution to (2) as it involves maximizing a continuous function over a non-empty compact set. If the associated payoff exceeds $\widehat{\pi}/(1-\delta)$ then such a solution is an OSSPE price path. If it does not exceed $\widehat{\pi}/(1-\delta)$ then an OSSPE price path is $\widehat{P}$ forever. ♦

Proof of Theorem 2: The proof is comprised of two steps. Suppose $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ is an OSSPE path. First, it is shown that if $P^0 \le \overline{P}^1 \le \cdots \le \overline{P}^{t'}$ then it is IC to keep price constant and thereby price at $\overline{P}^{t'}$ in $t'+1$. Note that the ICC when price is raised to $\overline{P}^{t'}$ and when it is kept constant at $\overline{P}^{t'}$ are identical in terms of current profit and the future payoff but differ only in terms of the current probability of detection. With B1, cheating on the cartel more favorably affects the probability of detection when price is raised to $\overline{P}^{t'}$ than when it is kept fixed at $\overline{P}^{t'}$. Thus, if it is IC to raise price to some level then it is IC to keep it at that level. Second, if, contrary to the theorem, this price path has a decreasing subsequence then, by the first step, one can substitute that decreasing subsequence with a constant price path which is IC and yields a strictly higher payoff. This produces the desired contradiction.

Given this OSSPE, let $V\left(\overline{P}^t\right)$ denote the associated payoff starting with period $t+1$.[20]

[20] Throughout this paper, $V(\cdot)$ denotes the payoff in period t from an OSSPE. This is not a value

In performing the first step, let us initially show that if $\overline{P}^{t'-1} \leq \overline{P}^{t'}$ and $V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/\left(1-\delta\right)$ then it is IC to keep price at $\overline{P}^{t'}$. There are two cases to consider: i) $V\left(\overline{P}^{t'-1}\right) > V\left(\overline{P}^{t'}\right)$; and ii) $V\left(\overline{P}^{t'-1}\right) \leq V\left(\overline{P}^{t'}\right)$. Starting with case (i) and recognizing that the lhs of (7) is $V\left(\overline{P}^{t'-1}\right)$, we have

$$\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right] + \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right] V\left(\overline{P}^{t'}\right) > V\left(\overline{P}^{t'}\right). \tag{7}$$

This implies

$$\frac{\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]}{1 - \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right]} > V\left(\overline{P}^{t'}\right). \tag{8}$$

Substituting the lhs of (8) for $V\left(\overline{P}^{t'}\right)$ in the expression for $V\left(\overline{P}^{t'-1}\right)$ on the lhs of (7), the following upper bound for $V\left(\overline{P}^{t'-1}\right)$ is derived:

$$\begin{aligned} V\left(\overline{P}^{t'-1}\right) &< \pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right] \\ &\quad + \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right]\left[\frac{\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]}{1 - \delta\left(1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right)}\right]. \end{aligned}$$

Re-arranging yields

$$V\left(\overline{P}^{t'-1}\right) < \frac{\pi\left(\overline{P}^{t'}\right) + \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]}{1 - \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right]} \tag{9}$$

which gives us an upper bound on $V\left(\overline{P}^{t'-1}\right)$.

Now consider a constant price path of $\overline{P}^{t'}$ starting in period $t'+1$. The payoff, denoted $W\left(\overline{P}^{t'}\right)$, is defined by:

$$W\left(\overline{P}^{t'}\right) = \pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right] + \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\right] W\left(\overline{P}^{t'}\right),$$

and solving for $W\left(\overline{P}^{t'}\right)$:

$$W\left(\overline{P}^{t'}\right) = \frac{\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]}{1 - \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\right]}. \tag{10}$$

$W\left(\overline{P}^{t'}\right) > V\left(\overline{P}^{t'-1}\right)$ follows from (9) and (10) since $\phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right) \leq \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)$ by A4. Given that, by supposition, $V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/\left(1-\delta\right)$ then $W\left(\overline{P}^{t'}\right) \geq \widehat{\pi}/\left(1-\delta\right)$. The

function and it is only required to be defined for values of the state variables on the OSSPE path.

next step is to show that this constant price path is IC. The ICC for period t' for the original OSSPE path is:

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\
&\quad+\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\right]V\left(\overline{P}^{t'}\right)\quad\geq \\
&\max_{P_i\in\Omega}\overline{\pi}\left(P_i,\overline{P}^{t'}\right)+\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\
&+\delta\left[1-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\right]V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right).
\end{aligned}\tag{11}$$

As $W\left(\overline{P}^{t'}\right)>V\left(\overline{P}^{t'-1}\right)>V\left(\overline{P}^{t'}\right)$ then (11) continues to hold if $W\left(\overline{P}^{t'}\right)$ replaces $V\left(\overline{P}^{t'}\right)$:

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\
&\quad+\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\right]W\left(\overline{P}^{t'}\right)\quad\geq \\
&\max_{P_i\in\Omega}\overline{\pi}\left(P_i,\overline{P}^{t'}\right)+\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\
&+\delta\left[1-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\right]V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right).
\end{aligned}\tag{12}$$

Now consider the ICC for a constant price path of $\overline{P}^{t'}$:

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\
&\quad+\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\right]W\left(\overline{P}^{t'}\right)\quad\geq \\
&\max_{P_i\in\Omega}\overline{\pi}\left(P_i,\overline{P}^{t'}\right)+\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\
&+\delta\left[1-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\right]V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right).
\end{aligned}\tag{13}$$

I want to show that (12) implies (13). Note that we only need to be concerned with $P_i<\overline{P}^{t'}$ as deviating with a price in excess of $\overline{P}^{t'}$ cannot yield a higher payoff than colluding as current profit is weakly lower by A1, the probability of detection is weakly higher, and, by B2, $W\left(\overline{P}^{t'}\right)\geq\widehat{\pi}/\left(1-\delta\right)$ implies $W\left(\overline{P}^{t'}\right)\geq V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)$ so that the MPE payoff is weakly lower than the future collusive payoff. Re-arranging (12) and (13), I want to show that:

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right)-\overline{\pi}\left(P_i,\overline{P}^{t'}\right)+\delta\left[W\left(\overline{P}^{t'}\right)-V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)\right] \\
\geq\ &\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left\{W\left(\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\} \\
&-\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\times \\
&\left\{V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\}
\end{aligned}\tag{14}$$

implies

$$\begin{aligned} & \pi\left(\overline{P}^{t'}\right) - \overline{\pi}\left(P_i, \overline{P}^{t'}\right) + \delta\left[W\left(\overline{P}^{t'}\right) - V_i^{mpe}\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right)\right] \\ \geq \;& \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\left\{W\left(\overline{P}^{t'}\right) - \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]\right\} \\ & -\delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'}\right) \times \\ & \left\{V_i^{mpe}\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right) - \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]\right\}, \\ \forall P_i \; < \;& \overline{P}^{t'}. \end{aligned} \tag{15}$$

As the lhs of (14) and (15) are identical, (14) implies (15) if the rhs of (14) is at least as great as the rhs of (15):

$$\begin{aligned} & \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left\{W\left(\overline{P}^{t'}\right) - \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]\right\} \\ & -\delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\left\{V_i^{mpe}\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right) - \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]\right\} \\ \geq \;& \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\left\{W\left(\overline{P}^{t'}\right) - \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]\right\} \\ & -\delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'}\right)\left\{V_i^{mpe}\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right) - \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]\right\}. \end{aligned}$$

Re-arranging this inequality,

$$\begin{aligned} & \left[\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right) - \phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\right]\left\{W\left(\overline{P}^{t'}\right) - \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]\right\} \quad \geq \\ & \left[\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right) - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'}\right)\right] \times \\ & \left\{V_i^{mpe}\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right) - \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]\right\}. \end{aligned} \tag{16}$$

Since

$$W\left(\overline{P}^{t'}\right) \geq \widehat{\pi}/\left(1-\delta\right) \geq V_i^{mpe}\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right) \geq \left(\widehat{\pi}/\left(1-\delta\right)\right) - F$$

and $\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right) \geq \phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)$ then (16) holds if:

$$\begin{aligned} & \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right) - \phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right) \quad \geq \\ & \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right) - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'}\right), \end{aligned}$$

or, equivalently,

$$\begin{aligned} & \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right) - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right) \quad \geq \\ & \phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right) - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'}\right). \end{aligned} \tag{17}$$

Given that $\overline{P}^{t'} \geq P_i, \overline{P}^{t'-1}$, (17) holds by B1. Having shown that a constant price path of $\overline{P}^{t'}$ starting from $t'+1$ is IC and yields a payoff strictly greater than $V\left(\overline{P}^{t'}\right)$, we have a contradiction that the original price path is an OSSPE path. Therefore, if $\overline{P}^{t'} \geq \overline{P}^{t'-1}$ and $V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/(1-\delta)$, it cannot be true that $V\left(\overline{P}^{t'-1}\right) > V\left(\overline{P}^{t'}\right)$.

Let us now examine case (ii): $V\left(\overline{P}^{t'-1}\right) \leq V\left(\overline{P}^{t'}\right)$. Consider keeping price at $\overline{P}^{t'}$ in period $t'+1$ but then continuing with the original OSSPE path. The ICC at $t'+1$ is:

$$\begin{aligned} & \pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\left[\left(\widehat{\pi}/(1-\delta)\right) - F\right] \\ & +\delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\right] V\left(\overline{P}^{t'}\right) \\ \geq \quad & \max_{P_i \in \Omega} \pi\left(P_i, \overline{P}^{t'}\right) + \delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'}\right)\left[\left(\widehat{\pi}/(1-\delta)\right) - F\right] \\ & +\delta\left[1 - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'}\right)\right] V_i^{mpe}\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right). \end{aligned} \tag{18}$$

Using the same series of steps as with case (i), (11) implies (18) if

$$\begin{aligned} & \left[\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right) - \phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\right]\left\{V\left(\overline{P}^{t'}\right) - \left[\left(\widehat{\pi}/(1-\delta)\right) - F\right]\right\} \quad \geq \\ & \left[\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right) - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'}\right)\right] \times \\ & \left\{V_i^{mpe}\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right) - \left[\left(\widehat{\pi}/(1-\delta)\right) - F\right]\right\}. \end{aligned}$$

The same argument is used to show that this inequality holds. The important point to note is that $V\left(\overline{P}^{t'}\right) \geq V_i^{mpe}\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right)$ because $V\left(\overline{P}^{t'}\right) \geq V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/(1-\delta)$. Hence, if $\overline{P}^{t'} \geq \overline{P}^{t'-1}$ then it is IC to keep price at $\overline{P}^{t'}$ and, in addition, $V\left(\overline{P}^{t'}\right) \geq \widehat{\pi}/(1-\delta)$.

To summarize, it has been shown that, on an OSSPE path, if $V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/(1-\delta)$ and $\overline{P}^{t'-1} \leq \overline{P}^{t'}$ then: i) it is IC to keep price at $\overline{P}^{t'}$; and ii) $V\left(\overline{P}^{t'}\right) \geq \widehat{\pi}/(1-\delta)$. Arguing by strong induction, I will show that if the price path is non-decreasing then it is IC to keep price constant in the future. First note that, by the conditions of an OSSPE, $V\left(P^0\right) \geq \widehat{\pi}/(1-\delta)$. Since, by supposition, $P^0 \leq \overline{P}^1$, it then follows that it is IC to keep price at $\overline{P}^1$. Also note that $V\left(\overline{P}^1\right) \geq \widehat{\pi}/(1-\delta)$. Now suppose $P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'}$. By strong induction, it follows from $P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'-1}$ that $V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/(1-\delta)$. Since $V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/(1-\delta)$ and, by supposition, $\overline{P}^{t'} \geq \overline{P}^{t'-1}$, it is IC to keep price at $\overline{P}^{t'}$. This shows that, on a non-decreasing price path, it is IC to keep price constant. Also note that as long as an OSSPE price path is non-decreasing then so is the value to colluding: if $P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'}$ then $V\left(P^0\right) \leq \cdots \leq V\left(\overline{P}^{t'}\right)$.

Armed with this property, the second step is to suppose that $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ is not non-decreasing and show that there exists another IC path which yields a strictly higher payoff. Suppose the price path declines at some time and let $t'+1$ be the first period in which it does so, $P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'} > \overline{P}^{t'+1}$. Define $t''+1$ as the first period after t' for which price is at least as great as in t': $\overline{P}^t < \overline{P}^{t'}$ $\forall t \in \{t'+1, \ldots, t''\}$ and $\overline{P}^{t''+1} \geq \overline{P}^{t'}$. t'' might be ∞. Now consider an alternative price path in which price equals $\overline{P}^{t'}$ for periods $t'+1, \ldots, t''$ and is identical to the original path starting at $t''+1$. First note that this alternative path yields a strictly higher payoff than the original path since it generates strictly higher profit in periods $t'+1, \ldots, t''$ (here I use the property that price does not exceed P^m so that a higher price means higher profit) and the same profit thereafter. Furthermore, by A4, it results in a weakly lower probability of detection in periods $t'+1, \ldots, t''+1$ because, with this alternative path, price doesn't change over $t'+1, \ldots, t''$ and, with respect to $t''+1$, the price rise is $\overline{P}^{t''+1} - \overline{P}^{t'}$ with the alternative path as opposed to a higher price rise of $\overline{P}^{t''+1} - \overline{P}^{t''}$ with the original path which means a weakly lower probability of detection.

Having established that this alternative price path yields a strictly higher payoff, let me argue that it is IC. Consider incentive compatibility over $t'+1, \ldots, t''$. If $t'=0$ then, since $P^0 = \widehat{P}$, a constant price path of $P^{t'}$ over $t'+1, \ldots, t''$ is certainly IC. If $t' \geq 1$ then $\overline{P}^{t'-1} \leq \overline{P}^{t'}$ and, by our previous analysis, a constant price of $\overline{P}^{t'}$ starting with period $t'+1$ is IC. It is also IC for periods after $t''+1$ since the previous period's price and the current period's price are the same as with the original path which, by supposition, is IC. The only remaining ICC is for period $t''+1$. The period $t''+1$ price is the same for both paths but with the original path the lagged price is $\overline{P}^{t''}$ and with the alternative path it is $\overline{P}^{t'}$ where $\overline{P}^{t'} > \overline{P}^{t''}$. The ICC for $t''+1$ for the original path is:

$$\begin{aligned}
& \pi\left(\overline{P}^{t''+1}\right) + \delta\phi\left(\overline{P}^{t''+1}, \overline{P}^{t''}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right] + \delta\left[1 - \phi\left(\overline{P}^{t''+1}, \overline{P}^{t''}\right)\right] V\left(\overline{P}^{t''+1}\right) \\
\geq \;& \pi\left(P_i, \overline{P}^{t''+1}\right) + \delta\phi\left(\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \overline{P}^{t''}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right] \\
& + \delta\left[1 - \phi\left(\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \overline{P}^{t''}\right)\right] V_i^{mpe}\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \\
\forall P_i \leq \;& P^{t''+1}
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
& \pi\left(\overline{P}^{t''+1}\right) - \pi\left(P_i, \overline{P}^{t''+1}\right) \\
& + \delta\left[V\left(\overline{P}^{t''+1}\right) - V_i^{mpe}\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right)\right] \geq \\
& \delta\phi\left(\overline{P}^{t''+1}, \overline{P}^{t''}\right)\left\{V\left(\overline{P}^{t''+1}\right) - \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) + F\right]\right\} \\
& - \delta\phi\left(\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \overline{P}^{t''}\right) \times \\
& \left\{V_i^{mpe}\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right) - \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]\right\}, \forall P_i \leq \overline{P}^{t''+1}.
\end{aligned} \tag{19}$$

The ICC for the alternative path at $t''+1$ is:

$$\begin{aligned}
&\pi\left(\overline{P}^{t''+1}\right)+\delta\phi\left(\overline{P}^{t''+1},\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]+\delta\left[1-\phi\left(\overline{P}^{t''+1},\overline{P}^{t'}\right)\right]V\left(\overline{P}^{t''+1}\right)\\
\geq\ &\overline{\pi}\left(P_i,\overline{P}^{t''+1}\right)+\delta\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\\
&+\delta\left[1-\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t'}\right)\right]V_i^{mpe}\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\\
\forall P_i\ \leq\ &P^{t''+1}
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&\pi\left(\overline{P}^{t''+1}\right)-\overline{\pi}\left(P_i,\overline{P}^{t''+1}\right)\\
&\quad+\delta\left[V\left(\overline{P}^{t''+1}\right)-V_i^{mpe}\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right)\right]\quad\geq\\
&\delta\phi\left(\overline{P}^{t''+1},\overline{P}^{t'}\right)\left\{V\left(\overline{P}^{t''+1}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\}\\
&-\delta\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t'}\right)\times\\
&\left\{V_i^{mpe}\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\},\forall P_i\leq\overline{P}^{t''+1}.
\end{aligned}\tag{20}$$

I then want to show that the rhs of (19) is at least as great as the rhs of (20):

$$\begin{aligned}
&\left[\phi\left(\overline{P}^{t''+1},\overline{P}^{t''}\right)-\phi\left(\overline{P}^{t''+1},\overline{P}^{t'}\right)\right]\left\{V\left(\overline{P}^{t''+1}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\}\\
\geq\ &\left[\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t''}\right)-\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t'}\right)\right]\times\\
&\left\{V_i^{mpe}\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\},\ \forall P_i\leq\overline{P}^{t''+1}.
\end{aligned}\tag{21}$$

Let me first argue that $V\left(\overline{P}^{t''+1}\right)\geq V_i^{mpe}\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right)$. As the OSSPE price path is non-decreasing over $1,\ldots,t'$ then, by our earlier argument, $V\left(\overline{P}^{t'}\right)\geq\widehat{\pi}/\left(1-\delta\right)$. Next note that since a constant price path of $\overline{P}^{t'}$ is IC - and recalling that $W\left(\overline{P}^{t'}\right)$ denotes the associated payoff - then the conditions of an OSSPE imply $V\left(\overline{P}^{t'}\right)\geq W\left(\overline{P}^{t'}\right)$. Since the expected income stream from the OSSPE path is less than that from the constant price path over $t'+1,\ldots,t''$ (recall that the former generates strictly lower profit and a weakly higher probability of detection in those periods), it must deliver a higher payoff stream after t''. Since $V\left(\overline{P}^{t''}\right)$ is the payoff associated with the stream after t'', it follows that $V\left(\overline{P}^{t''}\right)>V\left(\overline{P}^{t'}\right)$. We then have $V\left(\overline{P}^{t''}\right)\geq\widehat{\pi}/\left(1-\delta\right)$ and since $\overline{P}^{t''+1}\geq\overline{P}^{t''}$ implies $V\left(\overline{P}^{t''+1}\right)\geq\widehat{\pi}/\left(1-\delta\right)$, it follows that $V\left(\overline{P}^{t''+1}\right)\geq V_i^{mpe}\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right)$. Since $\phi\left(\overline{P}^{t''+1},\overline{P}^{t''}\right)\geq\phi\left(\overline{P}^{t''+1},\overline{P}^{t'}\right)$, a sufficient condition for (21) to hold is:

$$\begin{aligned}
&\phi\left(\overline{P}^{t''+1},\overline{P}^{t''}\right)-\phi\left(\overline{P}^{t''+1},\overline{P}^{t'}\right)\\
\geq\ &\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t''}\right)-\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t'}\right),
\end{aligned}$$

or, equivalently,

$$\begin{aligned} & \phi\left(\overline{P}^{t''+1},\overline{P}^{t''}\right)-\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t''}\right) \\ \geq\; & \phi\left(\overline{P}^{t''+1},\overline{P}^{t'}\right)-\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t'}\right). \end{aligned}$$

Since $\overline{P}^{t''+1} > \overline{P}^{t'} > \overline{P}^{t''}$ and $\overline{P}^{t''+1} > P_i$, this condition follows from B1. ♦

Proof of Lemma 3: The method of proof is to presume that $\exists t'$ such that $\overline{X}^{t'-1} < \overline{X}^{t'} > \overline{X}^{t'+1}$ and derive a contradiction. Associated with such a path of damages is a relatively high level of current damages (and, therefore, a high price) in t', $\overline{X}^{t'} - \beta\overline{X}^{t'-1}$, and a relatively low level, $\overline{X}^{t'+1} - \beta\overline{X}^{t'}$, in $t'+1$. However, as $\pi\left(\xi\left(\cdot\right)\right)$ is concave in damages then it is more profitable to have more incremental changes in damages. More specifically, it is shown that if current damages of $\overline{X}^{t'} - \beta\overline{X}^{t'-1}$ is preferred to $\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}$ in t' then it must be true that $\overline{X}^{t'} - \beta\overline{X}^{t'}$ is preferred to $\overline{X}^{t'+1} - \beta\overline{X}^{t'}$ in $t'+1$ which gives us a contradiction.

A critical property that will be used is that if, on an OSSPE path, the cartel prices at P' and the damage state variable at the end of the period is X' then pricing at P with end-of-period damages of X is also IC if $P \leq P'$ and $X \leq X'$. To see this, consider the ICC for $(P^t, X^t) = (P', X')$:

$$\begin{aligned} & \pi\left(P'\right)+\sum_{\tau=t+1}^{\infty}\delta^{\tau-t}\left(1-\phi^o\right)^{\tau-t}\left[\pi\left(P^t\right)-\Delta\gamma x\left(P^\tau\right)\right]+\left[\delta\phi^o/\left(1-\delta\left(1-\phi^o\right)\right)\right]\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\ & -\Delta\beta X' \geq \overline{\pi}\left(\psi\left(P'\right),P'\right)+\delta\left(\widehat{\pi}/\left(1-\delta\right)\right)-\theta X'+\theta\gamma x\left(P'\right)-\kappa F. \end{aligned}$$

Since, by deviating rather than colluding, a firm avoids current damages of $\gamma x\left(P'\right)$, if the end-of-period damages are X' when a firm colludes then they are $\left[X' - \gamma x\left(P'\right)\right]$ when it deviates. By C1-C2, the lhs decreases at a weakly faster rater with respect to X' than the rhs. Hence, if X' is replaced with a lower value for the damage variable, this condition still holds. By C3-C4, $\overline{\pi}\left(\psi\left(P\right),P\right)+\theta\gamma x\left(P\right)-\pi\left(P\right)$ is increasing in P. Hence, this ICC holds if P' is replaced with a lower price. I conclude that, on an OSSPE path, if (P^t, X^t) is replaced with a lower price and/or lower damage variable then the ICC at t still holds.

Since $X^0 = 0$, if $\overline{X}^1 = 0$ then, by the stationarity of the policy function, $\overline{X}^t = 0 \forall t$ and thus, trivially, damages are non-decreasing.[21] Next suppose that $X^0 < \overline{X}^1$. If Lemma 3 is not true then $\exists t' \geq 1$ such that $X^0 < \overline{X}^1 < \cdots < \overline{X}^{t'} > \overline{X}^{t'+1}$. (Note that if damages are constant from one period to the next then they are constant in all future periods by stationarity.) Given the path of damages on an OSSPE path, the associated prices in t'

[21] The assumption $X^0 = 0$ could be replaced with the condition that, on the optimal path, $X^0 < \overline{X}^1$.

and $t'+1$ are defined by $\overline{P}^{t'} = \xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right)$ and $\overline{P}^{t'+1} = \xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'}\right)$. That is, $\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right)$ is the price that results in damages of $\overline{X}^{t'}$ given inherited damages of $\beta\overline{X}^{t'-1}$.

Since, by supposition, $\overline{X}^{t'} > \overline{X}^{t'+1}$ and furthermore $\overline{X}^{t'+1} \geq \beta\overline{X}^{t'} > \beta\overline{X}^{t'-1}$ then $\overline{X}^{t'+1} \in \left[\beta\overline{X}^{t'-1}, \overline{X}^{t'}\right]$. Hence, it was feasible to set price in t' so that damages equalled $\overline{X}^{t'+1}$ at t' and the price that would have done this is $\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}\right)$. Since $\overline{X}^{t'} - \beta\overline{X}^{t'-1} > \overline{X}^{t'+1} - \beta\overline{X}^{t'-1}$ and ξ is increasing then $\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right) > \xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}\right)$. Given that, by supposition, charging a price of $\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right)$ with resulting total damages of $\overline{X}^{t'}$ is IC (as it is part of an OSSPE) then the price-damage pair $\left(\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}\right), \overline{X}^{t'+1}\right)$ is also IC as it involves a lower collusive price and lower damages. Since $\left(\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right), \overline{X}^{t'}\right)$ was selected in t' and, as just argued, the cartel could have chosen $\left(\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}\right), \overline{X}^{t'+1}\right)$, I conclude that the former yields at least as high a payoff. Letting $V(X)$ denote the payoff associated with the OSSPE when damages are X, the previous statement is then represented as:

$$\begin{aligned}
&\pi\left(\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right)\right) + \delta\phi^o\left[\left(\widehat{\pi}/(1-\delta)\right) - \overline{X}^{t'} - F\right] + \delta(1-\phi^o)V\left(\overline{X}^{t'}\right) \geq \\
&\pi\left(\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}\right)\right) + \delta\phi^o\left[\left(\widehat{\pi}/(1-\delta)\right) - \overline{X}^{t'+1} - F\right] + \delta(1-\phi^o)V\left(\overline{X}^{t'+1}\right) \quad \Leftrightarrow
\end{aligned}$$

$$\begin{aligned}
&\pi\left(\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right)\right) - \pi\left(\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}\right)\right) \geq \\
&\delta(1-\phi^o)\left[V\left(\overline{X}^{t'+1}\right) - V\left(\overline{X}^{t'}\right)\right] + \delta\phi^o\left(\overline{X}^{t'} - \overline{X}^{t'+1}\right).
\end{aligned} \tag{22}$$

Next note that $\left(\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right), \overline{X}^{t'}\right)$ being IC implies $\left(\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'}\right), \overline{X}^{t'}\right)$ is as well since $\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right) > \xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'}\right)$. Given that $\left(\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'}\right), \overline{X}^{t'+1}\right)$ was chosen in $t'+1$, it follows that $\left(\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'}\right), \overline{X}^{t'+1}\right)$ yields at least as high a payoff as $\left(\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'}\right), \overline{X}^{t'}\right)$:

$$\begin{aligned}
&\pi\left(\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'}\right)\right) + \delta\phi^o\left[\left(\widehat{\pi}/(1-\delta)\right) - \overline{X}^{t'+1} - F\right] + \delta(1-\phi^o)V\left(\overline{X}^{t'+1}\right) \geq \\
&\pi\left(\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'}\right)\right) + \delta\phi^o\left[\left(\widehat{\pi}/(1-\delta)\right) - \overline{X}^{t'} - F\right] + \delta(1-\phi^o)V\left(\overline{X}^{t'}\right) \quad \Leftrightarrow
\end{aligned}$$

$$\begin{aligned}
&\delta(1-\phi^o)\left[V\left(\overline{X}^{t'+1}\right) - V\left(\overline{X}^{t'}\right)\right] + \delta\phi^o\left(\overline{X}^{t'} - \overline{X}^{t'+1}\right) \geq \\
&\pi\left(\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'}\right)\right) - \pi\left(\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'}\right)\right).
\end{aligned} \tag{23}$$

(22)-(23) imply:

$$\begin{aligned}
\pi\left(\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right)\right) - \pi\left(\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}\right)\right) \quad &\geq \\
\pi\left(\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'}\right)\right) - \pi\left(\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'}\right)\right).&
\end{aligned} \tag{24}$$

Note that the difference in the arguments on the lhs of (24) is

$$\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right) - \left(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}\right) = \overline{X}^{t'} - \overline{X}^{t'+1},$$

and on the rhs is:

$$\left(\overline{X}^{t'} - \beta\overline{X}^{t'}\right) - \left(\overline{X}^{t'+1} - \beta\overline{X}^{t'}\right) = \overline{X}^{t'} - \overline{X}^{t'+1}.$$

By the concavity of $\pi(\xi(\cdot))$, it then follows from (24) that:

$$\overline{X}^{t'+1} - \beta\overline{X}^{t'-1} \leq \overline{X}^{t'+1} - \beta\overline{X}^{t'} \Leftrightarrow \overline{X}^{t'} \leq \overline{X}^{t'-1},$$

which is a contradiction. This proves that $\overline{X}^{t}$ is non-decreasing on an OSSPE path. ♦

Proof of Theorem 4: Let $\left\{\overline{X}^{t}\right\}_{t=1}^{\infty}$ denote the path of damages associated with $\left\{\overline{P}^{t}\right\}_{t=1}^{\infty}$. Recall that $\left(P^0, X^0\right) = \left(\widehat{P}, 0\right)$.[22] If $\overline{P}^1 \leq \widehat{P}$ then, since $X^0 = 0$, $\overline{X}^1 = 0$ (by C4). Hence, by stationarity, an OSSPE price path then involves pricing at $\overline{P}^1$ in period 2 and every period thereafter. As this contradicts the optimality of colluding, it is inferred that $\overline{P}^1 > \widehat{P}$ and, therefore, $\overline{P}^1 > P^0$.[23]

If Theorem 4 is not true then $\exists t' \geq 1$ such that $P^0 < \overline{P}^1 \geq \cdots \geq \overline{P}^{t'} < \overline{P}^{t'+1}$. For the OSSPE price path under consideration, let $\overline{P}^{t'} = P'$ and $\overline{P}^{t'+1} = P''$ where $P' < P''$. The analysis will involve comparing the original price path - $\{\overline{P}^1, \ldots, \overline{P}^{t'-1}, P', P'', \overline{P}^{t'+2}, \ldots\}$ - with an alternative price path - $\{\overline{P}^1, \ldots, \overline{P}^{t'-1}, P'', P', \overline{P}^{t'+2}, \ldots\}$ - which has the prices in t' and $t'+1$ switched. It'll be shown that if a price path has price rise from one period to the next then an alternative price path in which those two prices are switched yields a strictly higher collusive payoff and if the original price path was IC then so is this one. This contradicts the original price path being induced by an OSSPE and thus contradicts the supposition that an OSSPE price path has an increasing sub-sequence after period 1.

The first step is to show that an OSSPE price path is bounded from above by P^+ (which is defined in C6). Suppose not so that in some period price exceeds P^+. Consider an alternative price path which is identical except that it has a price of P^+ in those periods for which price exceeded P^+. By C6, the collusive payoff, which is expressed in (3), is strictly higher since $\pi(P^+) - \Delta\gamma x(P^+)$ exceeds the comparable expression when price exceeded P^+. By C4, accumulated damages are lower. As ICCs are loosened when damages are reduced, if the original price path is IC then so is this one. In that a price path has been constructed which generates a higher payoff and is IC, it contradicts the

[22] The assumption $P^0 = \widehat{P}$ can be replaced with $\overline{P}^1 > P^0$ on the optimal path.

[23] If $F > 0$ then colluding and pricing at or below $\widehat{P}$ is clearly inferior to not colluding. If $F = 0$ then it could be optimal to collude and price at $\widehat{P}$ though that is a non-generic result.

supposition that the original path was generated by an OSSPE. I conclude that an OSSPE price path is bounded from above by P^{+}.

Given $P' < P'' \leq P^{+}$, it follows from C6 that $\pi(P'') - \Delta\gamma x(P'') > \pi(P') - \Delta\gamma x(P')$. Inspection of (3) then reveals that, due to discounting, the alternative price path yields a strictly higher payoff as it has the cartel receive $\pi(P'') - \Delta\gamma x(P'')$ in period t' and $\pi(P') - \Delta\gamma x(P')$ in $t'+1$; which is the reverse of the original path. The remainder of the proof involves showing that if the original price path is IC then so is the alternative price path.

I begin with the supposition that the original path is IC in all periods. With the alternative path, the ICCs over periods $1, \ldots, t'-1$ are still satisfied since the collusive payoff is higher and the deviation payoff is unchanged. Next consider the period t constraint where $t \geq t'+2$. As the current and future price path is the same as with the original path, the only difference in the constraint is lagged damages. Note that accumulated damages at t, where $t \geq t'+2$, under the alternative path and under the original path are identical in all terms except for the damages incurred in periods t' and $t'+1$. The difference between the accumulated damages at t, where $t \geq t'+2$, under the alternative path and under the original path then equals:

$$\begin{aligned} & \left[\beta^{t-t'}\gamma x(P'') + \beta^{t-t'-1}\gamma x(P')\right] - \left[\beta^{t-t'}\gamma x(P') + \beta^{t-t'-1}\gamma x(P'')\right] \\ = \quad & -\beta^{t-t'-1}(1-\beta)\gamma\left[x(P'') - x(P')\right] < 0. \end{aligned}$$

Since, compared to the original path, the alternative path substitutes higher current damages in t' for lower ones in $t'+1$, accumulated damages are lower after $t'+1$. Given that damages are lower under the alternative price path, the path is IC for $t \geq t'+2$.

Next consider the ICC at $t'+1$. With the original price path, price is P'' and damages are $\beta^2\overline{X}^{t'-1} + \beta\gamma x(P') + \gamma x(P'')$ at $t'+1$. With the alternative price path, price is P' and damages are $\beta^2\overline{X}^{t'-1} + \beta\gamma x(P'') + \gamma x(P')$. As price is lower then, by C3, this loosens the ICC. As damages are lower, this also serves to loosen the ICC. I conclude that the ICC is satisfied at $t'+1$ for the alternative price path.

Finally, consider the ICC at t'. Using Lemma 3, it'll be shown that if the original price path is IC at $t'+1$ then the alternative path is IC at t'. As an initial step, compare the damages at $t'+1$ for the original path with those at t' for the alternative path. The latter is weakly smaller iff $\beta\overline{X}^{t'} + \gamma x(P'') \geq \beta\overline{X}^{t'-1} + \gamma x(P'')$. As $\overline{X}^{t'-1} \leq \overline{X}^{t'}$ by Lemma 3, this is then indeed true. Since then damages at t' for the alternative path are weakly lower than damages at $t'+1$ for the original path, *ceteris paribus*, if the original path is IC at

$t'+1$ then the alternative path is IC at t'. For the next step, recall that the collusive payoff at t' for the alternative path exceeds the collusive payoff at t' for the original path. Since $\overline{X}^{t'} \leq \overline{X}^{t'+1}$, it must then be true, for the original path, that $V\left(\overline{X}^{t'}\right) \geq V\left(\overline{X}^{t'+1}\right)$.[24] Holding fixed the level of accumulated damages, it follows that the collusive payoff at t' for the alternative path exceeds the collusive payoff at $t'+1$ for the original path. Still holding fixed the level of accumulated damages, since the price at t' for the alternative path is the same as the price at $t'+1$ for the original path, the deviation payoffs are the same. Finally, since the accumulated damages at t' for the alternative path are weakly lower than the accumulated damages at $t'+1$ for the original path, the ICC being satisfied at $t'+1$ for the original path then implies it holds at t' for the alternative path.

It has then been shown that the incentive compatibility of the original price path implies the incentive compatibility of the alternative price path. As the latter yields a strictly higher payoff, this contradicts the original path being generated by an OSSPE and thereby establishes that an OSSPE price path cannot have an increasing sub-sequence after period 1. ⧫

Proof of Theorem 5: If $\widetilde{P} = P^m$ then obviously $\sup\left\{\overline{P}^t \,|t = 1, 2, \ldots\right\} \leq \widetilde{P}$ so, for the remainder of the proof, assume $\widetilde{P} < P^m$. The ICCs can be re-arranged to be

$$\begin{aligned}
&\sum_{\tau=t}^{\infty} \delta^{\tau-t}\left[(1-\phi^o)^{\tau-t}\left(\pi\left(\overline{P}^{\tau}\right) - \Delta\gamma x\left(\overline{P}^{\tau}\right)\right) + \left(1-(1-\phi^o)^{\tau-t}\right)\widehat{\pi}\right] \\
&- \left[\left(\delta\phi^o/\left(1-\delta\left(1-\phi^o\right)\right)\right) - \kappa\right] F - (\Delta-\theta)\,\beta\overline{X}^{t-1} \\
\geq\ &\pi\left(\psi\left(\overline{P}^t\right), \overline{P}^t\right) + \delta\left(\widehat{\pi}/\left(1-\delta\right)\right), \forall t \geq 1,
\end{aligned}$$

Contrary to the theorem, suppose an OSSPE price path is not bounded below $\widetilde{P}$. First, suppose the price path reaches a maximum in finite time: $\exists t'$ such that $\overline{P}^{t'} \geq \overline{P}^t \forall t$ and $\overline{P}^{t'} \geq \widetilde{P}$. We then have

$$\begin{aligned}
&\pi\left(\psi\left(\overline{P}^{t'}\right), \overline{P}^{t'}\right) + \delta\left(\widehat{\pi}/\left(1-\delta\right)\right) \geq \pi\left(\overline{P}^{t'}\right)/\left(1-\delta\right) \geq \textstyle\sum_{\tau=t'}^{\infty} \delta^{\tau-t'}\pi\left(\overline{P}^{\tau}\right) \quad > \\
&\sum_{\tau=t'}^{\infty} \delta^{\tau-t'}\left[(1-\phi^o)^{\tau-t'}\left(\pi\left(\overline{P}^{\tau}\right) - \Delta\gamma x\left(\overline{P}^{\tau}\right)\right) + \left(1-(1-\phi^o)^{\tau-t'}\right)\widehat{\pi}\right] \\
&- \left[\left(\delta\phi^o/\left(1-\delta\left(1-\phi^o\right)\right)\right) - \kappa\right] F - (\Delta-\theta)\,\beta\overline{X}^{t-1}.
\end{aligned}$$

The first inequality follows from $\overline{P}^{t'} \geq \widetilde{P}$ and A6. The second inequality is because $\overline{P}^{t'} \geq \overline{P}^t \forall t$ and thus $\pi\left(\overline{P}^{t'}\right) \geq \pi\left(\overline{P}^t\right) \forall t$. The third inequality follows from C2, $\phi^o > 0$, and

[24] The reason is that, at t', the cartel can use the price path starting at $t'+1$ and, since damages are weakly lower in t', the collusive payoff must be weakly higher.

$\pi\left(\overline{P}^{t'}\right) > \widehat{\pi}$. We conclude that the ICC for t' is violated. This contradiction establishes that if $\exists t'$ such that $\overline{P}^{t'} \geq \overline{P}^{t} \forall t$ then $\overline{P}^{t'} < \widetilde{P}$ and thus $\sup\left\{\overline{P}^{t} \,|t = 1, 2, \dots\right\} < \widetilde{P}$.

Now suppose $\nexists t'$ such that $\overline{P}^{t'} \geq \overline{P}^{t} \forall t$ and, furthermore, $\sup\left\{\overline{P}^{t} \,|t = 1, 2, \dots\right\} \equiv P^{z} \geq \widetilde{P}$. Construct an infinite subsequence of prices comprised of the maximal price set up to that period: $\left\{\overline{P}^{t_1}, \overline{P}^{t_2}, \dots\right\}$ where $t_0 = 0 < t_1 < t_2 < \cdots$, $\overline{P}^{t_{i-1}} < \overline{P}^{t_i}$, and $\overline{P}^{t} \leq \overline{P}^{t_{i-1}} \forall t \in \{t_{i-1}, t_{i-1}+1, \dots, t_i - 1\}$. This sequence is obviously increasing and, since price is bounded, its limit exists and, in fact, equals P^{z}. First note that:

$$\begin{aligned}\pi\left(P^{z}\right)/\left(1-\delta\right) \quad > \quad & \lim_{i\to\infty} \sum_{\tau=t'}^{\infty} \delta^{\tau-t_i}\left[\left(1-\phi^{o}\right)^{\tau-t_i}\left(\pi\left(\overline{P}^{\tau}\right) - \Delta\gamma x\left(\overline{P}^{\tau}\right)\right) + \left(1-\left(1-\phi^{o}\right)^{\tau-t_i}\right)\widehat{\pi}\right] \\ & -\left[\left(\delta\phi^{o}/\left(1-\delta\left(1-\phi^{o}\right)\right)\right) - \kappa\right]F - \left(\Delta-\theta\right)\beta\overline{X}^{t_i-1}.\end{aligned}$$

Next note that:

$$\lim_{i\to\infty} \pi\left(\psi\left(\overline{P}^{t_i}\right), \overline{P}^{t_i}\right) + \delta\left(\widehat{\pi}/\left(1-\delta\right)\right) = \overline{\pi}\left(\psi\left(P^{z}\right); P^{z}\right) + \delta\left(\widehat{\pi}/\left(1-\delta\right)\right).$$

Since $P^{z} \geq \widetilde{P}$ and thus

$$\pi\left(\psi\left(P^{z}\right), P^{z}\right) + \delta\left(\widehat{\pi}/\left(1-\delta\right)\right) \geq \pi\left(P^{z}\right)/\left(1-\delta\right),$$

it follows from the preceding two equations that

$$\begin{aligned} & \lim_{i\to\infty} \overline{\pi}\left(\psi\left(\overline{P}^{t_i}\right), \overline{P}^{t_i}\right) + \delta\left(\widehat{\pi}/\left(1-\delta\right)\right) \\ > \quad & \lim_{i\to\infty} \sum_{\tau=t'}^{\infty} \delta^{\tau-t_i}\left[\left(1-\phi^{o}\right)^{\tau-t_i}\left(\pi\left(\overline{P}^{\tau}\right) - \Delta\gamma x\left(\overline{P}^{\tau}\right)\right) + \left(1-\left(1-\phi^{o}\right)^{\tau-t_i}\right)\widehat{\pi}\right] \\ & -\left[\left(\delta\phi^{o}/\left(1-\delta\left(1-\phi^{o}\right)\right)\right) - \kappa\right]F - \left(\Delta-\theta\right)\beta\overline{X}^{t_i-1}\end{aligned}$$

which means the ICC is violated. ♦

Proof of Theorem 6: Most of the proof works to show that if $\left\{\overline{P}^{t}\right\}_{t=1}^{\infty}$ is an OSSPE price path then it converges. Define $\mathcal{P}^{t} \equiv \max\left\{\overline{P}^{0}, \overline{P}^{1}, \dots, \overline{P}^{t}\right\}$ to be the maximum price set over the first t periods. As an initial step, it is shown that, on an OSSPE path, if the current period's price is at least as great as all past prices, $\overline{P}^{t'} \geq \mathcal{P}^{t'-1}$, then $\pi\left(\overline{P}^{t'}\right)/\left(1-\delta\right)$ is a lower bound on the value in that period: $V\left(\overline{P}^{t'}, \overline{X}^{t'}\right) \geq \pi\left(\overline{P}^{t'}\right)/\left(1-\delta\right)$; where $\overline{X}^{t}$ is the value of the state variable on the OSSPE path. Intuitively, if it was IC to change price to $\overline{P}^{t'}$ then it is IC to keep price at $\overline{P}^{t'}$ since the probability of detection is zero from doing so (by D2). The next step argues that, generally, $\pi\left(\mathcal{P}^{t}\right)/\left(1-\delta\right)$ is a lower bound on the equilibrium payoff. Since $\left\{\mathcal{P}^{t}\right\}_{t=1}^{\infty}$ and

$\{\pi(\mathcal{P}^t)/(1-\delta)\}_{t=1}^{\infty}$ are both non-decreasing bounded sequences (with the latter following from the former because the price space has an upper bound of P^m), they have a limit. From this we can argue that $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ has a limit. It is then straightforward to show that $\lim_{t\to\infty}\overline{P}^t = P^*$.

Assume $\overline{P}^{t'} \geq \mathcal{P}^{t'-1}$ in which case $\overline{P}^{t'} \geq \overline{P}^{t'-1}$. The ICC for period t' is

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[(\widehat{\pi}/(1-\delta)) - \beta\overline{X}^{t'-1} - \gamma x\left(\overline{P}^{t'}\right) - F\right] \\
&\quad + \delta\left[1-\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right] V\left(\overline{P}^{t'}, \beta\overline{X}^{t'-1} + \gamma x\left(\overline{P}^{t'}\right)\right) \quad \geq \\
&\max_{P_i\in\Omega}\overline{\pi}\left(P_i, \overline{P}^{t'}\right) + \delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\left[(\widehat{\pi}/(1-\delta)) - \beta\overline{X}^{t'-1} - F\right] \\
&\quad + \delta\left[1-\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\right] V_i^{mpe}\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \beta\overline{X}^{t'-1}\right).
\end{aligned} \tag{25}$$

We want to make two substitutions in (25). First, replace $\left[(\widehat{\pi}/(1-\delta)) - \beta\overline{X}^{t'-1} - \gamma x\left(\overline{P}^{t'}\right) - F\right]$ on the lhs with $\left[(\widehat{\pi}/(1-\delta)) - \beta\overline{X}^{t'-1} - F\right]$. Second, suppose, contrary to the claim that $\pi\left(\overline{P}^{t'}\right)/(1-\delta)$ is a lower bound on the collusive payoff, we have $V\left(\overline{P}^{t'}, \beta\overline{X}^{t'-1} + \gamma x\left(\overline{P}^{t'}\right)\right) <$ $\pi\left(\overline{P}^{t'}\right)/(1-\delta)$ and replace $V\left(\overline{P}^{t'}, \beta\overline{X}^{t'-1} + \gamma x\left(\overline{P}^{t'}\right)\right)$ with $\pi\left(\overline{P}^{t'}\right)/(1-\delta)$ on the lhs of (25). If (25) holds then it is still true after these two substitutions:

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[(\widehat{\pi}/(1-\delta)) - \beta\overline{X}^{t'-1} - F\right] \\
&\quad + \delta\left[1-\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right] \pi\left(\overline{P}^{t'}\right)/(1-\delta) \quad \geq \\
&\max_{P_i\in\Omega}\overline{\pi}\left(P_i, \overline{P}^{t'}\right) + \delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\left[(\widehat{\pi}/(1-\delta)) - \beta\overline{X}^{t'-1} - F\right] \\
&\quad + \delta\left[1-\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\right] V_i^{mpe}\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \beta\overline{X}^{t'-1}\right).
\end{aligned} \tag{26}$$

The objective is to show that pricing at $\overline{P}^{t'}$ from $t'+1$ onward is IC and thus $\pi\left(\overline{P}^{t'}\right)/(1-\delta)$ is a lower bound on $V\left(\overline{P}^{t'}, \overline{X}^{t'}\right)$ which gives us the desired contradiction.

As an alternative price path, consider the firm maintaining price at the t' level; that is, pricing at $\overline{P}^{t'}$ in period $t, \forall t \geq t'+1$. The ICC for period $t'+1$ is

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right) + \delta\left(\pi\left(\overline{P}^{t'}\right)/(1-\delta)\right) \quad \geq \\
&\max_{P_i\in\Omega}\overline{\pi}\left(P_i, \overline{P}^{t'}\right) + \delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\left[(\widehat{\pi}/(1-\delta)) - \beta\overline{X}^{t'} - F\right] \\
&\quad + \delta\left[1-\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\right] V_i^{mpe}\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \beta\overline{X}^{t'}\right),
\end{aligned} \tag{27}$$

where $\overline{X}^{t'} = \beta\overline{X}^{t'-1} + \gamma x\left(\overline{P}^{t'}\right)$. Note that $\overline{X}^{t'} \geq \overline{X}^{t'-1}$ since $\overline{P}^{t'}$ is the highest price charged thus far. In addition, damages are no longer present in the collusive payoff as, by D2, $\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)=0$. For both (26) and (27), the ICC holds when $P_i>\overline{P}^{t'}$ as pricing above $\overline{P}^{t'}$ weakly lowers current profit (by A1), weakly raises the probability of detection (by A4), and it'll be shown that the MPE payoff does not exceed the collusive payoff.

I want to show that (26) implies (27) which will establish that if the original price path was IC at t' then so is a price of $\overline{P}^{t'}$ at $t'+1$. Since the rhs of (27) is non-increasing in damages (using D4), a sufficient condition for (27) to hold is

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right)+\delta\left(\pi\left(\overline{P}^{t'}\right)/\left(1-\delta\right)\right) \quad \geq \qquad (28)\\
&\max_{P_i\in\Omega}\overline{\pi}\left(P_i,\overline{P}^{t'}\right)+\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta\overline{X}^{t'-1}-F\right]\\
&+\delta\left[1-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\right]V_i^{mpe}\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\beta\overline{X}^{t'-1}\right),
\end{aligned}$$

where $\beta\overline{X}^{t'}$ has been replaced with $\beta\overline{X}^{t'-1}$. Let us then show that (26) implies (28). This is true if the rhs minus the lhs of (28) is at least as great as the rhs minus the lhs of (26):

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right)+\delta\left(\pi\left(\overline{P}^{t'}\right)/\left(1-\delta\right)\right)-\overline{\pi}\left(P_i,\overline{P}^{t'}\right)\\
&-\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta\overline{X}^{t'-1}-F\right]\\
&-\delta\left[1-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\right]V_i^{mpe}\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\beta\overline{X}^{t'-1}\right)\\
\geq\ &\pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta\overline{X}^{t'-1}-F\right]\\
&+\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\right]\left(\pi\left(\overline{P}^{t'}\right)/\left(1-\delta\right)\right)\\
&-\overline{\pi}\left(P_i,\overline{P}^{t'}\right)-\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta\overline{X}^{t'-1}-F\right]\\
&-\delta\left[1-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\right]V_i^{mpe}\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\beta\overline{X}^{t'-1}\right),\\
\forall P_i\ <\ &\overline{P}^{t'}.
\end{aligned}$$

Eliminating common terms on both sides and re-arranging:

$$\begin{aligned}
&\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left\{\left(\pi\left(\overline{P}^{t'}\right)/\left(1-\delta\right)\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta\overline{X}^{t'-1}-F\right]\right\} \qquad (29)\\
\geq\ &\left[\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\right]\times\\
&\left\{V_i^{mpe}\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\beta\overline{X}^{t'-1}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta\overline{X}^{t'-1}-F\right]\right\}.
\end{aligned}$$

As D4 implies

$$\pi\left(\overline{P}^{t'}\right)/\left(1-\delta\right)\geq V_i^{mpe}\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\beta\overline{X}^{t'-1}\right)\geq\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta\overline{X}^{t'-1}-F$$

then (29) holds if

$$\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\geq\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right). \tag{30}$$

Suppose $\overline{P}^{t'} > P_i \geq \overline{P}^{t'-1}$. By A4,

$$\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right) \geq \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)$$

so that (30) holds. If instead $\overline{P}^{t'} \geq \overline{P}^{t'-1} > P_i$ then, by D3,

$$\begin{aligned} &\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right) + \left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right] \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'}\right) \quad \geq \\ &\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right) \end{aligned}$$

which implies

$$\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right) + \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'}\right) \geq \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)$$

so that (30) is true. We conclude that a constant price path of $\overline{P}^{t'}$ is IC in period $t'+1$. As far as $t > t'+1$, the ICC is as specified in (27) except that $\overline{X}^{t'}$ is replaced with a weakly higher level of damages. Since the rhs of (27) is decreasing in damages and the lhs is independent of them, the ICC holds. In summary, if pricing at $\overline{P}^{t'}$ is IC in t', where $\overline{P}^{t'}$ exceeds all past prices, then a constant price path of $\overline{P}^{t'}$ starting in period $t'+1$ is IC. This implies $V\left(\overline{P}^{t'}, \overline{X}^{t'}\right) \geq \pi\left(\overline{P}^{t'}\right)/(1-\delta)$ which gives us our desired contradiction. We have then show that if $\overline{P}^{t'} \geq \mathcal{P}^{t'-1}$ then $V\left(\overline{P}^{t'}, \overline{X}^{t'}\right) \geq \pi\left(\overline{P}^{t'}\right)/(1-\delta)$.

The next step is to show that, for all periods, a lower bound on the value function at the end of period t is $\pi\left(\mathcal{P}^t\right)/(1-\delta)$. The proof is by induction. Start with period t' and suppose that a lower bound on the value function is $\pi\left(\mathcal{P}^{t'}\right)/(1-\delta)$. Note that t' exists since

$$V\left(P^0, X^0\right) \geq \pi\left(P^0\right)/(1-\delta) = \pi\left(\mathcal{P}^0\right)/(1-\delta).$$

If $\overline{P}^{t'+1} \geq \mathcal{P}^{t'}$ then the result is immediate by the previous analysis. Next suppose $\overline{P}^{t'+1} < \mathcal{P}^{t'}$. By definition,

$$\begin{aligned} V\left(\overline{P}^{t'}, \overline{X}^{t'}\right) \quad = \quad & \pi\left(\overline{P}^{t'+1}\right) + \delta\phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)\left[\left(\widehat{\pi}/(1-\delta)\right) - \overline{X}^{t'+1} - F\right] \\ & +\delta\left[1 - \phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)\right] V\left(\overline{P}^{t'+1}, \overline{X}^{t'+1}\right). \end{aligned}$$

Since, by the inductive step, $V\left(\overline{P}^{t'}, \overline{X}^{t'}\right) \geq \pi\left(\mathcal{P}^{t'}\right)/(1-\delta)$ then

$$\begin{aligned} & \pi\left(\overline{P}^{t'+1}\right) + \delta\phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)\left[\left(\widehat{\pi}/(1-\delta)\right) - \overline{X}^{t'+1} - F\right] \\ & +\delta\left[1 - \phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)\right] V\left(\overline{P}^{t'+1}, \overline{X}^{t'+1}\right) \geq \pi\left(\mathcal{P}^{t'}\right)/(1-\delta). \end{aligned} \tag{31}$$

Given $\overline{P}^{t'+1} < \mathcal{P}^{t'}$ then $\pi\left(\overline{P}^{t'+1}\right) < \pi\left(\mathcal{P}^{t'}\right)$ (here we use the fact that the upper bound on the price space is P^m) which, using (31), implies

$$\begin{aligned}&\delta\phi\left(\overline{P}^{t'+1},\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\overline{X}^{t'+1}-F\right]\\&+\delta\left[1-\phi\left(\overline{P}^{t'+1},\overline{P}^{t'}\right)\right]V\left(\overline{P}^{t'+1},\overline{X}^{t'+1}\right)>\delta\pi\left(\mathcal{P}^{t'}\right)/\left(1-\delta\right).\end{aligned} \tag{32}$$

Given that

$$V\left(\overline{P}^{t'+1},\overline{X}^{t'+1}\right)\geq\left(\widehat{\pi}/\left(1-\delta\right)\right)-\overline{X}^{t'+1}-F$$

then (32) implies

$$V\left(\overline{P}^{t'+1},\overline{X}^{t'+1}\right)>\pi\left(\mathcal{P}^{t'}\right)/\left(1-\delta\right).$$

Since $\mathcal{P}^{t'+1}=\mathcal{P}^{t'}$ when $\overline{P}^{t'+1}<\mathcal{P}^{t'}$, we then have

$$V\left(\overline{P}^{t'+1},\overline{X}^{t'+1}\right)>\pi\left(\mathcal{P}^{t'+1}\right)/\left(1-\delta\right)$$

which is the desired result.

For an OSSPE path, $\pi\left(\mathcal{P}^t\right)/\left(1-\delta\right)$ is then a lower bound for $V\left(\overline{P}^t,\overline{X}^t\right)$. Since π is increasing in price (here we use the fact that the price path does not exceed P^m) and $\mathcal{P}^t$ is non-decreasing over time (being the maximum of all prices over the first t periods), this lower bound for the value function is a non-decreasing sequence. As it has an upper bound of $\pi\left(P^m\right)/\left(1-\delta\right)$, the sequence of lower bounds converges. Call $\overline{V}$ the value to which it converges.

Since $\mathcal{P}^t$ is non-decreasing and bounded, it converges and let $\mathcal{P}^\infty\equiv\lim_{t\to\infty}\mathcal{P}^t$. Thus, $\overline{V}=\pi\left(\mathcal{P}^\infty\right)/\left(1-\delta\right)$. An OSSPE price path is bounded from above by $\mathcal{P}^\infty$. If it does not converge to $\mathcal{P}^\infty$ then V^t is bounded below $\pi\left(\mathcal{P}^t\right)/\left(1-\delta\right)$ as $t\to\infty$ but this contradicts $\pi\left(\mathcal{P}^t\right)/\left(1-\delta\right)$ being a lower bound on the value function. Therefore, an OSSPE price path must converge to $\mathcal{P}^\infty$. For incentive compatibility to hold, it must then be true that

$$\lim_{t\to\infty}\left[\left(\pi\left(\mathcal{P}^t\right)/\left(1-\delta\right)\right)-\Lambda\left(\mathcal{P}^t\right)\right]\geq 0. \tag{33}$$

By the definition of P^* being the highest constant price path that is IC in the steady-state (that is, with damages equal to their steady-state value of $\gamma x\left(P^*\right)/\left(1-\beta\right)$), it follows from (33) that $\mathcal{P}^\infty\leq P^*$. The final step is to show $\mathcal{P}^\infty=P^*$.

If $\mathcal{P}^\infty<P^*$ then

$$\lim_{t\to\infty}\left[\left(\pi\left(\overline{P}^t\right)/\left(1-\delta\right)\right)-\Lambda\left(\overline{P}^t\right)\right]>0.$$

Recall that the cartel payoff is

$$\sum_{t=1}^{\infty} \delta^{t-1} \Pi_{j=1}^{t-1} \left[1-\phi\left(P^j, P^{j-1}\right)\right] \pi\left(P^t\right)$$
$$+\sum_{t=1}^{\infty} \delta^t \phi\left(P^t, P^{t-1}\right) \Pi_{j=1}^{t-1} \left[1-\phi\left(P^j, P^{j-1}\right)\right] \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta^t X^0 - \sum_{j=1}^{t} \beta^{t-j} \gamma x\left(P^j\right) - F\right].$$

Taking the derivative of it with respect to $P^{t'}$ and evaluating it at $P^{t'} = \overline{P}^{t'}$, if the ICC is not binding at t' then optimality requires that:

$$\pi'\left(\overline{P}^{t'}\right) + \delta\left(\frac{\partial\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)}{\partial P^t}\right)\left(\left(\frac{\widehat{\pi}}{1-\delta}\right) - \beta^{t'} X^0 - \sum_{j=1}^{t'} \beta^{t'-j} \gamma x\left(\overline{P}^j\right) - F\right) \tag{34}$$
$$+\delta^2\left[\left(\frac{\partial\phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)}{\partial P^{t-1}}\right)\left(1-\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right) - \phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)\left(\frac{\partial\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)}{\partial P^t}\right)\right] \times$$
$$\left[\left(\frac{\widehat{\pi}}{1-\delta}\right) - \beta^{t'+1} X^0 - \sum_{j=1}^{t'+1} \beta^{t'+1-j} \gamma x\left(\overline{P}^j\right) - F\right]$$

$$-\left[\left(\frac{\partial\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)}{\partial P^t}\right)\left(1-\phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)\right) + \left(1-\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right)\left(\frac{\partial\phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)}{\partial P^{t-1}}\right)\right] \times$$
$$\sum_{t=t'+2}^{\infty} \delta^{t-t'+1} \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right) \Pi_{j=t'+2}^{t-1}\left[1-\phi\left(\overline{P}^j, \overline{P}^{j-1}\right)\right]\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta^t X^0 - \sum_{j=1}^{t} \beta^{t-j} \gamma x\left(\overline{P}^j\right) - F\right]$$
$$-\sum_{t=t'}^{\infty} \delta^{t-t'+1} \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right) \Pi_{j=t'}^{t-1}\left[1-\phi\left(\overline{P}^j, \overline{P}^{j-1}\right)\right] \beta^{t-t'} \gamma x'\left(\overline{P}^{t'}\right) = 0.$$

As $t' \to \infty$, $\left(\overline{P}^{t'} - \overline{P}^{t'-1}\right) \to 0$ which implies, by D1-D2, that

$$\begin{aligned}
\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right) &\to 0, \phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right) \to 0 \\
\frac{\partial\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)}{\partial P^t} &\to 0, \frac{\partial\phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)}{\partial P^{t-1}} \to 0
\end{aligned}$$

Thus, (34) implies $\lim_{t'\to\infty} \pi'\left(\overline{P}^{t'}\right) = 0$. However, $P^* \leq P^m$ and, by supposition, $\lim_{t\to\infty} \overline{P}^t < P^*$ so that $\lim_{t'\to\infty} \pi'\left(\overline{P}^{t'}\right) > 0$. This contradiction proves that our original claim that $\mathcal{P}^\infty < P^*$ is false. I conclude that $\lim_{t\to\infty} \overline{P}^t = P^*$. ♦

7.2 Appendix B

The objective here is to provide sufficient conditions whereby infinite repetition of the single-period Nash equilibrium is a MPE and thereby show that A5 and D4 are not vacuous. The first case assumes the classic Bertrand price game: homogeneous products and constant marginal cost of c. The static Nash equilibrium then has firms price at cost: $\widehat{P} = c$. Also assume that the probability of detection is independent of the prices of firms with zero demand so that suspicions are triggered by changes in the prices that customers pay. Hence, the probability of detection depends only on the minimum price among firms, $\mathtt{P}^t \equiv \min \{P_1^t, \ldots, P_n^t\}$. Initially, suppose the state variable satisfies $\mathtt{P}^{t-1} \geq c$ and strategies are such that a firm prices at c. An individual firm that instead prices above c doesn't raise its current profit and doesn't impact the current probability of detection (since its demand is zero) nor its future payoff (since the state variables are left unaffected). Hence, it is not better off by pricing above c relative to pricing at c. Now suppose this firm instead prices below c. Relative to pricing at c, it lowers its current profit and weakly raises the probability of detection (by A4). I'll argue below that if the state variable is less than c then the future price path is bounded above by c and therefore the payoff is weakly less than $\widehat{\pi}/(1-\delta)$. Thus, if $\mathtt{P}^{t-1} \geq c$ then it is an equilibrium for all firms to price at c.

Now assume $\mathtt{P}^{t-1} < c$ and let us show that an equilibrium price path is bounded above by c. Consider a mixed-strategy equilibrium (which may be degenerate) in which the upper bound to the support of firm i's strategy is denoted $\overline{P}_i$. Suppose, contrary to the claim that the price path is bounded above by c, $\overline{P}_1 \geq \overline{P}_2 \geq \cdots \geq \overline{P}_n > c$ where the firms have been conveniently denumerated. First assume $\overline{P}_1 > \overline{P}_2$ and compare firm 1 pricing at $\overline{P}_1$ and at c in period t. Define $\mathtt{P}_{-i}^t \equiv \min \{P_1^t, \ldots, P_{i-1}^t, P_{i+1}^t, \ldots, P_n^t\}$. If $c \geq \mathtt{P}_{-1}^t$ then both $\overline{P}_1$ and c give firm 1 the same payoff as they generate zero current demand and the state variable is the same. If $\mathtt{P}_{-1}^t > c$ then, compared to pricing at $\overline{P}_1$, pricing at c yields the same profit stream - zero current profit and zero future profit since the future price path is c - but the probability of detection is lower today - compare $\phi\left(c, \mathtt{P}^{t-1}\right)$ and $\phi\left(\mathtt{P}_{-1}^t, \mathtt{P}^{t-1}\right)$ - and tomorrow - compare $\phi(c,c)$ and $\phi\left(c, \mathtt{P}_{-1}^t\right)$ - and is the same thereafter, at the level $\phi(c,c)$. As a price of c yields a strictly higher payoff than $\overline{P}_1$, this contradicts $\overline{P}_1$ being in the support of an equilibrium strategy from which I conclude $\overline{P}_1 = \overline{P}_2$. By the same argument, one can show that it is inconsistent with equilibrium for $\overline{P}_1 = \overline{P}_2 > \overline{P}_3$. It follows that all firms must have the same upper bound to their support, which I'll denote $\overline{P}$ and, by supposition, $\overline{P} > c$. If there is a mass point at $\overline{P}$ then a firm can do better by pricing marginally below $\overline{P}$. Doing so discretely increases current profit in the event that all other firms price at $\overline{P}$ and has only a marginal effect on the probability of detection and on the future payoff (where the presumed continuity in the last statement holds because the future price path is c). Now suppose there is no mass point at $\overline{P}$. Compare firm i pricing at $\overline{P}$ with pricing at $\overline{P}-\varepsilon > c$ where $\varepsilon > 0$ and $\overline{P}-\varepsilon$ receives positive density but is not a mass point. If $\mathtt{P}_{-i}^{t} < \overline{P}-\varepsilon$ then firm i's payoff is the same whether it prices at $\overline{P}-\varepsilon$ or $\overline{P}$. If instead $\mathtt{P}_{-i}^{t} > \overline{P}-\varepsilon$ then firm i's current profit is higher - compare $n\pi\left(\overline{P}-\varepsilon\right)$ and zero - and the current probability of detection is lower - compare $\phi\left(\overline{P}-\varepsilon, \mathtt{P}^{t-1}\right)$ and $\phi\left(\mathtt{P}_{-i}^{t}, \mathtt{P}^{t-1}\right)$. Furthermore, with either price, the future price path has all firms pricing at c so the future profit stream is the same. Finally, the probability of detection is lower with $\overline{P}-\varepsilon$ in the next period (and the same in all periods thereafter) since it is $\phi\left(c, \overline{P}-\varepsilon\right)$ compared to $\phi\left(c, \mathtt{P}_{-i}^{t}\right)$. This establishes that pricing at $\overline{P}-\varepsilon$ yields a strictly higher payoff than $\overline{P}$ and allows one to infer that the supposition that $\overline{P} > c$ is false. I then conclude that if $\mathtt{P}^{t-1} < c$ then the future equilibrium price path is bounded above by c which implies the future payoff is bounded above by $\widehat{\pi}/\left(1-\delta\right)$

The final step is to characterize an equilibrium for when $\mathtt{P}^{t-1} < c$. The problem here is that there is generally no symmetric pure-strategy equilibrium. For suppose there was and it entailed all firms pricing at $P' \in \left(\mathtt{P}^{t-1}, c\right)$ in period t. Each of these firms would instead prefer to price above P' as it raises their current profit from being negative to zero without altering the current probability of detection or the future price path (since both depend only on the minimum price that is charged). Fortunately, there does exist an asymmetric pure-strategy equilibrium in which one of the n firms takes the burden of gradually raising price when it lies below c. By "taking the burden," I mean that the other firms price higher and thus realize zero profit along this path. First define the optimal price path for such a firm:

$$\left\{\rho^{\tau}\left(P,X\right)\right\}_{\tau=1}^{\infty} \in \arg\max \sum_{\tau=1}^{\infty} \delta^{\tau-1} \Pi_{j=1}^{\tau-1}\left[1-\phi\left(p^{j}, p^{j-1}\right)\right] n\pi\left(p^{\tau}\right)$$
$$+\sum_{\tau=1}^{\infty} \delta^{\tau}\phi\left(p^{\tau}, p^{\tau-1}\right) \Pi_{j=1}^{t-1}\left[1-\phi\left(p^{j}, p^{j-1}\right)\right]\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta^{\tau}X-F\right]$$

subject to $p^{0} = \mathtt{P}^{t-1}$. Note that $\left\{\rho^{\tau}\left(P,X\right)\right\}_{\tau=1}^{\infty}$ exists. Consider the following asymmetric strategy profile. In period t, firm 1 prices at $\rho^{1}\left(P,X\right)$ and the other firms price at c. In period $t+\tau-1$, for $\tau \geq 2$, if $\mathtt{P}^{t+\tau-2} < c$ then the firm that charged $\mathtt{P}^{t+\tau-2}$ in $t+\tau-2$ prices at $\rho^{\tau}\left(P,X\right)$ and the other firms price at c.[25] If $\mathtt{P}^{t+\tau-2} \geq c$ then, according to the previous argument, firms price at c.

Let me show that this is a MPE. Suppose the previous period's minimum price is less than c and the history up to $t+\tau-1$ is such that firm i is to price at $\rho^{\tau}\left(P,X\right)$ and the

[25] If more than one firm charged $\mathtt{P}^{t+\tau-2}$ in $t+\tau-2$ then randomly select one of them to charge $\rho^{\tau}\left(P,X\right)$ and have all other firms price at c.

other firms are to price at c. Given the other firms' strategies, firm i's strategy is optimal given it prices at or below c. Now consider it pricing above c. The payoff is the same as pricing at c as, in both cases, current profit is zero, the current probability of detection is the same (as the change in the minimum price is unaffected), and the future price path is c. Thus, firm i's strategy is optimal. To consider firm $j \neq i$, first note that firm i's payoff is lower than that of the other firms. All suffer from the same expected present value of damages but firm i incurs losses by pricing below c. Also if firm j prices different from c but still above $\rho^{\tau}(P, X)$, the future outcome path is unaffected and thus so is the firm's payoff. Now consider firm j pricing below $\rho^{\tau}(P, X)$. In that case, firm j's problem is the same as firm i's problem as now firm j has set the minimum price and, according to the strategy profile, it is the one to price according to $\{\rho^{\tau}(P, X)\}_{\tau=1}^{\infty}$ in the future. Hence, conditional on pricing below $\rho^{\tau}(P, X)$, firm j's payoff is the same as firm i's payoff which we know is lower than that received from pricing at c.[26] Thus, it is optimal for firm j to price at c. This establishes that the asymmetric strategy profile is a Markov Perfect Equilibrium. This completes the proof that infinite repetition of the single-period Nash equilibrium is a MPE for the Bertrand price game.

The second case in which infinite repetition of the single-period Nash equilibrium is a MPE is when the probability of detection (when the cartel is inactive) is independent of an individual firm's price when that price is different from a common price charged by other firms. For example, this occurs when $n \geq 3$ and the probability of detection depends on the change in the median price or on the change in the average transaction price after discarding the greatest outlier. It is also the case when the probability of detection is fixed and thereby independent of all firms' prices. Suppose this assumption holds and consider a symmetric MPE path. Given all other firms price at some common level P, an individual firm's price does not influence the current probability of detection nor the next period's probability of detection. Thus, the future path and payoff are unaffected by its price. All this argues to a firm's optimal price being that which maximizes current profit. By symmetry, this implies that all firms charge a price of $\widehat{P}$.

7.3 Appendix C

The task is to provide some examples of functions for which B1 holds.

B1 If $P' \geq P$ and $P' > P^o$ then $\phi(P', P) - \phi((P', \ldots, P^o, \ldots, P'), P)$ is non-increasing

[26] If firm j's price equals $\rho^{\tau}(P, X)$ then its payoff is the average of firm i's payoff and firm j's payoff from pricing at c which, by the previous argument, is lower than firm j's payoff from pricing at c.

in P.

Suppose the probability of detection function is additively separable in the individual price changes where the weights can depend on the current price vector:

$$\phi\left(\underline{P}^t, \underline{P}^{t-1}\right) = \sum_{j=1}^{n} \omega_j\left(\underline{P}^t\right) \widetilde{\phi}\left(P_j^t - P_j^{t-1}\right).$$

Assume: i) $\omega_j : \Omega \rightarrow [0,1]$ and $\sum_{j=1}^{n} \omega_j\left(\underline{P}^t\right) = 1$; and ii) $\widetilde{\phi}$ is differentiable, $\widetilde{\phi}'(\varepsilon) \geq (\leq) 0$ when $\varepsilon \geq (\leq) 0$, and $\widetilde{\phi}''(\varepsilon) \geq 0$ when $\varepsilon \geq 0$. Thus, when the price change is negative, the probability of detection is non-increasing in the price change and, when the price change is positive, it is a weakly convex non-decreasing function of the price change. Given this class of functions, B1 takes the particular form,

$$\widetilde{\phi}(P' - P) - \theta\widetilde{\phi}(P' - P) - (1-\theta)\widetilde{\phi}(P^o - P) \text{ is non-increasing in } P,$$

where $\theta \equiv \sum_{j \neq i}^{n} \omega_j(P', \ldots, P^o, \ldots, P')$. Taking the first derivative with respect to P, the condition is

$$\begin{aligned} -\widetilde{\phi}'(P' - P) + \theta\widetilde{\phi}'(P' - P) + (1-\theta)\widetilde{\phi}'(P^o - P) &\leq 0 \Leftrightarrow \\ \widetilde{\phi}'(P^o - P) &\leq \widetilde{\phi}'(P' - P). \end{aligned}$$

If $P' > P^o > P$ then this condition holds since $\widetilde{\phi}$ is weakly convex in price increases. If $P' \geq P > P^o$ then $\widetilde{\phi}'(P^o - P) \leq 0 \leq \widetilde{\phi}'(P' - P)$ and thus it holds.

Turning to a second class of cases, suppose the probability of detection depends only on the movement in a summary statistic for the vector of prices. Define $f : \Omega^n \rightarrow \Omega$ to be this summary statistic and assume: i) $f(P, \ldots, P) = P$; and ii) if $P^o \leq P$ then $f(P, \ldots, P^o, \ldots, P) \leq P$. Next suppose that $\phi\left(\underline{P}^t, \underline{P}^{t-1}\right) = \widehat{\phi}\left(f\left(\underline{P}^t\right) - f\left(\underline{P}^{t-1}\right)\right)$, where $\widehat{\phi} : \Re \rightarrow [0,1]$ is differentiable and is non-decreasing and convex when $f\left(\underline{P}^t\right) \geq f\left(\underline{P}^{t-1}\right)$ and is non-increasing when $f\left(\underline{P}^t\right) \leq f\left(\underline{P}^{t-1}\right)$. Note that examples of f include the mean and median price and the minimum price. This last case is natural when products are homogeneous and detection is based on the price that customers pay. Defining $P'' \equiv f(P', \ldots, P^o, \ldots, P')$, B1 holds iff $\widehat{\phi}'(P' - P) \geq \widehat{\phi}'(P'' - P)$. If $P' > P'' > P$ then this condition is satisfied by convexity. If $P' > P > P''$ then this condition is satisfied because $\widehat{\phi}'(P' - P) \geq 0 \geq \widehat{\phi}'(P'' - P)$.

7.4 Appendix D

Lemma 7 *Assume A1-A2; i) $D(\cdot)$ is twice continuously differentiable, $D'(P) \leq 0$ and if $D(P) > 0$ then $D'(P) < 0$ and $D''(P) \leq 0$; ii) $\pi(P) = (P - c)D(P)$; iii) $x(P) =$*

$\max\left\{\left(P-\widehat{P}\right)D(P),0\right\}$; *and iv)* $\widehat{P}\geq c$. *Then* $\pi(\xi(d))$ *is strictly concave in* d $\forall d\in\left[\gamma x\left(\widehat{P}\right),\gamma x(P^m)\right)$.

Proof: First take the total derivative of $d=\gamma\left[\xi(d)-\widehat{P}\right]D(\xi(x))$ with respect to d:

$$\begin{aligned}1 &= \xi'(d)\gamma D(\xi(d))+\left[\xi(d)-\widehat{P}\right]\gamma D'(\xi(d))\xi'(x)\Leftrightarrow \\ \xi'(d) &= (1/\gamma)\left[D(\xi(d))+\left(\xi(d)-\widehat{P}\right)D'(\xi(d))\right]^{-1}>0.\end{aligned}\tag{35}$$

The second derivative is:

$$\begin{aligned}\xi''(d) &= -\frac{\left[2D'(\xi(d))+\left(\xi(d)-\widehat{P}\right)D''(\xi(d))\right]\xi'(d)}{\gamma\left[D(\xi(d))+\left(\xi(d)-\widehat{P}\right)D'(\xi(d))\right]^2} \\ &= -\gamma\left[2D'(\xi(d))+\left(\xi(d)-\widehat{P}\right)D''(\xi(d))\right]\left(\xi'(d)\right)^3>0.\end{aligned}\tag{36}$$

Taking the first two derivatives of $\pi(\xi(d))=[\xi(d)-c]D(\xi(d))$ with respect to d :

$$\frac{d\pi(\xi(d))}{dd}=[D(\xi(d))+(\xi(d)-c)D'(\xi(d))]\xi'(d)$$

$$\begin{aligned}\frac{d^2\pi(\xi(d))}{dd^2} &= [2D'(\xi(d))+(\xi(d)-c)D''(\xi(d))]\left(\xi'(d)\right)^2 \\ &\quad +[D(\xi(d))+(\xi(d)-c)D'(\xi(d))]\xi''(d).\end{aligned}$$

Substituting (35)-(36):

$$\begin{aligned}\frac{d^2\pi(\xi(d))}{dd^2} &= [2D'(\xi(d))+(\xi(d)-c)D''(\xi(d))]\left(\xi'(d)\right)^2 \\ &\quad -[D(\xi(d))+(\xi(d)-c)D'(\xi(d))]\times \\ &\quad \gamma\left[2D'(\xi(d))+\left(\xi(d)-\widehat{P}\right)D''(\xi(d))\right]\left(\xi'(d)\right)^3 \\ &= \left(\xi'(d)\right)^2\{[2D'(\xi(d))+(\xi(d)-c)D''(\xi(d))] \\ &\quad -[D(\xi(d))+(\xi(d)-c)D'(\xi(d))]\left[2D'(\xi(d))+\left(\xi(d)-\widehat{P}\right)D''(\xi(d))\right]\times \\ &\quad \left[1/\left(D(\xi(d))+\left(\xi(d)-\widehat{P}\right)D'(\xi(d))\right)\right]\}.\end{aligned}$$

Since $\xi(d)<P^m$ and $\widehat{P}\geq c$, A3 implies $D(\xi(d))+\left(\xi(d)-\widehat{P}\right)D'(\xi(d))>0$. The sign of $d^2\pi(\xi(d))/dd^2$ doesn't then change if we multiply through by $D(\xi(d))+\left(\xi(d)-\widehat{P}\right)D'(\xi(d))$. Doing so, $d^2\pi(\xi(d))/dd^2\leq 0$ iff $\chi\left(\widehat{P}\right)\leq 0$ where

$$\begin{aligned}\chi(P) &\equiv [D(\xi(d))+(\xi(d)-P)D'(\xi(d))][2D'(\xi(d))+(\xi(d)-c)D''(\xi(d))] \\ &\quad -[D(\xi(d))+(\xi(d)-c)D'(\xi(d))][2D'(\xi(d))+(\xi(d)-P)D''(\xi(d))].\end{aligned}$$

Next note that

$$\begin{aligned}\chi'(P) = \quad & [D(\xi(d)) + (\xi(d) - c) D'(\xi(d))] D''(\xi(d)) \\ & - D'(\xi(d)) [2D'(\xi(d)) + (\xi(d) - c) D''(\xi(d))] \quad < 0.\end{aligned}$$

Given that $\chi(c) = 0$ and $\widehat{P} \geq c$ then $\chi\left(\widehat{P}\right) \leq 0$. ♦

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