Small Noise Asymptotics for a Stochastic Growth Model

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In this paper we develop analytic asymptotic methods to characterize time series properties of nonlinear dynamic stochastic models. We focus on a stochastic growth model which is representative of the models underlying much of modern macroeconomics. Taking limits as the stochastic shocks become small, we derive a functional central limit theorem, a large deviation principle, and a moderate deviation principle. These allow us to calculate analytically the asymptotic distribution of the capital stock, and to obtain bounds on the probability that the log of the capital stock will differ from its deterministic steady state level by a given amount. This latter result can be applied to characterize the probability and frequency of large business cycles. We then illustrate our theoretical results through some simulations. We find that our results do a good job of characterizing the model economy, both in terms of its average behavior and its occasional large cyclical fluctuations.

1. INTRODUCTION

Modern macroeconomics is built on the foundation of nonlinear dynamic stochastic general equilibrium (DSGE) models. In particular, the stochastic growth model is one of the most widely used models in all of economics, and is the standard model for business cycle analysis. However because the model can only be solved in closed form under very restrictive assumptions (such as log utility and full depreciation of capital), analysis of the model must resort to approximations.[1] For example, a standard practice is to linearize the Euler equations which characterize the optimal solution around the deterministic steady state (or balanced growth path).[2] In many cases, there is little discussion of the quality of such approximations, particularly for the stochastic properties of the economy.[3] In this paper we provide some steps in this direction. We provide analytic asymptotic results which characterize the average behavior of the stochastic growth model and its occasional large fluctuations.

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[1] A summary of a variety of numerical methods can be found in Judd (1998).

[2] Papers which use linearizations are too numerous to list, but some notable applications in contexts similar to this paper include Magill (1977), Kydland and Prescott (1982), King, Plosser, and Rebelo (1988), and Cambpell (1994).

[3] An exception is the literature on perturbation methods, discussed in more detail below. However this literature has tended to focus on the analytic properties of decision rules instead of the stochastic properties of the resulting model economy.

In our analysis we consider the limits as the standard deviation σ of the stochastic technology shocks converges to zero. We show that the capital accumulation trajectories converge to the corresponding trajectories from a deterministic model. The limiting deterministic models are typically easier to analyze, particularly in the neighborhood of a steady state. The results provide analytic, theoretically justified approximations for stochastic models with small noise. There is less, if any, need for numerical methods and simulation. Further, the analytic expressions we obtain are useful for comparative statics and comparative dynamics, and can potentially be used as a means for estimation. Interestingly, we find that many asymptotic properties of the economy can be described by a linear approximation. However for larger fluctuations, and to describe potential asymmetries in the time series, nonlinear methods are needed. While we focus on a relatively simple and standard model, the methods we develop can be applied to more general nonlinear DSGE models which may provide a closer match to the data.

In our analysis below, we obtain three different characterizations of the rate at which the stochastic model converges to the deterministic one. We first formulate a functional central limit theorem which shows that at rate σ, the centered capital trajectories are asymptotically normal. While this result holds for both for the level of capital and its logarithm, for our other results we consider solely the log of the capital stock. We then apply a large deviation principle, which shows that the absolute differences of the log capital trajectories converge to zero exponentially fast. We also obtain estimates of the average time it takes the log capital stock to differ from its steady state level by a given amount. Finally, we present a moderate deviation principle which provides similar results, but for an intermediate range of asymptotics between the functional central limit theorem and the large deviation principle.

Both the functional central limit theorem and the moderate deviation principle are expressed in terms of a linear approximation to the deterministic model. These results thus suggest that in order to consider the average behavior of the economy and even to consider some "extreme" events, a linear approximation is sufficient. However the linear approximation does not fully capture the large deviation properties, and abstracts from certain asymmetries in the model due to nonlinearities.[4] By applying the large deviation results, we show both analytically and numerically that the model economy is slightly asymmetric and is slightly more likely to exhibit recessions (appropriately defined) than booms.

This appears to be the first paper to apply this type of convergence results for the stochastic growth model. However there are several of related papers in the literature. In a continuous time setting, Prandini (1994) used the large deviation results of Azencott (1980) to analyze a stochastic Solow growth model. He showed that the capital trajectories in the stochastic model converges uniformly on a finite time horizon to the corresponding trajectories from the standard deterministic model. We obtain similar results in our section on large deviations. However, unlike Prandini, we explic-

[4]There are higher order asymptotic results for continuous time diffusions in the literature, such as Fleming and Souganidis (1986) and Fleming and James (1992), which may capture some of these nonlinearities in an analytically tractable way. However to our knowledge there are no such results for our discrete time setting.

itly calculate the rate function and analyze the exit problem, which provides useful information in applications. Additionally, our analysis is complicated by the fact that there is explicit optimization in our model which requires us to prove the convergence of consumption policies. Furthermore, we cast our analysis in discrete time which is more relevant for many macroeconomic applications, and as we discussed, we obtain a much wider range of asymptotic results.

As we noted above, there is a closely related and extensive branch of the literature which focuses on the functional properties of approximate solutions. However the functional approximation results do not characterize the stochastic properties of the model economy, which is our focus. In the mathematics literature, key papers include Fleming (1971) and Fleming and Souganidis (1986) who prove the uniform convergence of the policy function in continuous time models when the noise goes to zero. In the economics literature, an early contribution was Magill (1977) who derived an asymptotic linear-quadratic approximation in a continuous time stochastic growth model. Judd (1998) contains a comprehensive overview of these perturbation methods and their applications in economics. Judd and Guu (1993, 1997) present numerical methods based on Taylor expansions to analyze stochastic and deterministic growth models respectively. Gaspar and Judd (1997) provide high order expansions for multivariate models. Most of these papers focus on the local analytic properties of the solution, with a few providing global numerical results. But again, none of these touch on the issues we analyze.

2. THE MODEL

In this section we lay out the benchmark model for the analysis. It is a specialized Brock-Mirman (1972) economy with production, capital accumulation, and stochastic productivity growth. To simplify the presentation and analysis, we assume that technology shocks are permanent, which implies that the model has a single state variable. This assumption can be relaxed. Further, in the development of the paper it should be evident that our results have applications to more general nonlinear stochastic models.

2.1. The Stochastic Growth Model

We assume that output is produced according to a standard constant returns to scale Cobb-Douglas production function with parameter α:

$$F(K, L) = K^{\alpha}(AL)^{1-\alpha},$$

where K is the capital stock, L is the labor supply and A is the labor-augmenting technology parameter. For simplicity, we fix the total labor supply at $L = 1$. We assume that A evolves exogenously as a unit root process in logarithms:

$$\log A_{t+1} = \kappa + \log A_t + \sigma W_{t+1} \tag{1}$$

where W is a standard normal random variable and $\kappa \geq 0$ is the mean rate of technology growth. The unit root assumption is made for simplicity, and is roughly consistent with US time series data. Let δ be the depreciation rate of capital, and C_t be consumption. Then the evolution equation for capital is given by:

$$K_{t+1} = A_t^{1-\alpha} K_t^{\alpha} - C_t + (1-\delta)K_t. \tag{2}$$

Although the technological process is nonstationary, the ratios of capital to technology, $k_t = K_t/A_t$, and that of consumption to technology, $c_t = C_t/A_t$, are stationary. We therefore represent the problem in terms of the stationary variables. Normalizing by the technology level, (2) becomes:

$$k_{t+1} = \theta Z_{t+1}^{\sigma} \left(k_t^{\alpha} - c_t + (1-\delta)k_t\right) \tag{3}$$

where we define the lognormal random variable Z^{σ} and constant θ as:

$$\begin{aligned} Z_{t+1}^{\sigma} &= \exp(-\sigma W_{t+1}), \\ \theta &= \exp(-\kappa). \end{aligned}$$

A representative agent has time-additively separable preferences over consumption, with CRRA period utility:

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma} = A^{1-\gamma}\frac{c^{1-\gamma}}{1-\gamma}.$$

The social planner's problem is to choose a consumption sequence to maximize the expected discounted utility of the representative agent. Thus we solve:

$$\sup_{\{C_t\}} E \sum_{t=0}^{\infty} \beta^t U(C_t) \tag{4}$$

subject to (2) and (1). Further, expressing utility in terms of c makes the effective subjective discount factor $\beta(\theta Z_{t+1}^{\sigma})^{\gamma-1}$, and thus introduces a form of preference shocks. Straightforward calculations, detailed in Appendix A, show that this Markov optimization problem has a solution which is a feedback control of the form: $c_t = c^{\sigma}(k_t)$. This implies that we can write the optimal capital evolution as:

$$\begin{aligned} k_{t+1} &= \theta Z_{t+1}^{\sigma} \left(k_t^{\alpha} - c^{\sigma}(k_t) + (1-\delta)k_t\right) \\ &\equiv \overline{f}^{\sigma}(Z_{t+1}^{\sigma}, k_t). \end{aligned} \tag{5}$$

The consumption policy function satisfies the stochastic Euler equation, derived in Appendix A:

$$c^{\sigma}(k)^{-\gamma} = \beta \int (\theta Z^{\sigma})^{\gamma} c^{\sigma}\left(\overline{f}^{\sigma}(Z^{\sigma},k)\right)^{-\gamma} \left[\alpha \overline{f}^{\sigma}(Z^{\sigma},k)^{\alpha-1} + 1 - \delta\right] dG^{\sigma}(Z^{\sigma}) \tag{6}$$

where G^{σ} is the relevant lognormal distribution function.

In our analysis to follow, it helps to split the capital evolution into its conditional expectation and its martingale component. Therefore we define the expectation of the right side of (5), conditioned on $k_t = k$ as:

$$f^\sigma(k) = \theta \exp\left(\frac{\sigma^2}{2}\right)(k^\alpha - c^\sigma(k) + (1-\delta)k), \tag{7}$$

and the random component is then:

$$\nu^\sigma(k) = \theta \exp\left(-\sigma W_{t+1} - \frac{\sigma^2}{2}\right)(k^\alpha - c^\sigma(k) + (1-\delta)k). \tag{8}$$

Thus we have an alternate expression for the capital evolution:

$$k_{t+1} = f^\sigma(k_t) + \nu^\sigma(k_t). \tag{9}$$

While the optimization problem is most naturally stated in terms of the level of the capital stock, for most of our analysis we will work with the logarithm of the capital stock. The multiplicative nature of the noise term in (5) makes the logs particularly easy to work with. Thus if we let $l_t = \log(k_t)$, we can take logs and rewrite:

$$l_{t+1} = g^\sigma(l_t) - \sigma W_{t+1}, \tag{10}$$

where we define:

$$g^\sigma(l) = \log\left[\exp(\alpha l) - c^\sigma(\exp(l)) + (1-\delta)\exp(l)\right] - \kappa. \tag{11}$$

The conditional normality of l_t greatly simplifies many of the results that follow.

2.2. A Deterministic Growth Model

Corresponding to the stochastic growth model, we can define a deterministic growth model by setting $\sigma = 0$ in the equations above. This yields a discrete time version of a standard Ramsey-Cass-Koopmans model in which the technology grows at a constant rate. The implied evolution for capital is therefore:

$$k_{t+1} = \theta[k_t^\alpha - c_t + (1-\delta)k_t]. \tag{12}$$

The optimal growth problem is then the deterministic analogue of (4), with law of motion (12). Again, straightforward calculations show that this problem has a feedback solution of the form $c_t = c^0(k_t)$. This implies that capital and log capital follow the deterministic difference equations:

$$k_{t+1} = f^0(k_t), \tag{13}$$

$$l_{t+1} = g^0(l_t), \tag{14}$$

where f^0 and g^0 are obtained by replacing c^σ with c^0 in (7) and (11), respectively. The optimal consumption policy satisfies the Euler equation analogous to (6):

$$c^0(k)^{-\gamma} = \beta\theta^\gamma c^0(f^0(k))^{-\gamma}\left[\alpha f^0(k)^{\alpha-1} + 1 - \delta\right]. \tag{15}$$

2.3. Comparisons of the Models

Let $\{k_t^\sigma\}$ be a realization of the capital stock trajectory from the stochastic growth model, $\{k_t^0\}$ be the capital stock trajectory from the deterministic model, and $\{l_t^\sigma\}$ and $\{l_t^0\}$ their respective logarithms. As $\sigma \to 0$ we expect that, starting from the same initial value, the sample paths of the solution of the stochastic growth model would approach that of the deterministic model. In the rest of the paper we show that this is indeed the case, and we obtain explicit characterizations of the asymptotics at different rates of convergence. These results provide us with approximate characterizations of the stochastic economy when the noise is small.

Our results consider the normalized differences, for $\rho \geq 0$:

$$X_t^{\sigma,\rho} = \frac{1}{\sigma^{1-\rho}}(l_t^\sigma - l_t^0). \tag{16}$$

In Section 3, we consider the case $\rho = 0$, and present a functional central limit theorem (FCLT). We show that at rate σ, the normalized differences converge to a Gaussian linear autoregressive process. In Section 4, we consider the case $\rho = 1$, and formulate a large deviation principle (LDP). Thus we obtain bounds on the probability that (on a given finite horizon) the stochastic capital trajectory differs from the deterministic one by a given amount. We also consider the exit problem, which provides estimates of how long it typically takes for the log capital stock to depart from its steady state level by a given amount. In Section 5, we consider a range of cases where $0 < \rho < 1$, and apply a so-called moderate deviation principle (MDP). These results are similar to the large deviation principle, but for σ-dependent neighborhoods. We also analyze the exit problem in this case. In Section 6, we illustrate our results through some explicit calculations and simulations in a calibrated model. Our results show that the theoretical predictions provide a good explanation of the behavior of the model as observed in simulations. We also illustrate some of the differences between the large deviation and moderate deviation results, which have implications for linear solution methods. Finally, Section 7 concludes. Throughout we make smoothness assumptions on the consumption policies, and we also assume that the stochastic policies converge to the deterministic one. In Appendix A we formally establish these results. Appendices B and C collect proofs of some of the results in the text.

3. A FUNCTIONAL CENTRAL LIMIT THEOREM

In this section we begin our analysis of the stochastic growth model. We present a functional central limit, which follows Klebaner and Nerman (1994), and shows that the normalized differences $X_t^{\sigma,0}$ from (16) converge to a Gaussian linear autoregression. We also show that a similar result holds for the capital levels. As noted above, in order to make clear which features of the model are required for each result we make direct assumptions on the policy functions. These assumptions are verified in our application in Appendix A. To cover both logs and levels, in the assumptions we use $h(x)$ as a stand-in for either $g(l)$ or $f(k)$, and we always use a subscript x for a derivative of a function. Klebaner and Nerman (1994) impose boundedness conditions on the derivatives of policy functions which cause some slight complications in our analysis. Due to the Inada condition on the utility function, the slopes of the policy functions (in levels) increase to infinity at zero capital. However on any compact set bounded away from zero, the boundedness conditions are satisfied. Therefore we extend the results by truncating the state evolution to a compact set, but relaxing the truncation in the limit. In what follows, we require $x \in \mathcal{X}$ with $\mathcal{X}$ a compact set. This notation is a stand in for $l \in \mathcal{L} \subset \mathbb{R}$ and $k \in \mathcal{K} \subset \mathbb{R}_{++}$ with $\mathcal{L}$ and $\mathcal{K}$ compact.

ASSUMPTION 3.1. *On any compact set $\mathcal{X}$, the function h^σ is continuous, twice continuously differentiable, and has bounded derivatives h_x^σ and h_{xx}^σ for all $\sigma \geq 0$.*

ASSUMPTION 3.2. *On any compact set $\mathcal{X}$, $h^\sigma \to h^0$ uniformly as $\sigma \to 0$.*

ASSUMPTION 3.3. *On a compact set $\mathcal{X}$, h^0 has a unique fixed point x^* which is stable, i.e. $|h_x^0(x^*)| < 1$, and whose domain of attraction includes all of $\mathcal{X}$.*

Then we have the following result on the convergence of the log capital stock.

THEOREM 3.1. *Suppose that Assumptions 3.1 and 3.2 hold for $h = g$. Then as $\sigma \to 0$, the normalized differences $\{X_t^{\sigma,0}\}$ converge weakly to a process $\{X_t\}$. The limit process follows the linear Gaussian autoregression, dependent on the deterministic process $\{l_t^0\}$:*

$$X_{t+1} = g_x^0(l_t^0) X_t + B_{t+1}, \tag{17}$$

where $\{B_{t+1}\}$ is an i.i.d. sequence of standard normal random variables.

Proof. See Appendix B. ∎

Since there is a unique steady state of the deterministic model, this result immediately implies that the differences from the steady state are asymptotically normal. As this result has implications for approximate solution methods, we find it useful to state it in the following.

COROLLARY 3.1. *Suppose that Assumptions 3.1-3.3 hold for $h = g$. Then as $\sigma \to 0$ and $t \to \infty$ the following holds asymptotically:*

$$l_{t+1}^\sigma - l^* = g_x^0(l^*)(l_t^\sigma - l^*) + \sigma B_{t+1}.$$

Therefore, as $\sigma \to 0$ and $t \to \infty$, $\{l_t^\sigma\}$ converges to a stationary Gaussian process with mean l^* and variance $\frac{\sigma^2}{1 - g_x^0(l^*)^2}$.

This result shows that the log capital stock asymptotically follows a Gaussian linear autoregression centered on the deterministic steady state. This has some interesting implications in practice. One simple and immediate application deals with the calibration of models. Typically, models are calibrated to match the first moments of key time series. However Corollary 3.1 allows the use of analytic asymptotic second moments as well. In particular, it provides a means of calibrating the risk aversion parameter (γ) by using the (asymptotic) autocorrelation of the capital stock, which equals the derivative of the policy function at the steady state. Notice that this calibration is completely analytic, and does not require simulation of the model. We use this result to calibrate the model in Section 6 below.

For the case of the levels, we substitute k_t for l_t in our definition of $X_t^{\sigma,0}$. We then have the following preliminary result.

LEMMA 3.1. *Let Assumptions 3.1 and 3.2 hold for $h = f$. Then for each fixed k, we have:*

$$\lim_{\sigma \to 0} \frac{1}{\sigma^2} E\nu^\sigma(k)^2 = f^0(k)^2.$$

Proof. This is a simple calculation, using the definitions of the functions and L'Hopital's rule. ∎

Then, making the appropriate substitutions, we have the following results for the capital stock levels.

COROLLARY 3.2. *Suppose that Assumptions 3.1 and 3.2 hold for $h = f$. Then as $\sigma \to 0$, the normalized differences converge weakly to the process:*

$$X_{t+1} = f_x^0(k_t^0) X_t + f^0(k_t^0) B_{t+1}.$$

Suppose in addition that Assumption 3.3 holds for $h = f$. Then the results of Corollary 3.1 hold with $\sigma f^0(k^*)$ in place of σ and the other obvious substitutions.

4. A LARGE DEVIATION PRINCIPLE

In the previous section we showed that the log capital stock (and the level) from the stochastic growth model converges to the deterministic capital stock as the technology shock standard deviation σ goes to zero. Further, the differences converge at rate σ and are asymptotically normal. In this section we provide further analysis by considering the asymptotics of the differences, without normalizing by σ. In other words, we study the process $\{X_t^{\sigma,1}\}$ from (16). We provide a large deviation principle which shows that, on a given finite horizon, the differences converge to zero exponentially fast and establishes the exponential convergence rate. We then analyze events in which the log capital stock differs from its steady state value by a given (positive, finite) amount. We show that the time period between such events increases exponentially, and we provide an estimate of the typical length. We also show that the curvature of the policy function determines whether large increases or decreases are more likely. These results have implications for the timing of business cycles. Economic booms and recessions are usually defined in terms of output and not the capital stock directly. However, we show below that large increases in the log capital stock typically correspond to troughs of economic activity while large booms are associated with large falls in the log capital stock. Therefore our results allow us to characterize the average time between large business cycles, and to determine whether booms or recessions occur more often.

As noted in the introduction, Prandini (1994) derived similar results in a continuous time setting without optimization. He applied Azencott's (1980) extension of the work of Freidlin and Wentzell (1984). Here our development is in discrete time, and draws on the results of Klebaner and Zeitouni (1994). We first develop a large deviation principle for the one-step Markov transitions of the log capital stock, using the well-known Gärtner-Ellis Theorem (see Dembo and Zeitouni, 1998). Then we essentially sum up each of the transitions to determine the large deviation principle for the entire discrete time paths.

We now formally define some terminology. Let a sequence $\{Z^\epsilon\}$ be defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a complete separable metric space $\mathcal{X}$. A *rate function* $S : \mathcal{X} \to [0, \infty]$ has the property that for any $M < \infty$ the level set $\{x \in \mathcal{X} : S(x) \leq M\}$ is compact.

Definition 4.1. A sequence $\{Z^\epsilon\}$ satisfies a *large deviation principle* on $\mathcal{X}$ with *rate function* S and *speed* ϵ if the following two conditions hold.

1. For each closed subset F of $\mathcal{X}$, Z^ϵ satisfies the large deviation upper bound:

$$\limsup_{\epsilon \to 0} \epsilon \log P\{Z^\epsilon \in F\} \leq - \inf_{x \in F} S(x).$$

2. For each open subset G of $\mathcal{X}$, Z^ϵ satisfies the large deviation lower bound:

$$\liminf_{\epsilon \to 0} \epsilon \log P\{Z^\epsilon \in G\} \geq - \inf_{x \in G} S(x).$$

Note that the sets F and G do not depend on ϵ. If the sequence converges to a limit not contained in F and G, the probability the sequence enters these sets converges to zero. Thus the theory deals with "rare events" with limit probability zero. If the sequence satisfies a large deviation principle, the convergence is exponential with the leading exponent determined by the rate function. The definitions also show how in large deviation theory the evaluation of a probabilistic statements is characterized by an optimization problem. This makes the theory amenable for analysis and leads to natural solution methods to apply it in practice.

We begin by looking at the one-step transitions of the log capital stock as in (10). We use the notation l_x^σ for the random variable whose distribution is identical to that of l_{t+1} conditioned on $l_t = x$:

$$l_x^\sigma \sim N(g^\sigma(x), \sigma^2).$$

Then we have our first intermediate result, which is an application of the Gärtner-Ellis Theorem. This result is in turn an extension of the well-known Cramér Theorem for non-i.i.d. random variables.

THEOREM 4.1. *Suppose that Assumptions 3.1 and 3.2 hold for $h = g$. Then for any given finite x, the sequence of random variables $\{l_x^\sigma\}$ satisfies a large deviation principle on $\mathcal{L} \subset \mathbb{R}$ with speed σ^2 and rate function:*

$$I(x, y) = \frac{1}{2}(y - g^0(x))^2. \tag{18}$$

Proof. Follows from the Gärtner-Ellis Theorem (see Dembo and Zeitouni (1998), Theorem 2.3.6). To apply the theorem, we need to calculate the limiting logarithmic moment generating function:

$$\begin{aligned} H(x, \lambda) &= \lim_{\sigma \to 0} \sigma^2 \log E \exp\left(\frac{\lambda l_x^\sigma}{\sigma^2}\right) \\ &= \lambda g^0(x) + \frac{\lambda^2}{2}, \end{aligned} \tag{19}$$

where the second equality follows from Assumption 3.2. Then the rate function is given by the Legendre transform of H:

$$I(x, y) = \sup_{\lambda \in \mathbb{R}} [\lambda y - H(x, \lambda)],$$

which gives the result. ▮

This result gives us a large deviation principle for one-step transitions. We now apply this result to develop a large deviation principle for finite time paths of the log capital stock. Following Klebaner and Zeitouni (1994), we let $[u]_T$ stand for a specific path $(u_0, u_1, ...u_T), T < \infty$. Then we define the cumulation of the one-step rate function:

$$S(T, [u]_T) = \sum_{t=0}^{T-1} I(u_t, u_{t+1}). \tag{20}$$

The next result shows that the log capital stock from the stochastic growth model converges uniformly on finite horizons to the deterministic log capital stock.

THEOREM 4.2. *Suppose that Assumptions 3.1 and 3.2 hold for $h = g$. Then on a given finite horizon $[0, T]$ the sequence $\{l_t^\sigma\}$ satisfies a large deviation principle on the product space $\mathcal{L}^T \subset \mathbb{R}^T$ (equipped with Euclidean metric) with rate function S and speed σ^2.*

Proof. Follows from Klebaner and Zeitouni (1994), Lemma 2.1. The necessary conditions of their theorem are easily verified given our assumptions and the form of the rate function I. In particular, their Lemma 2.5 holds under Assumption 3.2 and implies the uniformity of the large deviation principle. The continuity of I and the other technical conditions follow as in the proof of their Theorem 3.1 under our Assumptions 3.1 and 3.2. ▮

In particular, Theorem 4.2 implies that the log capital stock satisfies the large deviation upper bound:

$$\limsup_{\sigma\to 0} \sigma^2 \log P\left(\sup_{1\le t\le T} \left|l_t^\sigma - l_t^0\right| > \epsilon \,\middle|\, l_0^\sigma = l_0^0 = x \right) \le - \inf_{\left\{[u]_T:\ u_0 = x, |u_t - l_t^0| > \epsilon,\ 1\le t\le T\right\}} S(T, [u]_T)$$

where x is an arbitrary initial condition and $\epsilon > 0$. Therefore on $[0, T]$ as $\sigma \to 0$, the solution of the stochastic growth model converges uniformly to the solution of the deterministic growth model, provided they are initialized at the same point.

Thus Theorem 4.2 shows that the probability the stochastic and deterministic paths differ by a given amount converges to zero exponentially fast, and determines the rate of convergence. The rate function for the time paths is the cumulative rate function for each of the Markov transitions. The theorem also implies that once the capital stock reaches its deterministic steady state level, it has a small probability of exiting a neighborhood of the steady state. We now spell out this implication of the large deviation principle further by considering what is known as the exit or escape problem. The solution of this problem provides estimates on the typical time it takes for the log capital stock to depart from its deterministic steady state level by a given amount.

To begin with this analysis, we define:

$$V(x, y) = \inf_{\{[u]_T:\ u_0 = x, u_T = y,\ T < \infty\}} S(T, [u]_T). \tag{21}$$

Thus we define V as the minimized cost of moving from x to y in some finite horizon, where the cost of an arbitrary path is evaluated by the rate function S. Since the S function is continuous (as g^0 is continuous) and the space of paths connecting x and y in finite steps is compact, the minimum exists. A path $[u]_T$ which achieves the minimum is called a dominant escape path (from x to y). In our analysis, we are particularly interested in escapes from the deterministic steady state. Thus we fix an interval of length $2\epsilon > 0$ centered at the steady state level l^* and define:

$$\overline{V} = \inf_{\{y:\ |y - l^*| \ge \epsilon\}} V(l^*, y). \tag{22}$$

Again, the minimum exists but is not necessarily unique, so let Y^* be the set of minimizers from (22). Next define the exit time from the interval as:

$$\tau^\sigma = \inf\left\{t > 0 :\ |l_t^\sigma - l^*| \ge \epsilon,\ |l_0^\sigma - l^*| < \epsilon\right\}.$$

The following result shows that the exit times increase to infinity exponentially fast and determines the exponential rate of convergence. Furthermore, we determine the end of the interval where the exit will most likely take place.

Theorem 4.3. *Suppose that Assumptions 3.1-3.3 hold with $h = g$. Then for any $\eta > 0$ we have:*

$$\lim_{\sigma \to 0} P\left(\exp\left(\frac{\overline{V} - \eta}{\sigma^2}\right) \leq \tau^\sigma \leq \exp\left(\frac{\overline{V} + \eta}{\sigma^2}\right)\right) = 1.$$

For any exit which occurs at $l^\sigma_{\tau^\sigma}$ and for any $\eta > 0$ there exists a $y^* \in Y^*$ such that:

$$\lim_{\sigma \to 0} P(|l^\sigma_{\tau^\sigma} - y^*| < \eta) = 1.$$

Proof. Follows from Klebaner and Zeitouni (1994), Theorem 2.1 and the remark which follows it. The necessary conditions are shown to hold using the same arguments as in our Theorem 4.2. ∎

The key step in applying these large deviation results is to solve the minimization problem inherent in (21)-(22). Using (18) we can re-state this problem as:

$$\inf_{\{T<\infty, [u]_T\}} \frac{1}{2} \sum_{t=0}^{T-1} \left(u_{t+1} - g^0(u_t)\right)^2 \tag{23}$$

subject to $u_0 = l^*$, $u_T = l^* \pm \epsilon$. For a given horizon T and terminal value u_T, this is a discrete time minimum energy problem of the form analyzed in Chapter 2 of Lewis (1986). In general we cannot even solve for the policy function g^0 analytically, so we have little hope of obtaining an explicit solution to this problem. In Section 6 below we describe how to solve it numerically. Here we provide a local characterization of the solution, which helps to interpret our later results. In particular, we find that for small escape sets, the curvature of the policy function determines whether a positive or negative escape is more likely. The following Theorem adapts a result of Kasa (2001), who works in continuous time. This allows him to usee the results of Freidlin and Wentzell (1984) to obtain an explicit expression for the rate function V, making the local analysis more straightforward. The discrete time nature of our setting leads to a more involved derivation which is detailed in Appendix C.

Theorem 4.4. *Suppose that Assumptions 3.1- 3.3 hold for $h = g$. In a neighborhood of the stable equilibrium l^* the rate function in (21) has the expansion:*

$$V(l^*, y) = \frac{(l^* - y)^2}{2}(1 - g^0_x(l^*)^2) - \frac{(l^* - y)^3}{2} g^0_{xx}(l^*) g^0_x(l^*)^2 \frac{1 - g^0_x(l^*)^2}{1 - g^0_x(l^*)^3} + O(|l^* - y|^4).$$

Thus if g^0 is strictly convex (respectively, strictly concave), an exit from the interval $(l^* - \epsilon, l^* + \epsilon)$ happens at $l^* + \epsilon$ (respectively, $l^* - \epsilon$) with probability converging to one as $\sigma \to 0$ and $\epsilon \to 0$. If g^0 is linear the exit is equally likely to happen at either endpoint.

Proof. See Appendix C. ∎

5. A MODERATE DEVIATION PRINCIPLE

The previous sections have presented a functional central limit theorem and a large deviation principle to analyze the asymptotics of the stochastic growth model. In this section we derive some intermediate asymptotic results, which in effect combine the insights of the two previous sections. For the central limit theorem, we considered the sequence of processes $X_t^{\sigma,0}$, while the large deviation principle considered $X_t^{\sigma,1}$. In this section we present a moderate deviation principle, which considers the sequence $X_t^{\sigma,\rho}$ for the intermediate case $0 < \rho < 1$. A moderate deviation principle is simply a large deviation principle for the normalized process over a slower range of speeds. Our results in this section provide simpler, more explicit asymptotic characterizations of the behavior of the log capital stock.

Following Klebaner and Liptser (1999), we will first provide some of the heuristic arguments and then proceed more formally. For $\rho \in (0,1)$ we can write:

$$X_t^{\sigma,\rho} = \sigma^\rho X_t^{\sigma,0}.$$

Now by Theorem 3.1, we know that $X_t^{\sigma,0} \Rightarrow X_t$ where $\{X_t\}$ is the Gaussian linear autoregression in (17). Now suppose that the processes $\{X_t^{\sigma,\rho}\}$ and $\{\widetilde{X}_t^{\sigma,\rho}\} = \{\sigma^\rho X_t\}$ satisfy the same large deviation principle.[5] Recall from (17) that $\{\widetilde{X}_t^{\sigma,\rho}\}$ satisfies:

$$\widetilde{X}_{t+1}^{\sigma,\rho} = g_x^0(l_t^0)\widetilde{X}_t^{\sigma,\rho} + \sigma^\rho B_{t+1}.$$

Then by extending the results of Theorem 4.2, we could show that on an infinite horizon the sequence $\{\widetilde{X}_t^{\sigma,\rho}\}$ satisfies a large deviation principle with speed $\sigma^{2\rho}$ and rate function:

$$J([u]) = \frac{1}{2}\sum_{i=1}^{\infty}(u_i - g_x^0(l_{i-1}^0)u_{i-1})^2 \tag{24}$$

where $u_0 = 0$ and $[u] = (u_1, u_2, \ldots)$. As we note below, finite horizon results analogous to those for the large deviation principle above can also be established, but the infinite horizon results are stronger.

The conditional normality of our application simplified the form of the rate function S from (20) for the large deviation principle. However the proposed rate function J in (24) is even simpler, as it is a quadratic form. (Whereas S is quadratic in the nonlinear function g^0.) This makes applications particularly easy, and as we see below allows for some explicit analytic results. Further, the rate function in the moderate deviation principle is quadratic regardless of the shock distribution (although the conditions of the theorem below rule out the lognormal case which would make the analysis cover the capital stock levels k_t). Finally, the same asymptotics hold for all settings of $0 < \rho < 1$.

The following formal result verifies that the heuristics above do in fact hold.

THEOREM 5.1. *Let $0 < \rho < 1$, and assume that Assumptions 3.1 and 3.2 hold for $h = g$. Then the sequence $\{X_t^{\sigma,\rho}\}$ satisfies a moderate deviation principle on the*

[5] As emphasized by Kushner (1984), this will *not* hold in general, as there is only weak convergence.

infinite product space $\mathcal{L}^\infty \subset \mathbb{R}^\infty$ *equipped with metric*

$$d([u],[v]) = \sum_{n=1}^{\infty} 2^{-n} \frac{|u_n - v_n|}{1 + |u_n - v_n|}.$$

The MDP has speed $\sigma^{2\rho}$ *and rate function* J *given in (24).*

Proof. Follows from Klebaner and Liptser (1999), Theorem 2. Their necessary conditions are easily verified given our assumptions and the form of the H function defined in (19). ∎

As Klebaner and Liptser (1999) note, this MDP implies a corresponding MDP over any finite horizon. Analogous to the large deviation principle in Theorem 4.2, the finite horizon MDP is defined on a finite product space with Euclidean metric, with the rate function J restricted to finite sequences.

Theorem 5.1 implies a particularly easy characterization of the exit problem. If we now define the exit time:

$$\tau^{\sigma,\rho}(\epsilon) = \inf\left\{t > 0 : \; |X_t^{\sigma,\rho}| \geq \epsilon\right\},$$

which is the first time the process $\{l_t^\sigma\}$ exits from an window of width $2\epsilon\sigma^{1-\rho}$ centered on the path $\{l_t^0\}$. Then we have the following result characterizing the mean exit time. The idea is essentially the same as the linear case of Theorem 4.4, with the difference that the exit sets shrink with σ.

Theorem 5.2. *Let* $0 < \rho < 1$*, and assume that Assumptions 3.1-3.3 hold for* $h = g$*. Let* $l_0^0 = l^*$*. Then we have the following:*

$$\lim_{\sigma \to 0} \sigma^{2\rho} \log E\tau^{\sigma,\rho}(\epsilon) = \frac{\epsilon^2}{2}(1 - g_x^0(l^*)^2).$$

Proof. Under our assumptions, this follows from Theorem 5.1 and Klebaner and Liptser (1999), Theorem 4. ∎

6. AN APPLICATION

In this section we illustrate our theoretical results in a calibrated model. We choose the parameters of the model so that it matches certain features of US time series data. To calibrate the process for A_t, we used data on the cumulative Solow residual from Citibase, following the construction of Stock and Watson (1999). These residuals are scaled so that they can be interpreted as labor-augmenting technology. These data are quarterly from 1959:Q1 to 1999:Q2 and are constructed from output (GDP less farm, housing and government), capital (interpolation of annual values of fixed non-residential capital stock using quarterly investment), and labor (hours of employees on non-agricultural payrolls). As in Stock and Watson (1999), we then construct the technology shock process using a labor's share value of $\alpha = 0.65$. The mean growth rate κ and standard deviation σ of the A_t process are chosen to match the implied Solow technology process at an annual frequency.

TABLE 1.

The baseline model parameters. The values were obtained by fitting technology process to time series data on the Solow residual, and matching the mean log capital stock level of 1.396 and its autocorrelation of 0.943.

Parameter	Description	Annual Value
κ	technology growth	0.0176
σ	technology shock standard deviation	0.0492
α	labor's share	0.65
δ	depreciation	0.0517
β	subjective discount factor	0.9847
γ	relative risk aversion	4

The remaining parameters are chosen so that our theoretical predictions match certain features of the log capital stock time series. From the functional central limit theorem result in Corollary 3.1 we predict that for small σ the mean of the log capital stock should be l^* and its autocorrelation $g_x^0(l^*)$. In Appendix A we provide analytic expressions for these variables in terms of the model parameters. From our data sample, we find that the log capital stock has mean 1.396 and annual autocorrelation of 0.943. Given the values of α and $\theta = \exp(-\kappa)$ above, for any choice of the relative risk aversion parameter γ the expressions for $(l^*, g_x^0(l^*))$ determine pairs (β, δ) which are consistent with the data. However for arbitrary γ, we are not guaranteed that β and δ will be between zero and one. By experimenting, we found that setting γ to the plausible, if slightly high, value of 4 led to reasonable results for the subjective discount rate and deprecation. Our specific parameter choices are summarized in Table 1.

In accordance with our theory we analyze the model as we vary σ, but we keep the remainder of the parameters fixed. In order to carry out the analysis, we also need the optimal policy functions g^0 for the deterministic model and g^σ for the stochastic model. To find the policy functions, we solve the Euler equations (6) and (15) on a grid of 15,001 points centered on the steady state level, using log-linear interpolation. For the stochastic model, we approximate the conditional expectation using Gauss-Hermite quadrature with 51 nodes. For the functional central limit theorem results, we do not need the entire policy function but only its derivative at the steady state. Following Judd and Guu (1997), we determine this value exactly by differentiating the Euler equation (15) at the steady state, which leads to a quadratic equation. In Figure 1 we plot the log capital accumulation function $g^0(l)$ from the deterministic model along with its linear approximation $l^* + g_x^0(l^*)(l - l^*)$. Here we see that the policy function is nearly linear, but with a slight convexity which is apparent at the edges of the state space. This figure provides a preview of some of the results to follow: the linear approximation provides a good characterization of the policy function except

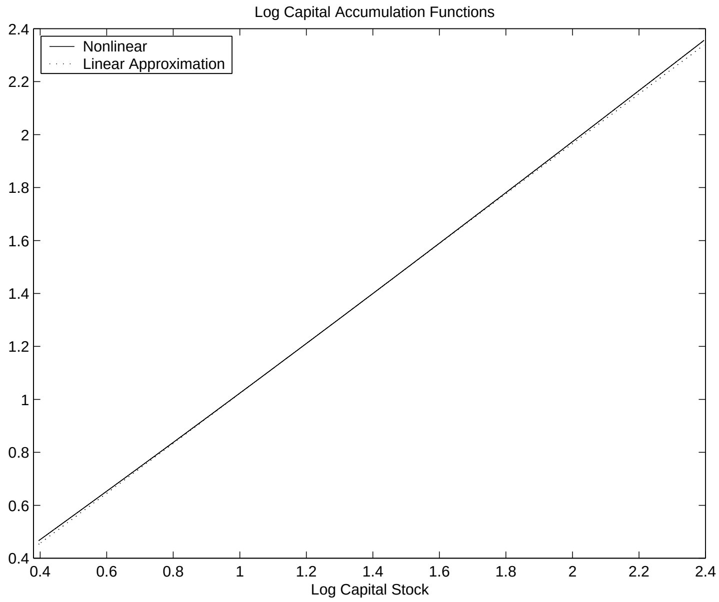

FIGURE 1. The log capital accumulation function from the numerical solution of the deterministic growth model and its linear approximation.

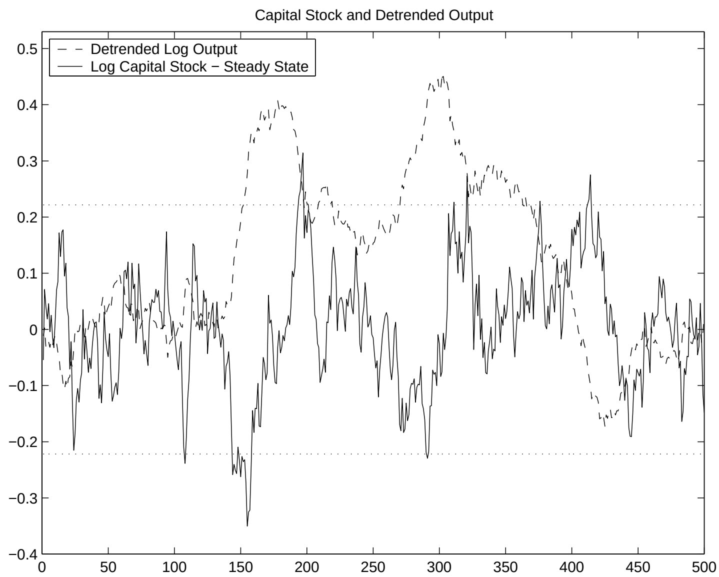

FIGURE 2. A simulation of the logarithm of the capital stock and detrended output. The dashed lines show the large deviation bands.

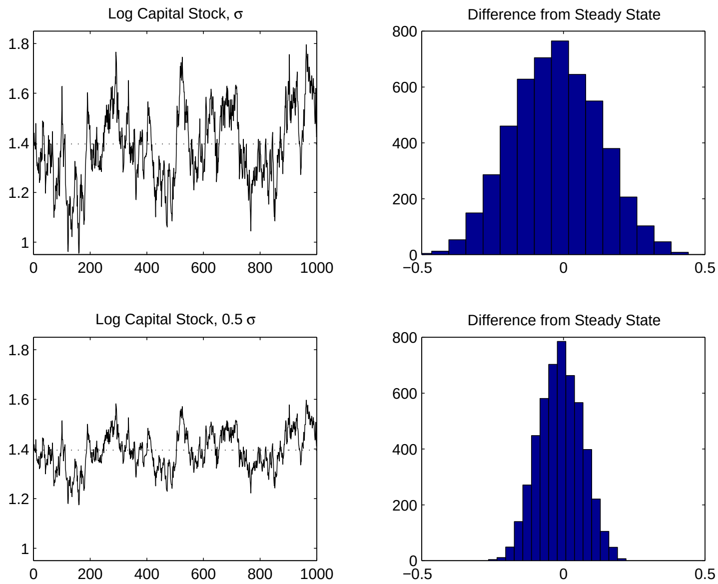

FIGURE 3. Simulations of the logarithm of the capital stock and for different shock variances, and histograms of the differences from the deterministic steady state (l^*).

far from the steady state. Furthermore by Theorem 4.4, the slight convexity suggests that positive escapes will be more likely.

As we noted above, our analysis identifies large movements in the log capital stock l_t with business cycles. Figure 2 provides the justification for this identification. The figure plots some simulated data from the baseline parameterization, showing the value of the log capital stock minus its deterministic steady state level $(l_t - l^*)$, and the detrended log output values. The figure clearly shows the negative correlation between the log capital stock and log output. However, more important for our purposes is the fact that large changes in output correspond to the large changes in the capital stock. The dotted lines in the figure plot the "escape sets" (bands of $\pm\epsilon$) that we will consider in our analysis of large and moderate deviations below. The significant escapes in the figure correspond to significant booms or recessions. For example, near period 150 there is a large fall in the log capital stock which accompanies a large increase in output relative to trend. Then around period 200 the log capital stock experiences a positive escape, which accompanies a decrease in output to back near trend. Another boom occurs in period 300, followed by a series of recession in periods 325, 375 and 425 as output falls from far above trend to below trend. In summary, we see the negative escapes correspond to booms (peaks of output relative to trend), while positive escapes correspond to recessions (falls in output relative to trend).

We now turn to examining the results of our three different asymptotic characterizations of the model. Table 2 summarizes the simulation results for different levels of the standard deviation of the technology shock process, which we index by $d\sigma$ where σ is as in Table 1 and we let d vary. Turning first to the functional central limit theorem predictions, the first two columns of the table give the autocorrelation and standard deviation of the log capital stock l_t from simulations of 5000 periods. From

TABLE 2.

Simulation results for different levels of the technology shock standard deviation.

Tech. Std. Dev.	FCLT Mean Level	FCLT Auto. Corr.	FCLT Std. Dev.	LDP Mean Time	LDP Positive (Percent)	MDP Mean Time	MDP Positive (Percent)
2 σ	1.32	0.946	0.301	10.1	49.6	18.9	49.8
1.5 σ	1.35	0.946	0.227	16.5	57.3	25.5	56.1
σ	1.37	0.947	0.152	42.3	51.6	42.3	51.6
0.75 σ	1.38	0.947	0.114	108.3	52.8	65.5	53.5
0.625 σ	1.38	0.947	0.095	241.0	54.4	88.7	52.3
0.5 σ	1.38	0.947	0.076	1008.1	58.2	132.0	53.1

Corollary 3.1 above, our theoretical predictions are that the mean should be l^*, the autocorrelation should be $g_x^0(l^*)$, and the standard deviation $d\sigma/\sqrt{1-g_x^0(l^*)^2}$, which in our baseline parameterization are 1.396, 0.943, and 0.148d, respectively. These are values very close to those in the table when $d=1$. Further, the standard deviation of l^* decreases proportionately to d in the simulations, just as the theory predicts. These results are further illustrated graphically in Figure 3. The left panels plot a portion of the simulated l_t series for σ and 0.5σ, while the right panels plot a histogram of $l_t - l^*$ from the full 5000 period simulations. The figure clearly illustrates that for small shock settings, the log capital stock series becomes concentrated around the deterministic steady state. Further, as our theoretical results suggest, the differences from the steady state look to be very Gaussian in character.

We now turn to our results based on the large deviation and moderate deviation principles, which characterize certain "rare" events in the log capital stock process. In both cases, the key to determining the probability of an escape and the mean escape times is given by the rate function. For the moderate deviation principle, our results in Theorem 5.1 analytically characterize the rate function. However for the large deviation principle, we must use numerical methods. Our solution strategy works with the first order conditions from the minimization problem in (21) which defines the function V and determine a dominant escape path. For any given horizon T, problem (23) is a two-point boundary value problem for a path $[u]_T$ which starts at l^* and ends at $l^* \pm \epsilon$ at date T. In principle, we could solve the problem for each T and then minimize over the horizon length. A simpler method, which we adopt, converts the boundary value problem to an initial value problem. The first order conditions from the minimization problem in (23), as given in (C.4) in Appendix C, can be rewritten so that they imply a recursion for the optimal path. In particular, we have for $1 \leq t \leq T-1$:

$$u_{t+1} = g^0(u_t) + \frac{u_t - g^0(u_{t-1})}{g_x^0(u_t)}. \tag{25}$$

With an arbitrary first step u_1, we can then iterate on (25) until we hit a terminal u_T which is at least ϵ units from l^*. Then we can evaluate the rate function along

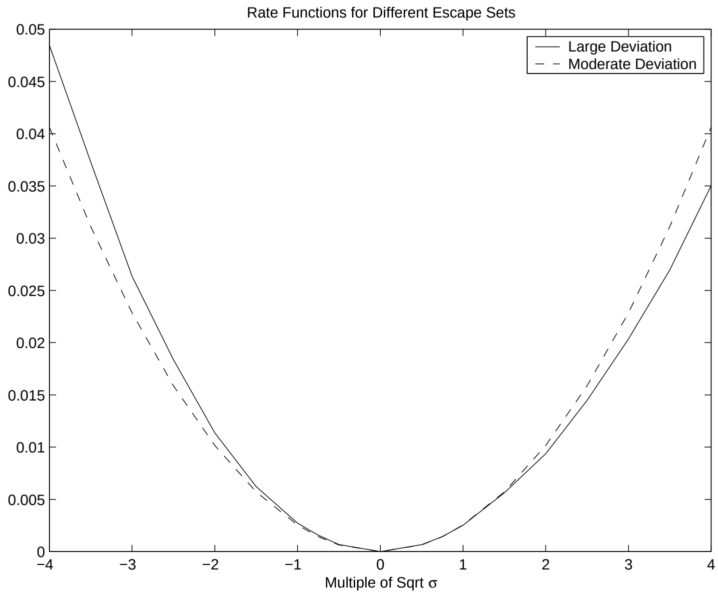

FIGURE 4. Comparisons of the rate functions for the large deviation principle and the moderate deviation principle, for varying escape sets (indexed by $\epsilon = n\sqrt{\sigma}$, for different n).

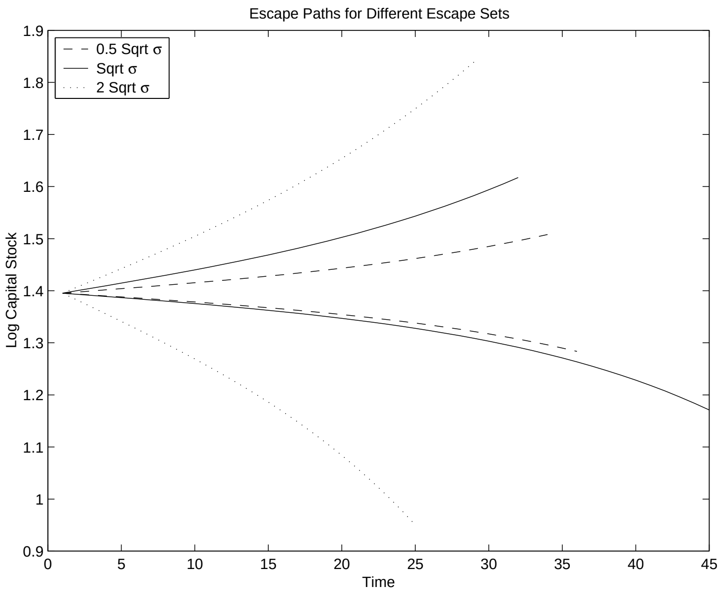

FIGURE 5. Comparisons of the escape paths for the large deviation principle, for varying escape sets (indexed by $\epsilon = n\sqrt{\sigma}$, for different n).

this proposed path, to obtain a value $S(T, (l^*, u_1, ..., u_T))$. Then minimizing over the initial step u_1, we obtain the optimized rate function $\overline{V}$. This algorithm determines T endogenously, and proved to be very fast in practice. Note that the entire algorithm must be carried out numerically, as (25) at each step uses the numerical policy function (shown in Figure 1) and its derivative.

The results of our calculations for the large and moderate deviation principles are shown in Figure 4. The figure plots the rate functions $\overline{V}$ and $\overline{J} = \frac{\epsilon^2}{2}(1 - g_x^0(l^*)^2)$ in our baseline parameterization for different values of the escape set ϵ. In particular, we index the escape sets by $\epsilon = n\sqrt{\sigma}$ and let n vary. We know from Corollary 5.2 that the MDP rate function is symmetric, while we find that the LDP rate function is slightly asymmetric. As Theorem 4.4 suggests, the convexity of the policy function g^0 implies that the rate function is slightly lower for positive escapes than for negative escapes. This implies that we expect large recessions to occur more frequently in the model than large booms. The corresponding dominant escape paths are shown in Figure 5. Interestingly, we find no overall pattern among the dominant escape paths for different size escape sets. In order to escape, we could either have a few very large shocks or a longer accumulation of small shocks in the same direction. The figure shows that larger escape sets are associated with paths that move larger distances in shorter amounts of time. However, the overall length of the dominant path may increase or decrease with the size of the escape set.

For relatively small escape sets, there is essentially no difference between the MDP and LDP rate functions. However for large escapes, the asymmetry becomes more pronounced and the rate functions differ more substantially . This implies that even though the MDP results only hold for events of order $\sigma^{1-\rho}$ with $0 < \rho < 1$, for small escapes they provide a good approximation of the LDP results which hold for events of order 1. Essentially this finding is due to the properties of the policy functions shown in Figure 1 above. The policy function is well-approximated by a linear function over much of the state space. Therefore even for the analysis of some events of order 1, the linear approximation may provide acceptable results. (This is also an implication of Theorem 4.4 for events of order ϵ^3.) Further, the MDP results which are based on the linear approximation have simple closed-form expressions which make them easy to apply and greatly simplify the analysis. However, just as the linear approximation breaks down far enough away from the steady state, for large enough escapes the MDP results differ substantially from the LDP results.

The preceding discussion was essentially about the magnitude of the rate functions in the LDP and MDP cases. However, it is also clear that the linear approximation misses the asymmetry in the policy function, and so the MDP misses the corresponding business cycle asymmetries. Even though this effect is most pronounced for large escape sets, it is still present for smaller escape sets. For example, for an escape of $\epsilon = +\sqrt{\sigma}$ we have $\overline{V} = 0.0025$, while for $\epsilon = -\sqrt{\sigma}$ we have $\overline{V} = 0.0027$. While the differences are small in absolute terms, recall that they are the exponents determining the exponential increase in the time between escapes. Therefore, for small enough σ even though both large booms and large recessions become increasingly unlikely, we expect recessions to occur more frequently than booms.

We now turn to some simulation results which illustrate our theoretical results. The last four columns of Table 2 provide a summary of 5000 simulated escape paths for different levels of the technology shock, which we index by $d\sigma$ for varying d. Our theoretical results show that as d goes to zero, the log capital stock converges to l^*,

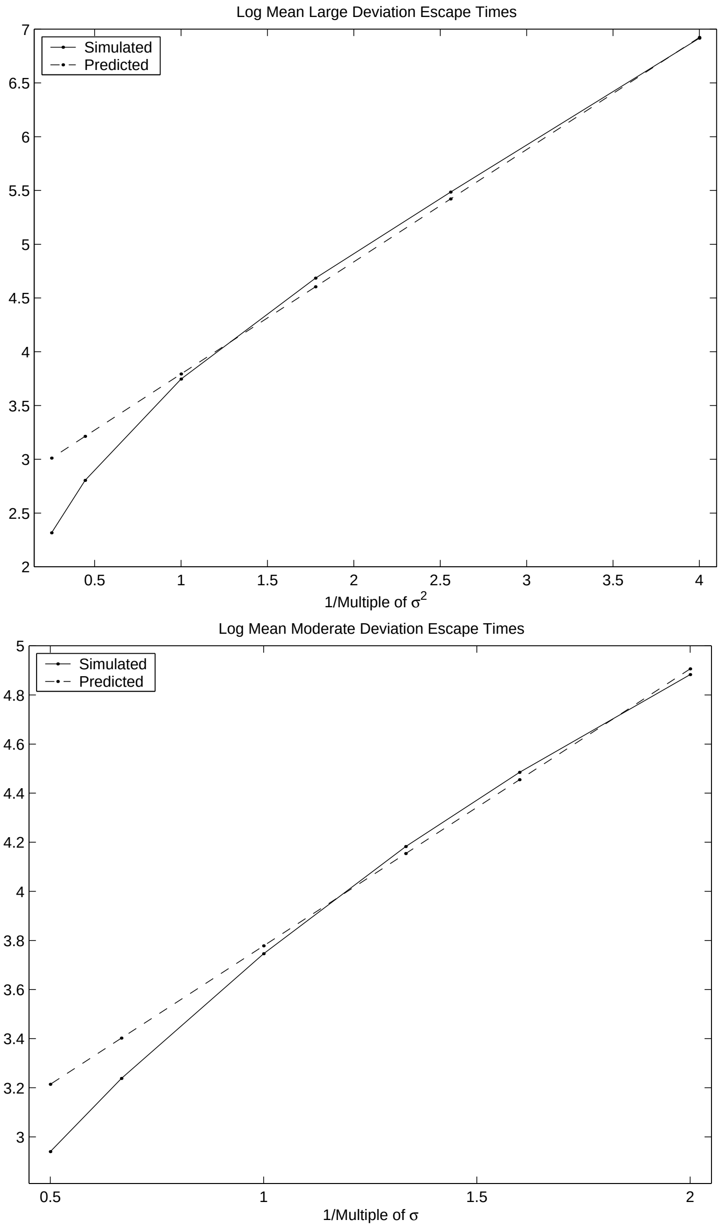

FIGURE 6. Logarithm of mean escape times for different standard deviations. The solid lines plot the results from simulations, while the dashed lines plot predicted results. The top panel plots the results for the large deviation principle, and the bottom panel plots the moderate deviation principle.

which is the fixed point of g^0. However for nonzero d the fixed point of g^σ, which we denote $\bar{l}^\sigma$, is different from l^* by an amount of order σ. In our simulation results, we thus look at escapes from the "unbiased" value of $\bar{l}^\sigma$ instead of from l^*. In practice, this has essentially no effect on the mean escape times, although it does affect the results regarding the asymmetry of escapes. As evidenced in the first column of Table 2, we find that $\bar{l}^\sigma$ converges to l^* from below. Thus by analyzing escapes from l^* we would be distorting the results in favor of negative escapes.

For each run of the simulation, we initialize the log capital stock at steady state $\bar{l}^\sigma$ and run the simulation on until either a positive or negative escape happens. For the large deviation principle, we let the size of the escape set be fixed at $\epsilon = \sqrt{\sigma} = 0.222$. Thus the events we consider correspond to nearly a 16 percent increase or decrease in the log capital stock, and thus represent sizable business cycles. For the moderate deviation principle, we let $\rho = 0.5$ and so analyze the process $X_t^{d\sigma,0.5} = \frac{l_t - l^*}{\sqrt{d\sigma}}$ with the same size escape set ϵ. This is equivalent to analyzing the process l_t but considering escape sets of size $\epsilon(d) = \sqrt{d\sigma}$, which thus shrinks with d. Therefore for $d = 1$, the LDP and MDP simulation results are the same, while for $d < 1$ the MDP escape set is smaller and thus more likely to be exited. The results clearly show the rapid increase in the mean escape times in both cases. Further, the asymmetry in the model is evident, especially in the LDP results. As predicted, for small d we observe more positive than negative escapes and thus find recessions to be more likely than booms. At the baseline parameters of $d = 1$, we find that recessions account for 52 percent of the escapes, while when $d = 0.5$ they account for 58 percent of the large deviation escapes (and over 53 percent of the moderate deviation escapes). Thus for small levels of the shock variance, the asymmetry of the large business cycle fluctuations becomes more apparent, especially in the large deviation results.

Our theoretical results above also provide accurate predictions of the rate at which the escape times increase. For the LDP, Theorem 4.3 shows that for small d the escape times are approximately

$$\begin{aligned} E\tau^{d\sigma} &\approx C_L \exp\left(\frac{\bar{V}}{(d\sigma)^2}\right), \quad \text{or} \\ \log E\tau^{d\sigma} &\approx \log C_L + \frac{\bar{V}}{(d\sigma)^2} \end{aligned}$$

for some constant C_L. Similarly, the MDP with $\rho = 0.5$, Theorem 5.2 shows that for small d the mean escape times are approximately:

$$\begin{aligned} E\tau^{d\sigma,0.5} &\approx C_M \exp\left(\frac{\bar{J}}{d\sigma}\right), \quad \text{or} \\ \log E\tau^{d\sigma,0.5} &\approx \log C_M + \frac{\bar{J}}{d\sigma} \end{aligned}$$

for some constant C_M, where again $\overline{J} = \frac{\epsilon^2}{2}(1 - g_x^0(l^*)^2)$. Thus the escape times increase exponentially with the inverse of d for the MDP and with the inverse of d^2 for the LDP, with the rate of increase determined by the rate function.

These results are shown graphically in Figure 6, where we plot the log of the mean escape times from Table 2 along with our predictions based on the rate function. Our analysis provides the rate functions, but does not determine the constants C_L and C_M. In the figure we choose the constants in order to get a good fit, and thus our results should only be evaluated on the slopes of the relevant lines. The top panel of the figure shows the results from the LDP, and so the x-axis is $1/d^2$, while the bottom panel plots the results from the MDP and so the x-axis is $1/d$. Although the simulations we have only six data points, the figure suggests that our theoretical predictions provide a good characterization of the rate of increase in the mean escape times. Both the LDP and MDP predictions are much closer for small d, as our theory predicts. Especially striking is how good the results are for all of the values of $d \leq 1$.

7. CONCLUSION

In this paper we have presented and applied a variety of asymptotic methods which characterize the properties of the stochastic growth model. We have shown that as the standard deviation of the technology shock process gets small, the log capital stock process converges to a Gaussian linear autoregression. Further, we have characterized the probability and frequency of large fluctuations in the log capital stock. Our results have shown that for small noise, the capital stock process converges to the deterministic steady state, and that the autocorrelation and variance of the log capital stock process can be characterized analytically. Additionally we have shown both theoretically and through simulations that for small noise large booms and recessions become increasingly unlikely, although recessions are more likely than booms. Finally, we showed that the average time between business cycles increased exponentially, and provided accurate predictions of the rate of increase.

For several reasons, in this paper we have focused on the stochastic growth model. It is one of the most widely used models, and is a standard test case for new methods. Further, the fact that there was a single state variable provided substantial technical simplifications. However it should be clear that our results apply much more broadly, and could be extended to multidimensional cases. Of particular interest may be our results on linearization methods. We show that for small noise, a linear approximation provides a characterization of the average behavior of the model and also certain occasional large fluctuations (although not events of order one). As linearization remains one of the most common solution methods, these results have many potential applications.

APPENDIX A

Properties of the Policy Functions

As described previously it is convenient to scale consumption and capital by the technology level. This allows us to derive Bellman equations that do not depend on A, but only on k, thus simplifying our analysis. As is well-known, the Bellman equation for the original problem is:

$$v(K, A) = \max_C \left\{ \frac{C^{1-\gamma}}{1-\gamma} + \beta \int v(K', A') dG(Z) \right\} \quad \text{s.t.} \quad (1), (2)$$

where G is the log-normal distribution function. We can then use the normalizations $k = K/A$ and $c = C/A$, then guess and verify that the Bellman equations take the form $v(K, A) = A^{1-\gamma} v(k)$. This leads to the reduced Bellman equation:

$$v(k) = \max_c \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta \int (\theta Z)^{\gamma-1} v(k') dG(Z) \right\} \quad \text{s.t.} \quad (3). \tag{A.1}$$

Note that the utility function satisfies the Inada conditions, so that we know solutions must be interior. The first order condition and an application of the envelope theorem then yield the Euler equation (6) in the text. Analogous results in the deterministic case lead to the Euler equation (15).

Standard results, as in Stokey, Lucas, and Precott (1989) show that in both the deterministic and stochastic cases the policy functions $k' = f^0(k)$ and $k' = f^\sigma(k)$ are continuous and bounded for finite k. Simple arguments also show that the policy functions are strictly increasing, and further that the consumption policies are also strictly increasing. Results from Araujo (1991) and Santos (1991) show that the policy functions are continuously differentiable in the deterministic case, and Santos (1991) provides a bound on the derivative. By adapting results from Amir (1997), we also have that the policy functions are twice continuously differentiable in the stochastic case (see also Blume et al, 1982), and are bounded on the interior of the state space. Finally, the results in Santos (1993) suggest that in our model the policy function in the deterministic case is also twice continuously differentiable. As an optimal path is always interior and, as we show below, there is a unique interior stable steady state, we can apply the imply the implicit function theorem to establish the higher order smoothness of the policy function.

We next turn to deriving analytic expressions for the deterministic steady state l^* and the derivative of the policy function at the steady state $g_x^0(l^*)$. These expressions form the basis of our calibrations. From the Euler equation (15), the unique interior deterministic steady state is easily seen to be:

$$k^* = \left(\frac{1 - \beta\theta^\gamma + \delta\beta\theta^\gamma}{\alpha\beta\theta^\gamma} \right)^{\frac{1}{\alpha-1}}.$$

Taking logs gives l^*.[1] Then from (3), we find the steady state consumption $c^* = (k^*)^\alpha + \frac{\theta - 1 - \delta\theta}{\theta} k^*$. This in turn allows us to evaluate the derivatives of the utility function at the steady state: $u' = (c^*)^{-\gamma}$, and $u'' = -\gamma (c^*)^{-\gamma-1}$. Following Judd and Guu (1997), we can apply the implicit function theorem to the Euler equation (15) to find a quadratic expression for the derivative of the consumption function at the steady state $c' = c_x^0(k^*)$:

$$(c')^2 \left[\beta\theta^{\gamma+1} u'' \left(\alpha (k^*)^{\alpha-1} + 1 - \delta \right) \right] + c' \left[u'' - \beta\theta^{\gamma+1} u'' \left(\alpha (k^*)^{\alpha-1} + 1 - \delta \right)^2 \right] - \beta\theta^\gamma u' \alpha(\alpha-1)(k^*)^{\alpha-2} = 0$$

Using the expression for k^*, this can be reduced to:

$$(c')^2 + D_1 c' + D_2 = 0$$

where:

$$D_1 = \frac{1}{\theta} - \frac{1}{\beta\theta^\gamma}, \quad D_2 = -\beta\theta^{\gamma+1} \frac{c^*}{\gamma} \alpha (1-\alpha)(k^*)^{\alpha-2}.$$

Since $D_2 < 0$, this insures that c' is real, and since we know c' is positive, we have that:

$$c' = -D_1 + \sqrt{D_1^2 - 4D_2} > -D_1.$$

With this expression, we can then evaluate $g_x^0(l^*)$:

$$g_x^0(l^*) = \frac{\alpha (k^*)^\alpha - k^* c' + (1-\delta) k^*}{(k^*)^\alpha - c^* + (1-\delta) k^*}.$$

[1] As always, there is also a trivial boundary steady state at zero. As long as the initial capital stock is positive, the zero steady state will not be reached.

As f^0 is strictly increasing, $g^0_x = kf^0_x/f^0$ is strictly positive. Further, using the expressions for c', c^*, and k^* we have that:

$$\begin{aligned} c' > -D_1 &= \frac{1}{\theta} - \frac{1}{\beta\theta^{\gamma}} \\ &= \alpha(k^*)^{\alpha-1} + 1 - \delta - \frac{1}{\theta} \\ &= \frac{c^*}{k^*} - (1-\alpha)(k^*)^{\alpha-1} \end{aligned}$$

and therefore $g^0_x(l^*) < 1$, so that the unique interior steady state is stable.

All that remains to be shown is that the stochastic policy function converges to the deterministic one. As in the assumptions, we confine our attention to a compact set $\mathcal{K} \subset (0,\infty)$. The proof adapts results in Santos and Vigo (1998). First note that the sequence $\{Z^\sigma\}$ converges weakly to the constant value of 1 as $\sigma \to 0$. Next consider the deterministic value function $v^0(k)$, and note that $(Z^\sigma)^{\gamma-1}v^0(Z^\sigma x)$ is continuous and uniformly integrable (since v^0 is bounded above). Therefore

$$\lim_{\sigma\to 0}\int (Z^\sigma)^{\gamma-1}v^0(Z^\sigma x)dG^\sigma(Z^\sigma) = v^0(x). \quad \text{(A.2)}$$

Furthermore, since $v(K, A)$ is a concave function for all $\sigma \geq 0$, $(Z^\sigma)^{\gamma-1}v^0(Z^\sigma x) \leq v^0(x)$, and therefore the convergence in (A.2) is monotone, and therefore by Dini's Theorem (see Dudley, 1989) uniform on compact sets. Next, we define the Bellman operator on the right side of (A.1) as T^σ, with the corresponding operator T^0 in the deterministic case. By our uniform convergence result, we then have that $T^\sigma v^0(k) \to T^0v^0(k)$. Additionally, since $T^\sigma v^0(k) \leq T^0v^0(k) = v^0(k)$, this convergence is again uniform on compact sets. Finally, letting $\|\cdot\|$ be the sup norm on the compact set $\mathcal{K}$, we have:

$$\begin{aligned} \|v^0 - v^\sigma\| &= \|T^0v^0 - T^\sigma v^\sigma\| \\ &\leq \|T^0v^0 - T^\sigma v^0\| + \|T^\sigma v^0 - T^\sigma v^\sigma\| \\ &\leq \|T^0v^0 - T^\sigma v^0\| + \beta\theta^{\gamma-1}\|v^0 - v^\sigma\| \end{aligned}$$

by the triangle inequality and the contraction property of T^σ. Thus:

$$\|v^0 - v^\sigma\| \leq \frac{1}{1-\beta\theta^{\gamma-1}}\|T^0v^0 - T^\sigma v^0\|$$

and so v^σ converges uniformly to v^0, which implies that c^σ converges pointwise to c^0. This in turn implies the pointwise convergence of g^σ to g^0.

To show that the convergence of the policy functions is uniform, for a given k define the function:

$$G(c) = \frac{c^{1-\gamma}}{1-\gamma} + \beta\theta^{\gamma-1}v^0(\theta(k^\alpha - c + (1-\delta)k)).$$

Further, let $c^0 = c^0(k)$ and $c^\sigma = c^\sigma(k)$. Then it is clear that $G(c^0) = v^0(k)$ and $G(c^\sigma) \geq T^\sigma v^0(k)$ so that $G(c^0) - G(c^\sigma) \leq T^0v^0 - T^\sigma v^0$. Then note that $G'(c^0) = 0$ and $G(c)$ is concave with a bounded second derivative on $\mathcal{K}$ (since the utility and value functions have finite second derivatives for $k > 0$). Call the bound η. Then we have:

$$\frac{\eta}{2}(c^0 - c^\sigma)^2 \leq G(c^0) - G(c^\sigma)$$

and therefore:

$$|c^0 - c^\sigma| \leq \sqrt{\frac{\eta}{2}|T^0v^0 - T^\sigma v^0|}.$$

Since this expression holds for all k, we can take the sup over k, and the uniform convergence follows by our results above.

Moreover the convergence of the value functions on $\mathcal{K}$ is exponential at rate σ^2, implying that the convergence of the policy functions is exponential at rate σ. This is evident from the following:

$$\begin{aligned}
\|T^\sigma v^0 - T^0 v^0\| &= \left\| \max_c \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta \int (\theta Z)^{\gamma-1} v^0(k') dG(Z) \right\} - \max_c \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta\theta^{\gamma-1} v^0(k') \right\} \right\| \\
&\leq \left\| \beta\theta^{\gamma-1} \max_{k'} \left\{ \int (\theta Z)^{\gamma-1} v^0(k') dG(Z) - v^0(k') \right\} \right\| \\
&\leq \left\| \beta\theta^{\gamma-1} \max_{x} \left\{ \int Z^{\gamma-1} v^0(Zx) dG(Z) - v^0(x) \right\} \right\| \\
&\leq \left\| \beta\theta^{\gamma-1} \max_{x} v^0_x(x) x E\left[\exp((\gamma-1)\sigma W) - \exp(\gamma\sigma W)\right] \right\| \\
&= \left| \beta\theta^{\gamma-1} \max_{x} v^0_x(x) x \left[\exp\left(\frac{1}{2}(1-\gamma)^2\sigma^2\right) - \exp\left(\frac{1}{2}\gamma^2\sigma^2\right)\right]\right|.
\end{aligned}$$

Here the third line uses the definition of k' in the stochastic model, the fourth uses the concavity of v^0, the boundedness of the derivatives of v^0 on $\mathcal{K}$, and the definition of Z, and the fifth uses the properties of lognormal random variables. On the compact set $\mathcal{K}$ the capital stock and the derivative of the value function are both bounded, yielding the result.

APPENDIX B

Functional Central Limit Theorem

In this appendix, we prove Theorem 3.1 by extending the results of Klebaner and Nerman (1994). The basis of our result is their Theorem 3, which requires that Assumption 3.1 hold globally, not just on restrictions to compact sets. We use a truncation argument following Kushner and Yin (1997) in order to extend the result and show that the truncation is not needed in the limit. As in most of the paper, we focus on the results in logarithms, i.e. for $l = \log k$. The results for the levels are similar.

In this section, since we focus on the case $\rho = 0$ we suppress the second superscript and write $X_t^\sigma = X_t^{\sigma,0}$. First we truncate the process. That is, for every integer M let q_M be a continuous function on the line satisfying: $-M \leq q_M(x) \leq M$, $q^M(x) = x$ for $|x| \leq M$, $q^M(x) = M$ for $x \geq M+1$, and $q^M(x) = -M$ for $x \leq -(M+1)$. Note that $q^M_x(x) = 1$ for $|x| \leq M$ and 0 for $|x| \geq M+1$. Then let $l_0^{\sigma,M} = l_0^\sigma$ and for $t \geq 0$ define the truncated process:

$$l_{t+1}^{\sigma,M} = g^\sigma(q^M(l_t^{\sigma,M})) - \sigma W_{t+1}. \tag{B.1}$$

Then we define the normalized differences: $X_t^{\sigma,M} = \frac{l_t^{\sigma,M} - l_t^0}{\sigma}$. Note that we do not need to truncate l_t^0 as long as $|l_0^0| < M$, which we assume. Under our assumptions, the truncated process (B.1) satisfies the conditions of Klebaner and Nerman (1994), Theorem 3. Since the truncation is not applied to the deterministic process, we have that truncated normalized differences $\{X_t^{\sigma,M}\}$ converge to weakly to the autoregression $\{X_t\}$ in (17).

We now show that the truncation is unnecessary in the limit by establishing the tightness of the untruncated normalized process. The results of Klebaner and Nerman (1994) establish weak convergence in the product space, so it is enough to verify tightness at each date. Thus we need to show that for any t we have:

$$\lim_{K\to\infty} \sup_\sigma P\left(|X_t^\sigma| \geq K\right) = 0.$$

By Prohorov's Theorem (see Kushner and Yin (1997), Theorem 7.3.1), tightness implies the existence of a weakly converging subsequence. As the truncation is not applied in the limit, the weak limit of the untruncated process must be the same as the truncated process.

Notice that from any initial condition $x \in \mathcal{L}$, we have:

$$X_1^\sigma = \frac{l_1^\sigma - l_1^0}{\sigma} = \frac{g^\sigma(x) - g^0(x)}{\sigma} + W_1.$$

The tightness of this sequence thus follows by the tightness of the normal random variable W_1. Then we proceed by induction. Thus for any t we have:

$$\begin{aligned}
X_{t+1}^{\sigma,0} &= \frac{l_{t+1}^\sigma - l_{t+1}^0}{\sigma} = \frac{g^\sigma(l_t^\sigma) - \sigma W_{t+1} - g^0(l_t^0)}{\sigma} \\
&= \frac{1}{\sigma}\left[g^\sigma(l_t^\sigma) - g^0(l_t^\sigma) + g^0(l_t^\sigma) - g^0(l_t^0)\right] - W_{t+1}.
\end{aligned}$$

Therefore we have:

$$P(|X_t^\sigma| \geq K) \leq P\left(\left|g^\sigma(l_t^\sigma) - g^0(l_t^\sigma)\right| \geq K\sigma\right) + P\left(\left|g^0(l_t^\sigma) - g^0(l_t^0)\right| \geq K\sigma\right) + P(|W_{t+1}| \geq K\sigma). \quad \text{(B.2)}$$

By the tightness of the normal distribution, the third term goes to zero with K. Then notice that we have the following:

$$\left|g^0(l_t^\sigma) - g^0(l_t^0)\right| = |g_x^0(l_t^0)||l_t^\sigma - l_t^0| + O(|l_t^\sigma - l_t^0|^2).$$

Since $x = l_0^0 \leq l_t^0 \leq l^*$ the derivative is bounded and by the induction hypothesis $|l_t^\sigma - l_t^0|$ converges to zero with σ, and the second term is of order σ^2. This implies that the second term in (B.2) converges to zero at rate σ, which in turn implies that it goes to zero with K. Finally, we split the first term in (B.2) depending on whether l_t^σ is in $\mathcal{L}$. Notice that:

$$\begin{aligned} P\left(\left|g^\sigma(l_t^\sigma) - g^0(l_t^\sigma)\right| \geq K\right) &= P\left(\left|g^\sigma(l_t^\sigma) - g^0(l_t^\sigma)\right| \geq K \| \, l_t^\sigma \in \mathcal{L}\right) P(l_t^\sigma \in \mathcal{L}) \\ &+ P\left(\left|g^\sigma(l_t^\sigma) - g^0(l_t^\sigma)\right| \geq K \| \, l_t^\sigma \in \mathcal{L}^c\right) P(l_t^\sigma \in \mathcal{L}^c). \end{aligned} \quad \text{(B.3)}$$

Notice that by the LDP results in Section 4, the probability of being in $\mathcal{L}$ at any date t is of order 1 as $\sigma \to 0$ and the probability of being in $\mathcal{L}^c$ goes to zero (exponentially fast) with σ^2. Further, we have previously established that on $\mathcal{L}$ the policy functions converge uniformly at rate σ. Thus the fist term on the right of (B.3) converges to zero at rate σ. To bound the term on $\mathcal{L}^c$ notice that $\lim_{k\to 0} f^\sigma(k)/f^0(k) = 1$ and $\lim_{k\to\infty} f^\sigma(k) = Hk$ for some $H > 0$ for all $\sigma \geq 0$. Thus $|g^\sigma(k) - g^0(k)|$ is bounded on $\mathcal{L}^c$. Putting this together implies that both terms on the right of (B.3) converge to zero at rate σ, and thus the first term in (B.2) converges to zero with K. This completes the proof of tightness, and so the result follows.

APPENDIX C

Local Expansion of the LDP Rate Function

In this appendix we derive the local expansion in Theorem 4.4. We expand the function $V(y) \equiv V(l^*, y)$ around $y = l^*$. At this point the optimal path is $u_t = l^*$ for all t. For a given y reached at date T, each step u_t of the optimal path will be an implicit function of y. However an application of the envelope theorem to problem (23) gives:

$$\frac{dV}{dy} = y - g^0(u_{T-1}), \quad \text{(C.1)}$$

$$\frac{d^2V}{dy^2} = 1 - g_x^0(u_{T-1})\frac{du_{T-1}}{dy}, \quad \text{(C.2)}$$

$$\frac{d^3V}{dy^3} = -g_x^0(u_{T-1})\frac{d^2u_{T-1}}{dy^2} - g_{xx}^0(u_{T-1})\left(\frac{du_{T-1}}{dy}\right)^2. \quad \text{(C.3)}$$

Note that the first derivative (C.1) equals zero when evaluated at $y = u_{T-1} = l^*$.

To evaluate the higher order derivatives, we use the first order conditions from problem (23):

$$u_t - g^0(u_{t-1}) - (u_{t+1} - g^0(u_t))g_x^0(u_t) = 0, \quad \text{(C.4)}$$

for $t = 1, \ldots, T-1$. We obtain implicit expressions for $\frac{du_t}{dy}$ by differentiating (C.4). When evaluated at l^* this gives:

$$(1 + g_x^0(l^*)^2)\frac{du_t}{dy} - g_x^0(l^*)\frac{du_{t-1}}{dy} - g_x^0(l^*)\frac{du_{t+1}}{dy} = 0, \quad \text{(C.5)}$$

with boundary conditions $\frac{du_0}{dy} = 0$ and $\frac{du_T}{dy} = 1$. For use below, note that (C.5) is a homogenous second order difference equation, whose characteristic equation has roots $g_x^0(l^*)$ and $1/g_x^0(l^*)$. Therefore the general solution can be written:

$$\frac{du_t}{dy} = c_1 g_x^0(l^*)^{T-t} + c_2 g_x^0(l^*)^{t-T}$$

for some c_1, c_2. In order to satisfy the boundary conditions (for $T \to \infty$) we must have $c_1 = 1$ and $c_2 = 0$, so that

$$\frac{du_t}{dy} = g_x^0(l^*)^{T-t}. \quad \text{(C.6)}$$

Substituting this expression into (C.2) gives:

$$\left.\frac{d^2V}{dy^2}\right|_{l^*} = 1 - g_x^0(l^*)^2. \quad \text{(C.7)}$$

Note that this agrees with the expressions in Lewis (1986), who on pp. 32-35 explicitly solves this problem with a linear state evolution. This follows since a problem with linear evolution whose slope is $g_x^0(l^*)$ yields the identical difference equation (C.5).

Using similar logic, we evaluate the second derivative with respect to the end point y by implicitly differentiating (C.4) again. After simplifying and evaluating at l^* we obtain:

$$\begin{aligned}(1+g_x^0(l^*)^2)\frac{d^2u_t}{dy^2} \;&-\; g_x^0(l^*)\frac{d^2u_{t-1}}{dy^2} - g_x^0(l^*)\frac{d^2u_{t+1}}{dy^2} \\ &+\; 3g_x^0(l^*)g_{xx}^0(l^*)\left(\frac{du_t}{dy}\right)^2 - g_{xx}^0(l^*)\left(2\frac{du_t}{dy}\frac{du_{t+1}}{dy} + \left(\frac{du_{t-1}}{dy}\right)^2\right) = 0.\end{aligned}$$

Using (C.6), this simplifies to:

$$(1+g_x^0(l^*)^2)\frac{d^2u_t}{dy^2} - g_x^0(l^*)\frac{d^2u_{t-1}}{dy^2} - g_x^0(l^*)\frac{d^2u_{t+1}}{dy^2} + g_{xx}^0(l^*)g_x^0(l^*)^{2(T-t)-1}\left(3g_x^0(l^*)^2 - 2 - g_x^0(l^*)^3\right) = 0, \quad \text{(C.8)}$$

with boundary conditions $\frac{d^2u_0}{dy^2} = \frac{d^2u_T}{dy^2} = 0$. Thus (C.8) is an inhomogeneous second order linear difference equation of the same form as (C.5). We transform it to a homogeneous equation by defining the variable $z_t = \frac{d^2u_t}{dy^2} - \mu_t$, where

$$\mu_t = \frac{g_{xx}^0(l^*)\left(3g_x^0(l^*)^2 - 2 - g_x^0(l^*)^3\right)}{g_x^0(l^*)^4 - g_x^0(l^*)^3 - g_x^0(l^*) + 1} g_x^0(l^*)^{2(T-t)}. \quad \text{(C.9)}$$

Then z_t follows the same difference equation as (C.5), but now with boundary conditions $z_0 = -\mu_0 = 0$, $z_T = -\mu_T$. This means that we have the solution:

$$z_t = -\mu_T g_x^0(l^*)^{T-t}$$

and therefore:

$$\begin{aligned}\frac{d^2u_t}{dy^2} &= z_t + \mu_t \\ &= \mu_T\left(g_x^0(l^*)^{2(T-t)} - g_x^0(l^*)^{T-t}\right). \quad \text{(C.10)}\end{aligned}$$

Evaluating (C.10) at $T-1$ and using (C.9) gives:

$$\frac{d^2u_{T-1}}{dy^2} = \frac{g_{xx}^0(l^*)g_x^0(l^*)\left(3g_x^0(l^*)^2 - 2 - g_x^0(l^*)^3\right)}{g_x^0(l^*)^4 - g_x^0(l^*)^3 - g_x^0(l^*) + 1}(g_x^0(l^*) - 1) \quad \text{(C.11)}$$

Then substituting (C.6) and (C.11) into (C.3) and simplifying gives:

$$\left.\frac{d^3V}{dy^3}\right|_{l^*} = -3g_{xx}^0(l^*)g_x^0(l^*)^2\frac{1-g_x^0(l^*)^2}{1-g_x^0(l^*)^3}. \quad \text{(C.12)}$$

Collecting (C.7) and (C.12) along with Taylor's theorem gives the expansion stated in Theorem 4.4. The conclusions follow from Theorem 4.3, where we note that by Assumption 3.3 the sign of the third derivative term is determined by g_{xx}^0.

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