Shocking Escapes*

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Abstract

We modify Sargent's model of inflation (Sargent, 1999) to include real oil prices, and use it as a laboratory to study the impact of exogenous supply shocks on the behavior of the inflation time-series. We are particularly interested in whether these shocks can trigger escape-like episodes, or *empirical escapes*.[1] We consider two types of shocks: unobserved permanent shocks to the natural rate of unemployment; and observed permanent shocks to the mean real oil price. Using simulations, we find that favorable shocks to the natural rate of unemployment significantly decrease the expected time to empirical escape; that is, in the shocked economy, empirical escapes tend to occur much sooner than they do in the economy without shocks. On the other hand, we find that observed permanent shocks to the mean real oil price cause the economy to quickly move to its new equilibrium. We show that the activation of empirical escapes is well explained by analyzing the system's mean dynamics.

Keywords: Supply Shocks, Adaptive Learning, Self-Confirming Equilibrium, Mean Dynamics, Escape Route

JEL Classification: E31, E32, E37, D83

1 Introduction

In "The Conquest of American Inflation", (Sargent, 1999), Sargent, building off the work of Sims (1988), constructs a model designed to study the implications of mis-specification. He assumes that the real economy is governed by a neo-classical natural

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[1]The term empirical escape is used here to distinguish this behavior from the more formal notion put forth by Williams (2002a).

rate theory; whence there is no long term trade-off between unemployment and inflation. However, the government is not aware (or not convinced) of this natural rate theory, and instead believes that the economy is well described by a statistical Phillips curve obtained by regressing unemployment on current and lagged inflation and lagged unemployment. This belief potentially implies a long run trade-off, depending on the perceived coefficients in this regression. Associated to this combination of truth (natural rate theory) and misspecification (statistical Phillips curve) is a natural notion of equilibrium, which Sargent calls a self-confirming equilibrium (SCE). Informally, in an SCE, agents beliefs, as summarized by the regression coefficients, are consistent with the true data generating process as determined by the natural rate theory; a more formal definition will be given later. Sargent shows that the government's inflation target in an SCE is the same as would be obtained by the associated Kydland-Prescott model. In particular, the result is the suboptimal Nash outcome.

To include adaptation in the model, Sargent assumes that policy makers estimate the coefficients of the statistical Phillips curve using a recursive least squares algorithm. He find that if they use ordinary least squares as their estimation technique then the economy will eventually converge to the SCE; in particular, the SCE is globally stable under least squares learning. However, appealing to historical data, Sargent has his agents think that the statistical Phillips curve may move around some. More specifically, policy makers believe that the parameters of their empirical model may vary with time. To account for this, policy makers use a constant gain learning rule to form their estimates of the regression coefficients.

This minor modification has important implications for the behavior of the inflation time-series. Using simulations, Sargent shows that the economy will endogenously escape the SCE and the inflation rate will be driven to near zero, where it will remain for some time before slowly returning to the SCE level. He calls this behavior the activation of an escape route. The intuition is straightforward: while in SCE, the government perceives a long run trade-off between inflation and unemployment. Occasionally, while in or around an SCE, a sequence of unusual shocks will occur which discredit this long run trade-off and hence cause the government to reduce inflation. As the government reduces inflation, the belief that the trade-off does not exist is reinforced, ultimately causing the mean inflation rate to fall to near zero. Once near zero, the appearance of a correlation between inflation and unemployment returns and the government slowly starts ramping up the inflation rate in an attempt to exploit this trade-off.

Formal analysis of the escape routes appeared intractable in the general case until Williams, in a remarkable technical feat, solved the problem. Using results from the theory of large deviations, he was able to show that the escape routes may be described as solutions to a certain type of optimal control problem, which, in effect, computes the most likely sequence of unusual shocks. We will discuss these results briefly below, but for complete details, see Williams (2002a). Cho, Williams, and Sargent (2003) then applied Williams' results to Sargent's model of inflation, thus analytically describing the escape routes in Sargent's model, as well as the asymptotic probability of their occurrence.[2]

In the last part of his book, armed with an elegant model yielding both low inflation and endogenous fluctuations, Sargent takes his case to the data. He finds that his model fits poorly.[3] There are at least two reasons for this result: one is the rigid form of the government's objective function, a problem that we intend to address in a future paper; the other reason is that surely the actual US inflation time-series is affected by both endogenous effects, such as Sargent's escapes, and exogenous effects, both observable and latent. For example, no serious attempt to empirically model inflation should ignore the oil shocks of the seventies, and yet Sargent's model does not account for these shocks.

It is here our work begins. With the ultimate future goal of carefully fitting Sargent's model to the data, we attempt to address the implications of exogenous structural shocks; our idea being that it may be possible for exogenous shocks to activate escapes. Informally, the intuition behind this idea is simple: perhaps a permanent exogenous shock can, in some sense, play the role of the unusual sequence of temporary shocks, by causing the government to disavow a belief in a long run trade-off. If so, it is natural to expect escape behavior to result. Because this escape behavior will not, in a formal sense, be the activation of an escape route, we refer to this behavior as an *empirical escape*.

It is quite natural to consider in the context of Sargent's model the impact structural changes, for indeed the adaptation mechanism of the policy makers was specified with structural change in mind. The usual rationale for constant gain estimation is that the associated estimators remain alert to structural shifts. Cho, Williams, and Sargent (2003) found that the constant gain specification itself led to escapes; we simply address whether the structural shifts can result in similar behavior.

We begin by considering a "static" model as a baseline, that is, a model in which the policymakers regress current unemployment on current inflation and a constant. This model is particularly useful because the Phelps's problem becomes a static optimization problem and hence can be solved analytically, thus adding greatly to the insight provided. We then proceed to change the constant term in the natural rate theory, thus modeling an unobserved permanent shock to the natural rate of un-

[2]Williams has since successfully applied his results on escapes to other types of models of endogenous fluctuations, showing how, for example, constant gain learning can imply endogenous switching between Cournot and competitive outcomes in a model of duopoly. See Williams (2002a) and Williams (2002b) for more details.

[3]The empirical result of a good fit was not the goal of Sargent's program, and is certainly not the criterion by which it should be judged. However, given the potential importance of his contribution, it is reasonable to address the model's fit.

employment. Using simulations, we find that given an initial natural rate of 5%, decreasing the rate to 4.75% greatly increases the likelihood that an empirical escape will occur soon. In contrast, increases in the natural rate of unemployment cause the economy to slowly converge to the new SCE.

It is well known that the mean dynamics of a constant gain algorithm may well describe the algorithm's behavior, provided a large deviation, such as escape, does not occur; see Cho, Williams, and Sargent (2003), and Evans and Honkapohja (2001), for details. We find that the mean dynamics also well explains the empirical escapes. In particular, we find that though the SCE is a stable rest point of the mean dynamics, this stability is very local in the following sense: there is a small neighborhood of the SCE outside of which the mean dynamics may first force inflation to near zero before slowly returning to the SCE level. In fact, the path of inflation along the mean dynamics well describes the path of the empirical escapes. Thus we find that if the exogenous shock moves the beliefs of the government outside this small neighborhood of stability (and only very small shocks are required for this to occur), and if the movement is in the right direction, then the mean dynamics will drive the economy toward an empirical escape, and in particular, a rare event, such as a large deviation, is not required for an empirical escape to occur.[4]

After analysis of the static model we turn to a more realistic "dynamic" model. We first specify a natural rate theory which includes a component accounting for the effects of changes in the real price of oil.[5] The policy makers are assumed to regress current unemployment on current and lagged inflation, lagged unemployment, and current and lagged real oil prices. Note this model includes Sargent's model as a special case. We use simulations to analyze the impact on the escape behavior of unobserved changes in the natural rate of unemployment, and of observed changes in the mean real price of oil. We find, in case of the shocks to the natural rate of unemployment, that the economy behaves much in the same way as it did in the static case: even small decreases in the natural rate greatly increase the chances that an empirical escape will occur in near future. On the other hand, when the mean real price of oil is shifted either up or down, we find that the economy quickly moves to the new equilibrium. This result is well explained by the observation that the SCE values of the regression coefficients are not altered when shocks of this type occur and are observed.

The qualitative results obtained by analyzing the dynamic model allow for an interesting interpretation of the US data. The seventies and early eighties saw the simultaneous occurences of high oil prices and stagflation. And this is consistent with the behavior predicted by our model. A rise in the real price of oil raises the self-

[4]It may well be the case that a true escape occurs after the exogenous shock, thus forcing inflation to zero more rapidly than predicted by the mean dynamics. It may also be the case that the exogenous shock increases the likelihood of a true escape occuring.

[5]See McGough (2000) for a first principles derivation of such a natural rate theory.

confirming equilibrium levels of both mean unemployment and mean inflation, and the dynamics of the model imply that this new equilibrium will be attained almost immediately; thus stagflation is predicted. Also, the nineties saw a more rapid increase in productivity, a decrease in unemployment, and very low inflation. Interpreting the increase in productivity as resulting from a favorable technology shock, and assuming this shock caused a corresponding decrease in the natural rate of unemployment, our model precisely predicts both lower unemployment, as directly implied by the neoclassical theory underlying the model, and dramatic decrease in inflation caused an empirical escape, which was itself activated by the decrease in the natural rate of unemployment.

2 The Static Model

We begin our investigations with a static model. While empirically unrealistic, this version of the model will allow for explicit computation of the solution to the government's control problem as well as a careful description and understanding of the self-confirming equilibrium and of the mean dynamics. Furthermore, as has been demonstrated by Sargent (1999), and Cho, Williams, and Sargent (2003), and as we will find here, the inflation time-series of the static model is qualitatively very similar to that of the dynamic model.

2.1 Sargent's Static Model of Inflation

Because this model is identical to the static model explored by Sargent (1999), our description of it will be somewhat brief. Sargent's model has five principle components as described below:

1. A Natural Rate Theory representing the real behavior of the economy;
2. A Statistical Phillips Curve representing the government's perception of the relationship between inflation and unemployment;
3. The Phelps Problem, which describes the control problem of the government;
4. Self Confirming Equilibrium, a natural notion of equilibrium given the model's structure;
5. Adaptive Policy Makers, and an associated algorithm describing how the government updates its beliefs.

We discuss each of these in turn.

2.1.1 The Natural Rate Theory

Sargent assumes the real economy – specifically, the unemployment rate – to be governed by an expectations augmented Phillips curve consistent with a neo-classical vision of the economy. He writes this curve as

$$u_t = u^* - \frac{\theta}{1-\rho_2 L}(\pi_t - \pi_t^e) + \frac{\nu_{1t}}{1-\rho_1 L}, \tag{1}$$

where L is the lag operator, ν_{1t} is white noise, and π_t^e represents the public's expectation of the inflation rate. Such a curve may be derived from first principles using a model employed by Sargent (1987). Following Sargent, for simplification, we assume $\rho_i = 0$. Note that, assuming agents are rational, the natural rate of unemployment implied by this theory is given by u^*.

2.1.2 A Statistical Phillips Curve

The government's perception of the economy is described by a standard Phillips curve. In the static model, this curve takes the decidedly simple form

$$u_t = \gamma_1 \pi_t + \gamma_2 + \varepsilon_t. \tag{2}$$

Notice that the government's beliefs may be summarized by the coefficients of this curve, γ.[6]

2.1.3 The Phelps Problem

The government sets inflation up to white noise to maximize its utility, subject to its perceived constraint, the statistical Phillips curve. The government's problem is written

$$\max_{\hat{\pi}_t}(1-\beta)E\sum_{t=1}^{\infty} -\frac{\beta^{t-1}}{2}(u_t^2 + \pi_t^2)$$

$$\begin{aligned} s.t.\ u_t &= \gamma_1 \pi_t + \gamma_2 + \varepsilon_t \\ \pi_t &= \hat{\pi}_t + \nu_{2t} \end{aligned}$$

This is the Phelps's problem. We assume that ν_{1t} and ν_{2t} are independent.

[6] Sargent calls this version of the statistical curve, that is, with current inflation as a regressor, the classical identification. There are non-trivial dynamic implications of this identification; see Sargent (1999) for details.

In the more general model studied in the next section, the perceived Phillips curve will contain lags of several variables. However, here, we see that the constraint imposes no dynamic implications of today's decisions. In particular, the government is free to choose its target $\hat{\pi}_t$ to minimize $E(\pi_t^2 + u_t^2)$. A straightforward computation shows

$$\hat{\pi}_t \equiv h(\gamma) = -\frac{\gamma_2 \gamma_1}{1+\gamma_1^2}. \tag{3}$$

One more point is worth noting. Since agents are rational and are aware of the target $\hat{\pi}_t$, and since $\pi_t = \hat{\pi}_t + \nu_{2t}$, it follows that the unemployment rate is serially uncorrelated and unaffected by government policy.

2.1.4 Self-Confirming Equilibrium

The economy is in a self-confirming equilibrium (SCE) provided the government's beliefs are consistent with the true data generating process. More formally, the time-series must satisfy the following orthogonality condition:

$$E\left(u_t - \gamma'[\pi_t, 1]'\right)[\pi_t, 1] = 0.$$

This simply implies that the government's beliefs coincide with the true population regression coefficients.

We may define a map $T : \mathrm{R}^2 \to \mathrm{R}^2$ by

$$E\left(u_t - T(\gamma)'[\pi_t, 1]'\right)[\pi_t, 1] = 0,$$

that is, for given perceptions γ, $T(\gamma)$ is the vector of actual regression coefficients. Note, then, that a fixed point of the T-map defines a SCE. A straightforward computation shows that

$$T(\gamma) = \begin{bmatrix} -\theta \\ u^* - \theta h(\gamma) \end{bmatrix},$$

thus implying that the government's beliefs in an SCE are $\gamma_1 = -\theta$, and $\gamma_2 = u^*(1 + \theta^2)$.

2.1.5 Adaptive Policy Makers

Closing the model requires specifying how the government forms its beliefs. Following Sargent, we assume the government uses a constant gain updating algorithm to track potententially drifting coefficients in the statistical Phillips curve. Specifically, let $x_t = [\pi_t, 1]'$. Then

$$\begin{aligned} \gamma_t &= \gamma_{t-1} + \varepsilon P_t^{-1} x_t (u_t - \gamma_{t-1}' x_t) && (4) \\ P_t &= P_{t-1} + \varepsilon (x_t x_t' - P_{t-1}). && (5) \end{aligned}$$

Here ε is the gain set by the government.

2.2 Escapes and Mean Dynamics

The model is now completely described by equations (1), (4), (5), together with

$$\pi_t = h(\gamma_{t-1}) + \nu_{2t},$$

where $h(\gamma_{t-1})$ is given by (3). Sargent used $u^* = 5$, $\theta = 1$, and $var(\nu_i) = .09$ in his calibration and simulated the model using the SCE values of γ and P as his initial conditions.[7] The resulting dynamic behavior was astounding. He found that while the inflation rate tended to stay near the SCE mean of 5, it would periodically plunge to near zero and stay low for long periods of time. A sample time path displaying this behavior is given in Figure 1. Note the time-scale is altered to be consistent with the mean-dynamics as described below. The constant gain here and everywhere is taken to be $\varepsilon = .01$.

FIGURE ONE HERE

He called this drastic type of behavior the activation of an escape route, and informally, its cause is not difficult to understand. In an SCE the government perceives γ_1 to be -1, thus indicating a significant long term trade-off between unemployment and inflation. As Kydland and Prescott (1977) taught us, the government will try to exploit this trade-off thus keeping the inflation rate at a sub-optimally high level. Occasionally however, a sequence of shocks will occur, which, due to the tracking algorithm (4) and (5), will cause the government to doubt the trade-off, that is, the sequence of shocks will decrease the value of γ_1. When this occurs, the government will be inclined to lower inflation. The model's structure (and in particular, the mean dynamics: see below) will then cause the beliefs parameter γ_1 to be further reduced thus encouraging the government to lower inflation even more. This process will continue until the inflation target is set to near zero. With the target near zero, the inflation rate becomes ν_{2t}, which then implies a significant correlation between u_t and π_t. The government's tracking algorithm will slowly pick up on this correlation and thus encourage the government to begin exploiting this trade-off. The inflation rate then begins to rise toward the self-confirming equilibrium value of 5, and will continue to do so until another unusual series of shocks again activates an escape route.

To better understand the behavior exhibited by this model, we turn to the mean dynamics. The mean dynamics is a system of differential equations associated to the updating algorithm (4) and (5). Informally, this system describes the expected time-path of the beliefs parameters (and hence, in the static case, the inflation target), provided nothing unusual happens. More formally, we proceed as follows. We may rewrite (4) as

$$\gamma_t = \gamma_{t-1} + \varepsilon P_t^{-1} x_t((T(\gamma_{t-1}) - \gamma_{t-1})' x_t + \nu_{1t}). \tag{6}$$

[7] The SCE value of P is given by the covariance matrix of x at the SCE value of γ.

Let $M(\gamma) = E(x_t x_t')$, Williams (2002a) shows that as $\varepsilon \to 0$, for a given initial condition, the process (γ_t, P_t) converges weakly to the corresponding solution of the following system of differential equations, which he refers to as the mean dynamics:

$$\dot{\gamma} = P^{-1}M(\gamma)(T(\gamma) - \gamma) \tag{7}$$
$$\dot{P} = M(\gamma) - P \tag{8}$$

Note this convergence is *across* sequences, and not along them.

The unique stationary point of the above system is precisely the SCE of the model. Furthermore, it is not difficult to show that this fixed point is locally asymptotically stable for the calibrated model. However, there is small neighborhood of the SCE, so that paths associated to initial conditions not chosen in this neighborhood may wander far from the SCE even if they ultimately converge to it. This type of behavior has been noted by Cho, Williams and Sargent in this model and by Evans and Honkapohja (2001) in a different model, and tells part of the escape story. Once a series of unusual shocks causes the government to question a long term trade-off, the mean dynamics take over, further encouraging the government to reduce inflation. A mean dynamics time-path is the smooth curve plotted in Figure 1. While technically the mean dynamics determine the time-path of the belief coefficients, because we are in a static model, it is straighforward to interpret from this a time-path for inflation. Specifically, we simply take the associated inflation rate to be the inflation target given the beliefs determined by the mean dynamics. The static nature of the Phelps problem makes this inflation target independant of lagged inflation and unemployment.

The initial condition for the mean dynamics time-path plotted in Figure 1 was taken from the simulated time-series just after an escape was noted to have occured; in this case at time $t = 3.2$. This initial condition is indicated by the large dot. The mean dynamics time path is so true to the simulation that it is difficult to even distinguish it during the escape. Notice the mean dynamics drive the inflation rate to near zero before slowly encouraging a return to the SCE. Also notice that over time, the mean dynamics and simulated time-series diverge.[8]

Of course, the mean dynamics tell only part of the story. Explaining the occurence of these types of endogenous jumps requires a full understanding of the unusual sequence of shocks required to trigger such an escape. Williams (2002a) showed how to use the theory of large deviations to determine both the most likely sequence of unlikely shocks, which he called the dominant escape path, and the probability with which this sequence of shocks will occur. In Cho, Williams, and Sargent (2003), this theory was applied to Sargent's model and found to predict very well the outcomes of the simulations.

[8] Here and through out the paper, the time scale is transformed to be consistent with the mean dynamics; specifically, the step size in the simulations is taken to be ε.

2.3 Exogenous Shocks

We are now in a position to be begin conducting our experiments with exogenous shocks. In this section we restrict our attention to shocks affecting the natural rate of unemployment u^*; the idea being to analyze the impact on the dynamics of the models of a sudden permanant change in the natural rate of unemployment, due, say, to a policy shift or technology shock. To implement such a shock we first calibrate the model as Sargent did and choose the initial conditions of the model to correspond to the associated SCE. Then, before the simulation is begun, the calibrated value of u^* is changed to $u^* + \Delta u^*$. Note that the government is unaware of the natural rate theory in general, and unaware of the shock to u^* in particular. The simulation then reveals the time-series corresponding to a model which experiences an exogenous change to the natural rate of unemployment while in equilibrium. For future reference, we note that in the static model, the mean inflation rate in SCE is equal to u^*.

We are particularly interested in whether this type of exogenous shock can produce escape-like behavior, that is, behavior similar to that observed in Figure 1. However, a true escape, as defined by Cho, Williams, and Sargent, occurs in an economy initially in an SCE. Our economy is not initially in an SCE; the belief parameters are consistent with the SCE associated to the model with non-shocked parameters. Thus it is formally incorrect to identify any escape-like behavior we witness with an escape route. To deal with this, we informally define an *Empirical Escape* as the event that the inflation rate is driven near zero; and formally, an empirical escape occurs whenever the inflation rate falls below 2%. We note that any true escape will be captured by this definition.

2.3.1 Negative Shocks

We begin by analyzing the impact of a negative permanent exogenous shock to the natural rate of unemployment. An example simulation is presented in Figure 2. In this case the unemployment rate is taken to fall by .5% (that is, u^* falls to 4.5). The top panel represents the time-series of the shocked economy and the bottom panel represents the time-series of the non-shocked economy. The exogenous noise terms have exactly the same realizations for both time series. Notice that while both economies exhibit empirical escapes (and of course the second economy is actually exhibiting a true escape), the shocked economy escapes to zero inflation much sooner than non-shocked economy. Also plotted on the top panel are two mean dynamics paths: one (the solid line) initialized just as the simulation is, and the other (the dashed line) initialized with belief parameters obtained from the simulation after 10 periods. Notice that both mean dynamics paths indicate an empirical escape should be expected with out any need for an unusual sequence of shocks.[9]

FIGURE 2 HERE

The behavior exhibited by the mean dynamics is indicative of the local nature of the SCE's asymptotic stability. It takes but a slight deviation in the correct direction from the SCE's parameter values for the time-path of the mean dynamics to predict an empirical escape will occur. Given the calibration above, we found that a reduction in the natural rate by as little as .068, that is, u^* falls to 4.932, results in the mean dynamics predicting the activation of an empirical escape.

It is interesting to analyze this further. Assume we begin in a SCE and u^* changes by Δu^*. Fix $P = M(\gamma)$, as it would be in a self-confirming equilibrium, so that the mean dynamics then reduces to the differential equation

$$\dot{\gamma} = T(\gamma) - \gamma,$$

which is easily recognized as the standard E-stability differential equation. Since $\gamma_1 = -\theta$, it follows that $\dot{\gamma}_1 = 0$. Also, $\gamma_2 = (1+\theta^2)/\theta^2 u^*$ so that

$$\dot{\gamma}_2 = u^* + \Delta u^* - \frac{1}{1+\theta^2}\gamma_2.$$

We conclude that the E-stability differential equation is not only locally and globally stable (so that the SCE is stable under learning) but further that the distance to the fixed point is always diminishing. Thus it is the evolution of P (held constant in the above analysis) that directs the mean dynamics move so far away from the SCE.

While the mean dynamics seem to explain the behavior of our simulation in particular and of the impact of exogenous shocks in general, one simulation is not particularly convincing. Thus, to tell a more complete story, we turned to repeated simulations and density estimation. We begin by defining the random variable $\Phi(\Delta u^*)$ to be first escape time associated to the model as calibrated above and with shock Δu^*. Then, for a given shock value, $n = 400$ simulations are run and the values of Φ recorded. These are used to obtain histograms, descriptive statistics, as well as non-parameteric density estimations; for a description of the non-parameteric method employed, see Appendix B. For a complete listing of the descriptive statistics obtained, see Table 1.

As a benchmark, we obtain a sample of realizations of $\Phi(0)$. The estimated density is given in Figure 3. We then estimate the density of $\Phi(-.5)$: see Figure 4.

[9]The fact that the actual empirical escape occured slightly before time predicted by the mean dynamics can be attributed to the fact that the mean dynamics represent an approximation, and the farther along a fixed realization, the more likely the mean dynamics will diverge from that realization. It is for this reason two mean dynamic paths are plotted. Other simulations not reported show the simulation escaping after the time predicted by the mean dynamcis.

Due to the complicated nature of the non-parametric estimator, formal comparision of the estimated densities of $\Phi(0)$ and $\Phi(-.5)$ is not made; however, qualitatively, the difference between the two estimated densities is large: the introduction of the exogenous shock appears to affect the shape of the escape density, both shifting it left and narrowing its width.

To more carefully test that the escape densities of $\Phi(0)$ and $\Phi(-.5)$ are not the same, we perform the simplest possible mean comparison: we test the null hypothesis that the mean escape time of the shocked economy is the same as the mean escape time of the non-shocked economy. The sample mean of $\Phi(0)$ is 2.81 and of $\Phi(-.5)$ is .79. The null hypothesis is rejected at the one percent level, indicating that the exogenous shock altered the escape densitity.[10]

While the formal hypothesis tests do indicate a changed density, we believe that the estimated densities tell a more intriguing story. These estimations show us how a negative exogenous shock influences escape time densities, and in particular, strongly suggest that a negative shock to the natural rate of unemployment increases the probability that an empirical escape will soon occur.

TABLE ONE HERE

FIGURES 3 AND 4 HERE

Smaller shocks yield similar results. As indicated by the table, the mean escape time of $\Phi(-.25)$ is 1.33, which is significantly smaller than that of $\Phi(0)$. The density, which we do not report, is altered in a similar fashion to that of $\Phi(-.5)$, though not quite as dramatically.

As indicated by the previous discussion, even in the presence of a small negative shock, the mean dynamics predict an escape will occur. It is thus tempting to conclude the mean dynamics fully explain the witnessed behavior. And, in fact, this conclusion is strongly supported by analyzing the mean dynamics when $\Delta u^* = -.5$. If the mean dynamics is entirely responsible for the escape behavior, we would expect the mean of our sample to be approximately the same as the escape time predicted by the mean dynamics. And, as indicated in Figure 2 and in the Table, the escape time predicted by the mean dynamics and the sample mean almost exactly coincide. On the other hand, this support is not given in case $\Delta u^* = -.25$. When solved with respect to a

[10] Through out the sequel, we will refer to "rejection of the null hypotheses", and to mean escape times being "significantly different." In all cases, we will be refering to the hypothesis that the shocked and non-shocked economies have the same mean escape time. Also, in all cases, and as indicated in the Table, this hypothesis will be rejected at the one percent level, thus indicating that shocks always alter the distribution of escape times. In fact, saying that the rejection obtains at the one percent level is misleading. The associated p-scores are often much lower that .01, with the lowest being on the order of 10^{-100}.

shock of $-.25$, the mean dynamics indicate an escape should occur at approximately $t = 2.2$; however, we see that the sample mean is much lower, suggesting another mechanism may also be at work. A simple thought experiment presents a compelling explanation. Consider the case of $\Delta u^* = 0$: the mean dynamics predict no change from SCE levels, and therefore give no information on the density of the escape time. The density, therefore, must depend entirely on the escape dynamics and thus large deviation calculations. For $\Delta u^* = 0$ small enough the mean dynamics remain in the local basin of attraction and this argument continues to hold. This thinking suggests that for Δu^* small, but large enough to place the beliefs just outside the local basin of attraction, the density of empirical escape times should reflect both mean dynamics and escape dynamics, while for larger Δu^* the mean dynamics will dominate.

2.3.2 Positive Shocks

In case of positive shocks to the natural rate of unemployment, the subsequent behavior of the inflation time-series is more straightforward. The mean dynamics in this case indicate the model will slowly move toward its new SCE, and this is precisely what we see happen in repeated simulations.[11] We find that while the density, which we do not report, is not drastically different from that of $\Phi(0)$, it does appear shifted slightly to the right, indicating that the mean escape time has increased. And this is supported by descriptive statics: the mean escape time is 3.95. Larger positive shocks appear to further increase the mean escape time.

2.3.3 Static Discussion

Analysis of this simple static model allows us to draw several interesting conclusions. First, even small negative shocks to the natural rate of unemployment tend to greatly decrease the expected time to an empirical escape; and the larger the shock, the greater the decrease. Thus negative shocks are doubly beneficial to the economy; they may lead to (an admittedly not permanent) nearly optimal inflation policy, and the new SCE of the economy is pareto superior. On the other hand, positive shocks to the natural rate of unemployment appear to increase the expected time to escape. Thus, positive shocks are doubly detrimental in that they put off escaping to a nearly optimal policy, and the new SCE of the economy is pareto inferior. We have also learned that the mean dynamics goes a long way toward explaining the witnessed behavior, though clearly there are other mechanisms at play.

[11]We have also observed that the simulated economies do not tend toward the new equilibrium as quickly as the mean dynamics would predict.

3 The Dynamic Model

We now turn to a more realistic model of the economy. This model is based on the dynamic model presented in Sargent (1999), with one key difference: we include real oil prices in the aggregate supply curve. After constructing the model below, we turn to analyzing the dynamic impacts of certain types of exogenous shocks. The more complicated nature of the model makes analysis of the mean dynamics less instructive; the time-path of the mean dynamics still well approximates the time-path of the estimators, but the associated inflation path is less meaningful because of its dependence on lagged terms. Thus we rely on density estimation and descriptive statistics to tell our story.

3.1 Amending Sargent's Dynamic Model of Inflation

Sargent's dynamic model has the same structure as the static model, with the principle difference being the government's beliefs. In the dynamic model, the government's statistical Phillips curve includes as regressors lagged unemployment and inflation. We alter this dynamic model by including real oil prices in both the natural rate theory and in the government's regressors, under the assumption that the government views the current real oil price before setting inflation. This impacts the description as well as the analysis of the Phelps problem and of SCE computation, as described below.

3.1.1 The Natural Rate Theory

Sargent obtained his natural rate theory from first principles using a model described in Sargent (1987). In McGough (2000) we modified this model to include oil as a factor of production. The resulting aggregate supply can then be constructed and, appealing to Okun's law, transformed into an expectations augmented Phillips curve which includes a real oil price term. The resulting natural rate theory is given by

$$u_t = u^* - \frac{\theta}{1-\rho_2 L}(\pi_t - \pi_t^e) + \frac{g}{1-\rho_1 L}R_t + \frac{1}{1-\rho_3 L}\nu_{3t}, \tag{9}$$

where R_t represents real oil price at time t, and is assumed to follow the process $R_t = \bar{R} + \mu_t$, with μ_t white noise. Consistent with Sargent's dynamic model, we make the simplifying assumption that $\rho_i = 0$, thus eliminating the lag terms. Notice that the natural rate of unemployment implied by this model is now given by $u^* + g\bar{R}$.

3.1.2 A Statistical Phillips Curve

The government believes real oil prices influence the statistical Phillips curve. Set

$$\begin{aligned} X_t &= [\pi_t, \pi_{t-1}, u_t, u_{t-1}, 1]' \\ \Lambda_t &= [R_t, R_{t-1}]' \end{aligned}$$

The government's perceived Phillips curve is given by

$$u_t = \gamma_1 \pi_t + \gamma_{-1}' X_{t-1} + \delta' \Lambda_t + \varepsilon_t. \tag{10}$$

Notice government's beliefs are summarized by the vector $[\lambda, \delta]$.

3.1.3 The Phelps Problem

The government sets inflation up to white noise to maximize its criterion, subject to its perceived constraint, the statistical Phillips curve, as well as the process followed by real oil prices. The government's problem is written

$$\max_{\hat{\pi}_t} (1-\beta) E \sum_{t=1}^{\infty} -\frac{\beta^{t-1}}{2} (u_t^2 + \pi_t^2)$$

$$\begin{aligned} s.t.\ u_t &= \gamma' \begin{bmatrix} \pi_t \\ X_{t-1} \end{bmatrix} + \delta' \Lambda_t + \varepsilon_t \\ \pi_t &= \hat{\pi}_t + \nu_{2t} \\ R_t &= \overline{R} + \mu_t \end{aligned}$$

A noted earlier, this is the Phelps's problem. The solution to this dynamic programming problem is a feedback rule of the form

$$\hat{\pi}_t = h(\gamma, \delta)' \begin{bmatrix} X_{t-1} \\ \Lambda_t \end{bmatrix}.$$

Analytic computation of h is intractable; a numerical algorithm is presented in Appendix A.

3.1.4 Self-confirming Equilibrium

The economy is in a self-confirming equilibrium provided the data generated by the natural rate theory and the Phelps problem confirm the beliefs of the government concerning the slope of the Phillips curve. Formally, the time-series must satisfy the following orthogonality condition:

$$E\left(u_t - [\gamma', \delta'] \begin{bmatrix} \pi_t \\ X_{t-1} \\ \Lambda_t \end{bmatrix}\right)\left[\pi_t, X'_{t-1}, \Lambda'_t\right] = 0.$$

Note that the self-confirming equilibrium is characterized by the associated beliefs $[\gamma, \delta]$.[12]

Following Sargent, define the map $T : \mathrm{R}^8 \to \mathrm{R}^8$ by

$$E\left(u_t - T(x)' \begin{bmatrix} \pi_t \\ X_{t-1} \\ \Lambda_t \end{bmatrix}\right)\left[\pi_t, X'_{t-1}, \Lambda'_t\right] = 0.$$

A self-confirming equilibrium is a fixed point of this map. Fixed points may be found using a recursive algorithm such as

$$x_{i+1} = \kappa x_i + (1 - \kappa) T(x_i),$$

provided the map T may be evaluated for arbitrary x. Here, $\kappa \in [0, 1]$. Analytic computation of T is intractable. An algorithm yielding its numerical computation is provided in Appendix A.

3.1.5 Adaptive Policy Makers

Policy makers form their beliefs using the same algorithm as before. Set $\phi_t = \left[\pi_t, X'_{t-1}, \Lambda'_t\right]'$. The algorithm is given by

$$\begin{aligned} \begin{bmatrix} \gamma_t \\ \delta_t \end{bmatrix} &= \begin{bmatrix} \gamma_{t-1} \\ \delta_{t-1} \end{bmatrix} + \varepsilon P_t^{-1} \phi_t \left(u_t - \left[\gamma'_{t-1}, \delta'_{t-1}\right] \phi_t\right), \\ P_t &= P_{t-1} + \varepsilon(\phi_t \phi'_t - P_{t-1}). \end{aligned}$$

3.2 The Government's Propensity to Escape

The striking feature of Sargent's model is its prediction that the economy will periodically escape the self-confirming equilibrium and inflation will be driven down to very low levels. Sargent explains this behavior as depending on the comovements of the sum of the coefficients on inflation (SCI) and the constant term in the beliefs parameters of the government. The intuition is quite simple: The stochastic nature of the model together with the weighted estimation mechanism of the government will

[12] Of course the specific behavior of unemployment and inflation in equilibrium will also depend on the values of the parameters charaterizing the natural rate theory.

occasionally cause the government to believe the trade-off between unemployment and inflation has worsened. The manifestation of this belief is an increase in the SCI toward zero. (In SCE, the SCI will be negative, indicating a long run Phillips curve.) If the government believes the trade-off has worsened, it will lower inflation. The new data generated by the model will likely show the resulting trade-off has further worsened and this process will continue until inflation is set near zero. The role of the constant term may be seen as follows: Suppose $SCI = 0$. If the constant term in the perceived Phillips curve is positive and, say, fairly large, then high long run unemployment is indicated. An option of the government is to set inflation equal to zero for all t. This is not optimal however, because there are no first order costs to raising inflation slightly, but because of its linear effect on unemployment there are first order benefits. Also, the smaller the constant term, the smaller the benefit. We conclude that for the solution of the Phelps problem to advise setting inflation to zero, both the SCI and the constant term must be near zero.

Experiments involving supply shocks will display behavior not seen when Sargent's original model is considered. In particular, we will find that the economy may stay near the SCE even when the comovements of the SCI and the constant term indicate an escape should occur. To explain this behavior, we must look a little more closely at the advice given by the Phelps problem. For simplicity, we consider the model without oil, though the argument, and resulting measure are analogous if oil is included.

Set $X_t = [\pi_t, \pi_{t-1}, u_t, u_{t-1}, 1]'$. The government's perceived Phillips curve may then be written

$$u_t = \gamma_1 \pi_t + \gamma_{-1} X_{t-1} + \varepsilon_t.$$

Given beliefs γ, the government sets the inflation target at $\hat{\pi}_t = h(\gamma)' X_{t-1}$. The resulting inflation rate is some zero mean shock of this target. The associated process (u_t, π_t) is asymptotically stationary. Denote the mean by $(\overline{u}, \overline{\pi})$. Then

$$\overline{u} = \left[\frac{\gamma_1 + \gamma_2 + \gamma_3}{1 - \gamma_4 - \gamma_5} \right] \overline{\pi} + \frac{\gamma_6}{1 - \gamma_4 - \gamma_5}.$$

The coefficient modifying $\overline{\pi}$, which we call the government's propensity to escape, or $GPE(\gamma)$, represents the government's perceived long run trade-off between mean unemployment and mean inflation. Notice the numerator of this measure is the SCI, so that when the SCI is near zero, then, provided the denominator is not too small, the trade-off will be small and the government will set inflation to zero. This is consistent with Sargent's argument. However, the above measure also allows the government to set inflation relatively high even with an SCI near zero if the denominator is near zero as well. And this is exactly what happens in the presence of large positive shocks to the natural rate.

3.3 Shocks to the Natural Rate of Unemployment

We begin our analysis by considering shocks to u^*. The model is calibrated to remain consistent with Sargent. Also, since we are not considering shocks to real oil price, the results presented in this section correspond to the simplified dynamic model in which real oil price is ignored. Thus we take $u^* = 5$, $\theta = 1$, and the variances of the noise terms again equal to .09. The shocks to the natural rate will be implemented just as before.

3.3.1 Negative Shocks

Simulations of the dynamic model yield time-paths that are very similar to those obtained from the static model. The SCE mean level of inflation is again 5%, and we find that in the non-shocked economy, the inflation rate tends to stay near its SCE level, but, as before, endogenous escapes do occur which send the inflation rate plummeting to near zero. Also, as before, we find that even a small negative shock greatly reduces the expected time to empirical escape, as is demonstrated now.

Again, denote by $\Phi(\Delta u^*)$, the random variable representing the time of first escape. Begin by taking, as a benchmark, the case of no shock. The estimated density of $\Phi(0)$ is given in Figure 5. The sample mean is 1.38. Note that the expected escape time is considerably sooner in the dynamic case than it is in the static case. This is consistent with results obtained by Cho, Williams, and Sargent.

FIGURE 5 HERE

Consider now the estimated density of $\Phi(-.5)$, as shown in Figure 6. We see that the shape of the density is greatly affected by this shock. The sample mean is .74, but the size of this sample mean reflects the extent to which the distribution is right tailed. In particular, the median is .28.

FIGURE 6 HERE

We also see that, unlike before, this estimated density appears to be bimodal, with a small symmetric hump centered at about 1.8. The reason for this second hump is due to some interesting behavior, which reveals itself in the static model as well, if the variance of the noise terms is reduced, and if larger shocks are considered. It turns out that, in a relatively small proportion of simulations, instead of instigating an escape, the negative shock will cause the inflation rate to jump to a new, higher level, and remain there until a true escape occurs.[13] The second hump is capturing the

[13]The movement to a higher level of inflation is, in fact, predicted by the mean dynamics. To conserve space, we do not report an example simulation.

distribution of true escapes times for the simulations in which this type of behavior occured.

The estimated density of $\Phi(-.25)$ is shaped similarly to that of $\Phi(-.5)$, though somewhat less distorted from that of $\Phi(0)$. Also, it does not display the bimodal feature of $\Phi(-.5)$. This is consistent with the observation that large shocks are required for the previously mentioned jump in the inflation rate to occur. The sample mean is .85. We conclude that even small negative shocks to the natural rate of unemployment yield changes in the escape time densities and significantly lower expected escape times.

3.3.2 Positive Shocks

In the dynamic model, simulations suggest that small positive shocks have dynamic effects similar to those in the static model; the inflation rate tends to rise slowly to its new SCE level. The effect of a small shock on the escape time density is not dramatic. The sample mean of $\Phi(.25)$ is 1.75, thus indicating that the expected escape time is increased somewhat. To conserve space, we do not include estimated densities here. Larger positive shocks have similar descriptive statics.

Some interesting behavior is observed when simulating large positive shocks. Figure 7 displays a simulation in which a shock of $\Delta u^* = 1$ occurs at time $t = .3$, that is, the model is initialized in SCE and allowed to run for 30 periods; then u^* is altered and the model continues. Notice that the shocked economy (first panel) exhibits no noticeable effect at the time of this shock, though, longer simulations suggest that the time-series may be slowly rising toward its new SCE level. On the other hand, the third panel, which displays the sum of the coefficients on inflation, shows an instant reaction to the shock; in particular, this sum heads toward zero and stays near there. This is in apparent conflict with the intuition that as the SCE goes to zero, the government's perceived trade-off diminishes and hence the government should lower inflation.[14] However, this conflict is resolved by observing the bottom panel, which displays the government's propensity to escape. This panel shows that, even though the SCI heads toward zero, the GPE stays near -1, thus advising the government of a persistent long run trade-off and inclining them to keep inflation high. Note that movements in the coefficients on lagged unemployment are responsible for keeping the GPE near -1 while the SCI is near zero. We also see that an escape occurs precisely when the GPE spikes above zero and then quickly decreases back to zero thus indicating the elimination of the long run trade-off, and hence the activation of the induction hypothesis.

FIGURE 7 HERE

[14] To simplify exposition, we are ignoring the impact of the constant term.

3.4 Shocks to Real Oil Prices

Now we turn to analyzing the impact of changes in the real price of oil. The model is calibrated to be consistent with Sargent's parameter values, as well as roughly consistent with real data, by using the following observation: During the time-period 1955 to 1970 the mean real oil price was about \$10 and the mean unemployment rate was about 5%. During the time-period 1970 to 1985, the mean real oil price was about \$20 and the mean unemployment rate was about 7%. For the purposes of simulation, we will assume a doubling of real oil price leads to 2% increase in the natural rate of unemployment, and choose parameter values consistent with this assumption.[15] Thus, we set $u^* = 3$, we normalize $\bar{R} = 1$, and set $g = 2$ so that the corresponding natural rate of unemployment remains at 5%.

A shock to the real oil price is modeled in a way analogous to shocks to the natural rate of unemployment. The model is initialized in an SCE corresponding to $\bar{R} = 1$ in case of positive oil shocks, and $\bar{R} = 2$ in case of negative oil shocks. Then, before the simulation is begun, $\bar{R}$ is shocked by $\Delta\bar{R}$. We denote by $\Phi(\Delta\bar{R})$ the random variable representing the time of first empirical escape.

We begin with the estimated density in case of no shocks to $\bar{R}$. While intuitively this density should coincide with the density represented in Figure 5, the model here is different in that the government includes real oil prices in its regression; thus we estimated the density for completeness: See Figure 8.[16] We then estimated the densities of $\Phi(1)$ and $\Phi(-1)$. We find that the impact of the shock on the estimated density's shape appears much less pronounced than it was when the shock was unobserved. (Because the estimated densities do not tell an important story here, we only report the estimated density of $\Phi(0)$.)[17]

FIGURE 8 HERE

On the other hand, shocks to real oil prices do appear to alter mean escape times (and hence escape time densities). The sample means of both $\Phi(-1)$ and $\Phi(1)$ are significantly smaller than the sample mean of $\Phi(0)$. The reason for this are not clear to us.

[15] These estimates are back of the envelope and we are well aware that the variance of real oil price in the second time period was much greater than in the first. We do not intend our model to be empirically accurate. We use these statistics to remain in touch with, if not true to, reality.

[16] It is interesting to note that, in fact, the estimated densities in Figure 5 and Figure 8 do not appear to coincide, thus suggesting that the model specification is relevant, even when parameters are chosen in a consistent fashion.

[17] It may be possible to address rigorously whether the estimated densities are significantly different by appealing to statistical analysis of the non-parametric estimators. However, due to the apparent difficulty of choosing the optimal bandwidth (see Appendix B), we avoid this issue for now.

The fact that the impact of the oil shocks on escape time densities is relatively small is easily understood once a sample simulation is observed. Consider Figure 9. This graph represents the time-series of a simulation in which a positive shock to real oil price occurs at time $t = .3$. We see from this Figure that, at the time of the shock, the inflation rate appears to jump from its old SCE mean of 5% to its new SCE mean of 7%. More generally, in case of a permanent shock to the mean real oil price, the economy appears to jump instantly to the new SCE. And this is not surprising; the change in $\bar{R}$ has no impact on the SCE value of the government's beliefs γ. When $\bar{R}$ increases, the mean value of u_t increases as well. But since R_t is observed by the government and used as a regressor, the government is already accounting for the impact on u of a change in real oil price. The government is not surprised by the change in u and thus has no reason to alter its beliefs. Since the shock does not cause the government to alter its beliefs, the parameters of its policy response are not changed; hence, if the economy is in SCE when the shock occurs, the economy is also in SCE after the shock occurs, and now the mean SCE level of inflation corresponds to a higher mean unemployment rate. Thus, in the simulation, we witness the inflation rate jumping instantly to its new mean level at the time of the shock.

While the shock to the real oil price does not appear to jar the economy out of SCE, the decrease in mean escape time indicates a destabilizing effect. As noted above, this effect is not captured by immediate movements in the regression coefficients, but can perhaps be attributed to the dynamics of the covariance matrix P. The oil shock alters some of the covariance terms and thus may make the regression estimates more responsive to changes in forecast errors.

FIGURE 9 HERE

3.5 Dynamic Discussion

In case of shocks to the natural rate of unemployment, the dynamic model appears to produce some behavior consistent with the static model, though in case of a negative shock, the magnitude of the shift in the mean escape time is even more pronounced. The economic implications are quite similar; because it lowers SCE inflation and unemployment and may promote escape to Ramsey, a negative shock may be doubly good, and because it raises SCE inflation and unemployment and may postpone escape to Ramsey, a positive shock may be doubly bad. We also learned that measuring the long run trade-off between inflation and unemployment as the government's propensity to escape may explain the behavior of the inflation target better than just analyzing the sum of the coefficients on inflation. Finally, we found that, provided the economy is in an SCE, observed shocks to the real price of oil cause the economy to immediately move to the new SCE level. In case of positive shocks to the real oil price, the predicted result is thus stagflation.

3.6 Conclusion

Sargent (1999), and Cho, Williams, and Sargent (2003), developed models in which misspecification leads to astonishing dynamics in the form of escape routes. These authors further show that the activation of these escapes is caused by a sequence of unlikely realizations of the model's intrinsic i.i.d. shocks. In this paper, we offer an alternative explanation for the occurrence of escape-like behavior. We find that this behavior, which we call empirical escapes, can result from non-i.i.d exogenous supply shocks modeled as permanent shifts to structural parameters.

We considered two types of shocks: unobserved shocks to the natural rate of unemployment; and observed shocks to the real price of oil. Oil price shocks shift the self-confirming equilibrium, and this shift is tracked by the policy makers if they include real oil price in their regression model. Unobserved favorable shocks in the natural rate of unemployment shift the SCE favorably (i.e. downward), but are also likely to trigger escape dynamics, resulting in a dramatic fall in the rate of inflation. We found that this type of behavior is well explained by the mean dynamics. We also found that a measure called the Government's Propensity to Escape (GPE) provides a simple and intuitive way to summarize the likelihood of an empirical escape path in the dynamic specification.

Further work is suggested by this paper. It may be possible to obtain an analytic description of the escape time densities, similar in vein to those of Williams (2002a), though it is difficult to see how such an analysis would capture the double-humped nature of some of the densities. Also, there is clearly some empirical work to be done. Specifically, one could, in principle, test for the activation of an empirical escape following a sudden shift in the natural rate of unemployment. Furthermore, the results of this paper suggest that including real oil prices, as well as relaxing the government's objective function by parameterizing the relative weights placed on unemployment and inflation, may greatly improve the fit of Sargent's model. We intend to address this in future research.

Appendix A: The Dynamic Phelps Problem and SCE Computation

The Phelps Problem

Solutions to specific linear-quadratic dynamic programming problems are well known. However, we were unable to transform our problem into a form who solution is presented in the literature. Thus we turn to first principles to obtain a solution.

Consider a problem of the form

$$\max_{\bar{\pi}_t} -E \sum_{t=0}^{\infty} \beta^t \left(X_t' R X_t + \bar{\pi}_t^2 Q + 2X_t' W \bar{\pi}_t + f\left(\omega_{t+1}\right)\right) \tag{11}$$

$$\begin{aligned} s.t. \quad X_{t+1} &= AX_t + B\bar{\pi}_t + C\omega_{t+1} \\ K &= E_t\left(f\left(\omega_{t+1}\right)\right) \end{aligned}$$

with initial condition X_0. Let $V\left(X_0\right)$ be the value function of the above problem. Then

$$V\left(X_0\right) = \max_{\pi} \begin{pmatrix} X_0' R X_0 + \bar{\pi}_0^2 Q + 2X_0' W \bar{\pi}_0 \\ +E_0\left(f\left(\omega_1\right)\right) + \beta E_0\left(V\left(X_1\right)\right) \end{pmatrix}, \tag{12}$$

and the solution $\hat{\pi}_t$ to problem (11) must solve

$$\hat{\pi}_t = \arg\max_{\bar{\pi}_t} \begin{pmatrix} X_t' R X_t + \bar{\pi}_t^2 Q + 2X_t' W \hat{\pi}_t \\ +E_t\left(f\left(\omega_{t+1}\right)\right) + \beta E_t\left(V\left(X_{t+1}\right)\right) \end{pmatrix}.$$

These problems are representative of Bellman's functional equations.

To solve (11), then, we first attempt to find a function V satisfying (12). We guess a solution of the form

$$V\left(X\right) = X' P X + d,$$

where P is symmetric.[18] Then

$$\begin{aligned} V\left(X_1\right) &= X_0' A' P A X_0 + \pi B' P B \pi + \omega_1 C' P C \omega_1 \\ &\quad +2X_0' A' P B \pi + 2X_0' A' P C \omega_1 + 2\pi B' P C \omega_1. \end{aligned}$$

Thus

$$X_0' P X_0 + d = \max_{\pi} \begin{pmatrix} X_0'\left(R + \beta A' P A\right) X_0 \\ +\pi\left(Q + \beta B' P B\right)\pi + 2X_0'\left(W + \beta A' P B\right)\pi + \hat{d} \end{pmatrix}$$

where

$$\hat{d} = \beta d + E_0 f\left(\omega_1\right) + \beta tr\left(B' P C Var\left(\omega_1\right)\right).$$

[18] See Sargent and Hansen (2001), for this trick.

Setting $\frac{\partial}{\partial \pi} = 0$ yields

$$\pi = -\left(Q + \beta B'PB\right)^{-1}\left(W' + \beta B'PA\right)X_0.$$

We may now substitute this back into the objective function to obtain a relation for P :

$$\begin{aligned} X_0'PX_0 + d &= X_0'\left(R + \beta A'PA\right)X_0 + \hat{d} \\ &\quad - X_0'\left(W + \beta A'PB\right)\left(Q + \beta B'PB\right)^{-1}\left(W' + \beta B'PA\right)X_0. \end{aligned}$$

Thus we find that P must satisfy the following equation:

$$P = R + \beta A'PA - \left(W + \beta A'PB\right)\left(Q + \beta B'PB\right)^{-1}\left(W' + \beta B'PA\right). \tag{13}$$

Analytic solutions to this Ricatti equation are in general not tractable. However, for specific values of the relevant matrices, equation (13) converges under iteration. Note that the same argument shows that the solution for $\hat{\pi}_t$ is given by

$$\hat{\pi}_t = -\left(Q + \beta B'PB\right)^{-1}\left(W' + \beta B'PA\right)X_t.$$

Thus, the solution to the dynamic programming problem is a contingency plan depending on realizations of the stochastic process.

To solve our specific problem, it then remains to show that it may be placed in the form of (11). Recall we are to solve

$$\max_{\hat{\pi}_t}\left(1 - \beta\right)\sum_{t=1}^{\infty}\frac{-\beta^{t-1}}{2}\left(u_t^2 + \pi_t^2\right)$$

$$\begin{aligned} s.t. \quad u_t &= \gamma'\begin{bmatrix} \pi_t \\ X_{t-1} \end{bmatrix} + \delta'\Lambda_t + \varepsilon_t \\ \pi_t &= \hat{\pi}_t + \nu_{2t} \\ R_t &= \alpha + \hat{\beta}R_{t-1} + \nu_{3t} \end{aligned}$$

where

$$X_t = [\pi_t, \pi_{t-1}, u_t, u_{t-1}, 1]', \Lambda_t = [R_t, R_{t-1}]'$$

and R_t is assumed known at time t. To transform this model into the correct form we set $\hat{X}_t = [X_{t-1}', \Lambda_t']'$ and

$$\omega_{t+1} = [\varepsilon_t, \nu_{2t}, \nu_{3t+1}]'.$$

Note that by information assumptions, $E_t(\omega_{t+1}) = 0$. Define the following matrices:

$$B = \begin{bmatrix} 1 & 0 & \gamma_1 & 0 & 0 & 0 & 0 \end{bmatrix}'$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \delta_1 & \delta_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 & o \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}'$$

Then

$$\hat{X}_{t+1} = AX_t + B\hat{\pi}_t + C\omega_{t+1}.$$

Now set $\hat{\gamma} = [\gamma', \delta']'$ and

$$\begin{aligned} f(\omega_t) &= \left(1+\hat{\gamma}_1^2\right)\omega_{2t}^2 + \omega_{1t}^2 + 2\hat{\gamma}_1\omega_{1t}\omega_{2t} \\ &\quad +2\left(\left(\hat{\gamma}_1^2+1\right)\hat{\pi}_{t-1} + \hat{\gamma}_1\left(\hat{\gamma}_{-1}'\hat{X}_{t-1}\right)\right)\omega_{2t} \\ &\quad +2\left(\hat{\gamma}_1\hat{\pi}_{t-1} + \hat{\gamma}_{-1}'\hat{X}_{t-1}\right)\omega_{1t}. \end{aligned}$$

Then, with $W = \hat{\gamma}_{-1}\hat{\gamma}_1$, $R = \hat{\gamma}_{-1}\hat{\gamma}_{-1}'$, and $Q = 1 + \hat{\gamma}_1^2$ we have

$$u_t^2 + \pi_t^2 = \hat{X}_t' R \hat{X}_t + \hat{\pi}_t^2 Q + 2\hat{X}_t' W \hat{\pi}_t + f(\omega_{t+1}).$$

Thus we may apply the above solution to our Phelps problem.

Computing Self-confirming Equilibria

For a given set of parameter values, the associated self-confirming equilibrium may be computed using the T-map as described above. Recall that the $T-$ map is given by

$$E\left(u_t - T(\hat{\gamma})' \begin{bmatrix} \pi_t \\ X_{t-1} \\ \Lambda_t \end{bmatrix}\right)[\pi_t, X_{t-1}, \Lambda_t] = 0,$$

where $\hat{\gamma} = [\gamma', \delta']'$. Set $\hat{X}_t = \left[\pi_t, X_{t-1}', \Lambda_t'\right]'$. Then

$$T(\hat{\gamma}) = E\left(\hat{X}_t\hat{X}_t'\right)^{-1} E\left(\hat{X}_t u_t\right).$$

Thus we must obtain numerical computations of $E\left(\hat{X}_t\hat{X}_t'\right)^{-1}$ and $E\left(\hat{X}_tu_t\right)$. To do this, we use the technique suggested by Sargent. Because of our assumptions on the parameters of the model, we may write

$$\begin{aligned} u_t &= g_0 + g_2\nu_{2t} + g_4\nu_{1t} \\ \pi_t &= h(\hat{\gamma})'[X_{t-1}', \Lambda_t']' + \nu_{2t} \\ R_t &= \overline{R} + \nu_{3t} \end{aligned}$$

Let $\tilde{X}_t$ be X_t without the constant term. Define the following matrices:

$$Y_t = [\nu_{2t}, \nu_{3t}, \tilde{X}_{t-1}, R_t, R_{t-1}]'.$$

Using the true processes of the relevant variables, we may write

$$Y_t = A + BY_{t-1} + C\hat{\nu}_t.$$

It follows that

$$\begin{aligned} E\left(Y_t\right) &= \left(I - B\right)^{-1} A \\ Var\left(Y_t\right) &= BVar\left(Y_t\right)B' + CVar\left(\hat{\nu}_t\right)C'. \end{aligned}$$

The second equation is a Sylvester equation and may be solved using a period doubling algorithm. Specifically, set

$$\alpha_0 = B, \quad \beta_0 = B', \text{ and } \xi_0 = W.$$

Then iterating the system

$$\begin{aligned} \xi_k &= \xi_{k-1} + \alpha_{k-1}\xi_{k-1}\beta_{k-1} \\ \beta_k &= \beta_{k-1}^2 \\ \alpha_k &= \alpha_{k-1}^2 \end{aligned}$$

causes β_k to converge to $Var\left(\bar{X}_t\right)$. See Anderson et al (1996) for details. Notice the we take $Var(\nu_t)$ as given.

From $Var(Y_t)$ and $E(Y_t)$ we may compute EY_tY_t'. To calculate the desired moments, proceed as follows. It is easy to write

$$\begin{aligned} \hat{x}_t &= \xi_1 + \xi_2 Y_t, \\ u_t &= \xi_3' Y_t. \end{aligned}$$

Then

$$\begin{aligned} E\left(\hat{X}_t\hat{X}_t'\right) &= \xi_1\xi_1' + \xi_1 E(Y_t)'\xi_2' + \xi_2 E(Y_t)\xi_1' + \xi_2 E(Y_tY_t')\xi_2' \\ E\left(\hat{X}_tu_t\right) &= \xi_1\xi_3' E(Y_t) + \xi_2 E(Y_tY_t')\xi_3 \end{aligned}$$

and are thus computable.

Appendix B: Non-Parametric Density Estimation

To aid our understanding of the impact of exogenous shocks we use non-parametric estimation of escape time densities as well as producing simple histograms. Let x_i, $i = 1, \cdots, n$, be a random sample from a given distribution. Let f be the true density of the distribution. The standard kernel estimator of f is

$$\hat{f}(x) = \frac{1}{hn} \sum_i K(\frac{x - x_i}{h}),$$

where K is a kernel, usually taken to be a symmetric density (the normal density is a standard choice) and h a choice parameter indicating the window size.

As Chen (2000), points out, this estimator with symmetric kernel is inappropriate in case f has support on $[0, \infty)$, as is the case with escape time densities. He proposes using a gamma kernel, and we employ his method here. Set

$$K_{x/b+1,b}(t) = \frac{t^{x/b} e^{-t/b}}{b^{x/b+1} \Gamma(x/b + 1)}.$$

The associated estimator is given by

$$\hat{f}(x) = \frac{1}{n} \sum_i K_{x/b+1,b}(x_i).$$

For this estimator, the bandwidth b is a choice parameter and plays the role of h.

It remains to choose an appropriate value of b for a given density estimation. A standard method is to employ Least Squares Cross Validation, which effectively chooses the value of b that minimizes the integrated squared error. Unfortunately, besides having an astoundingly slow rate of convergence, in our case the objective function in the optimization process (i.e. the estimate of the integrated squared error) appears to be very flat, making numerical optimization inaccurate and intractable.

If we wanted to perform statistical analysis of the density estimator, this failure to choose an optimal bandwidth would pose a serious problem. However, our intention here is to tell a qualitative story, and to use non-parametric estimation, in conjuction with histograms, to aid in the telling of that story. Therefore, we chose the bandwidth (which is reported at the top of each Figure containing a density estimation) simply by inspection.

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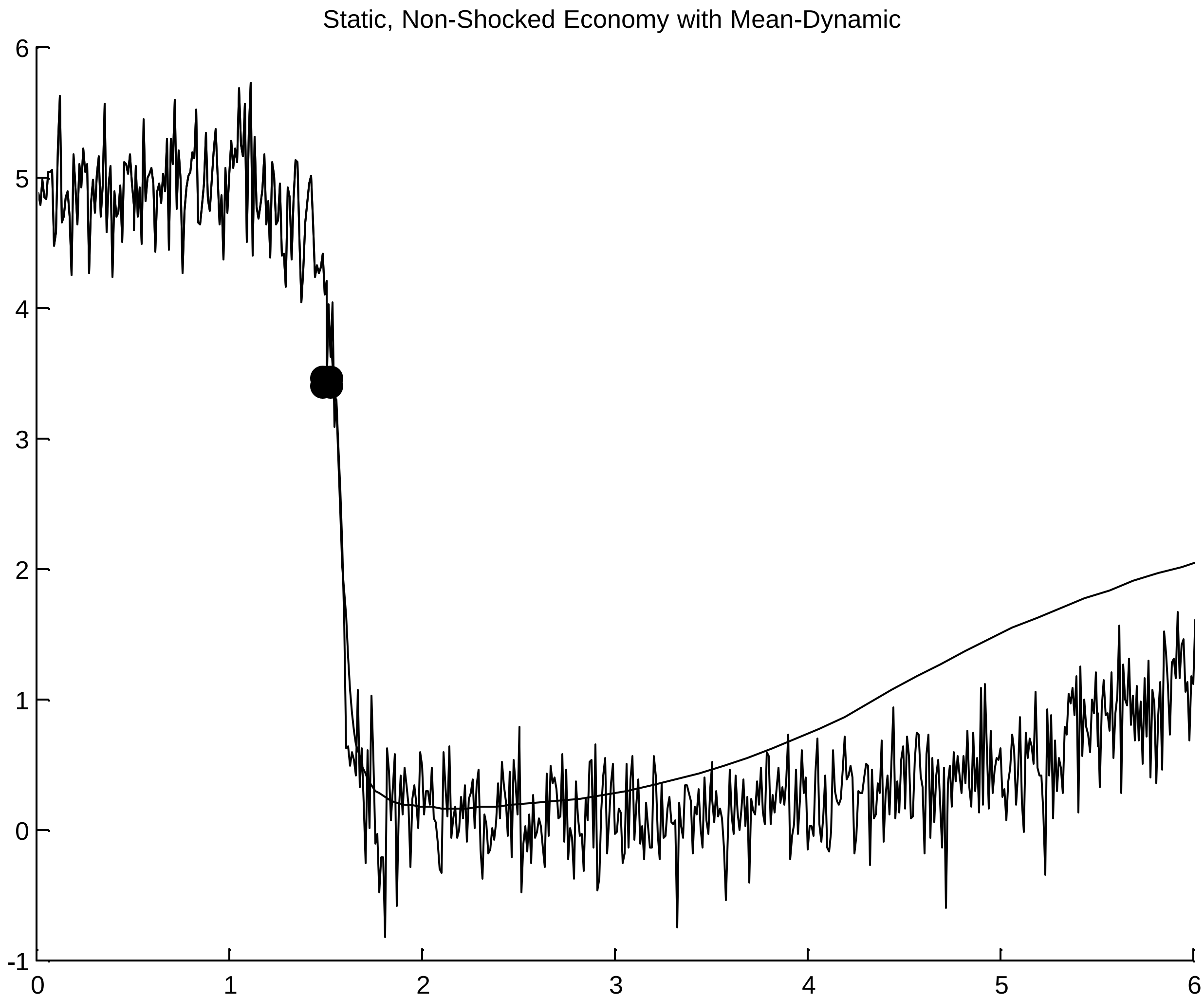
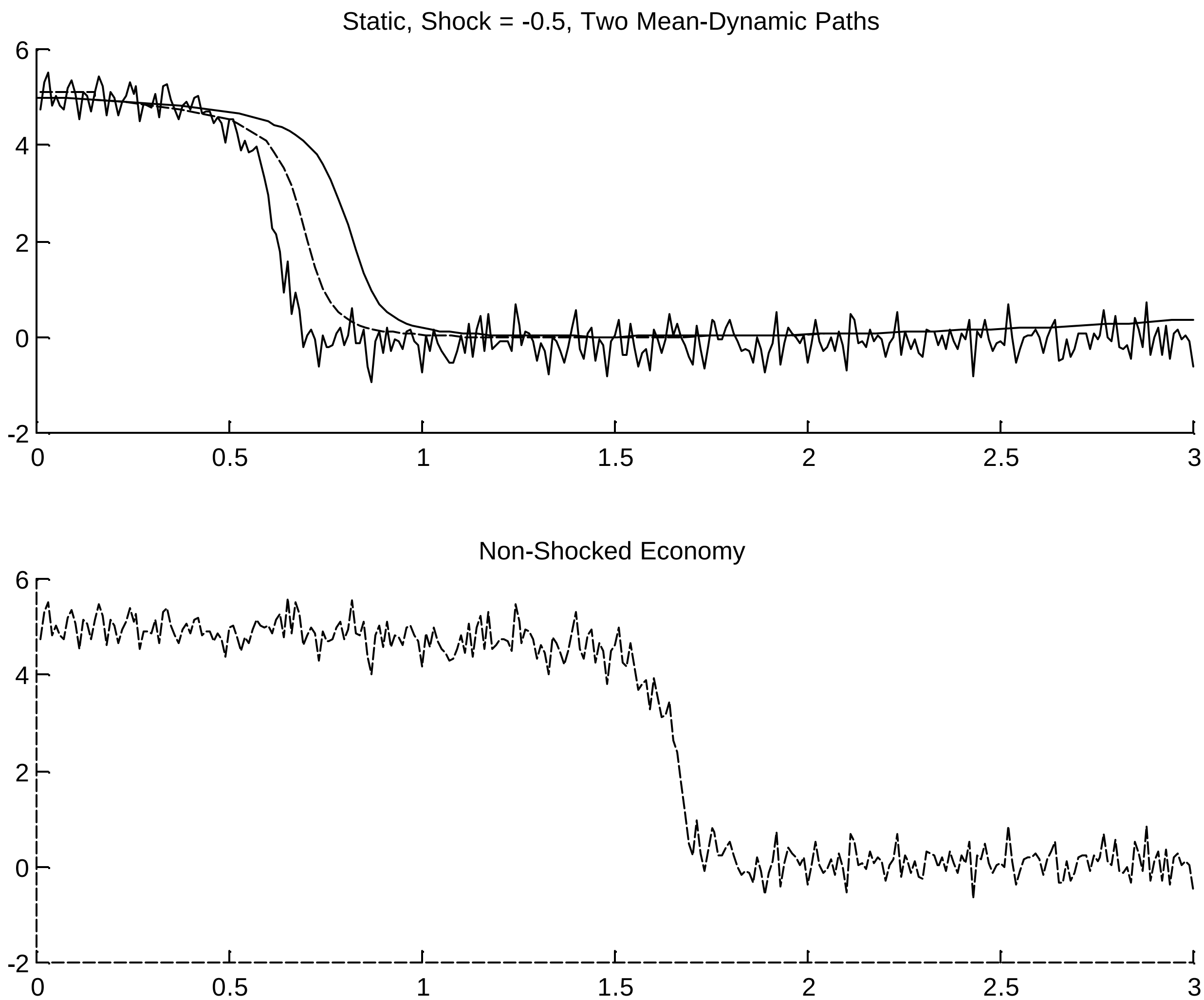
Table 1

Static Model		Dynamic Model, Shock to U*		Dynamic Model, Oil Shock	
Variable	*Mean*	*Variable*	*Mean*	*Variable*	*Mean*
$\phi(0)$	2.81 (1.35)	$\phi(0)$	1.38 (.7)	$\phi(0)$	.65 (.19)
$\phi(-.25)$	1.33* (.65)	$\phi(-.25)$	.85* (.71) .	$\phi(-1)$	.6* (.18)
$\phi(-.5)$	.79* (.4)	$\phi(-.5)$	74* (.89)	$\phi(1)$	.53* (.16)
$\phi(.25)$	3.95* (1.64)	$\phi(.25)$	1.75* (.68)	--	--

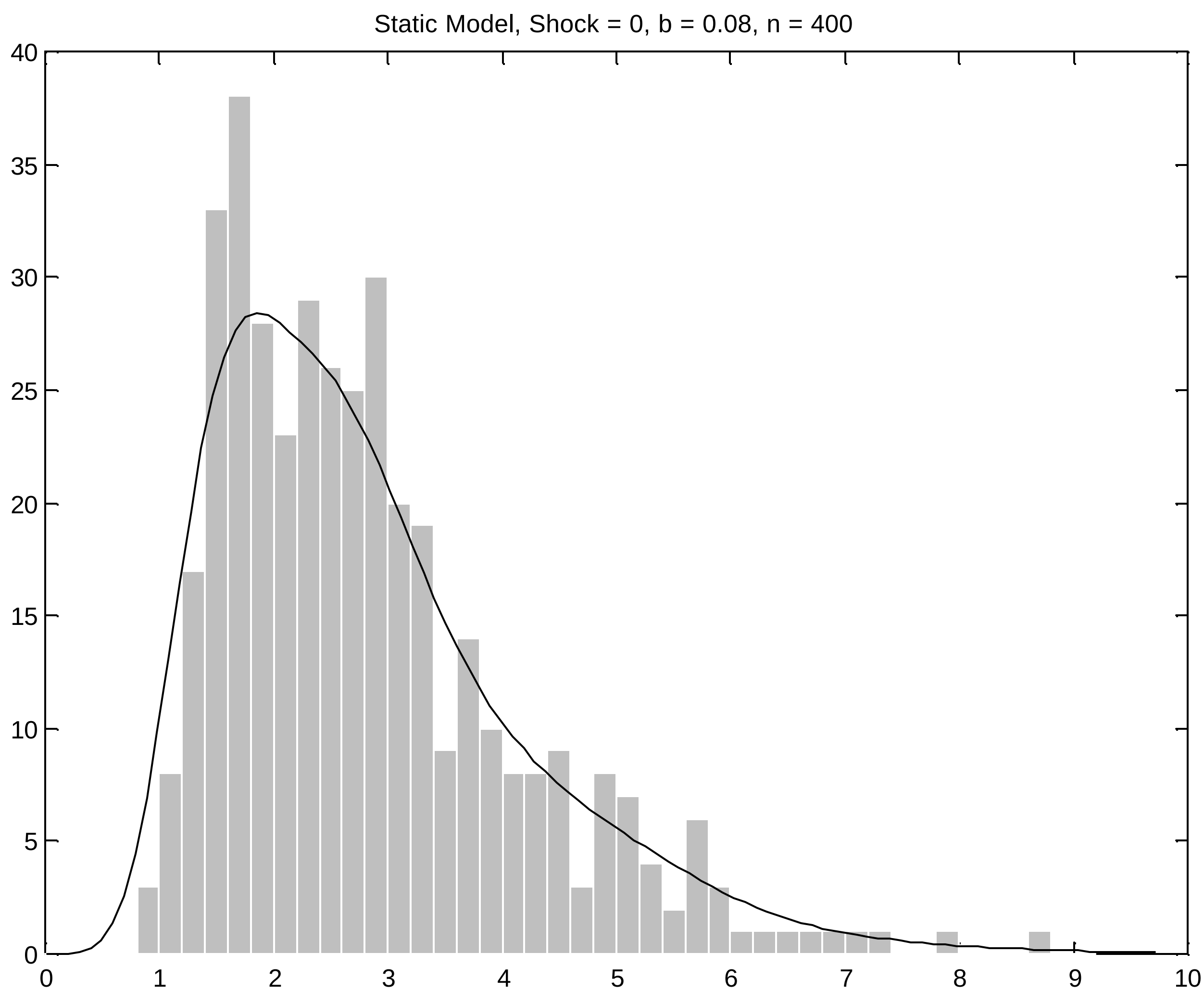
The asterisk modifying the sample mean indicates rejection at the 1% level of the null hypothesis that mean of the shocked economy is the same as the mean of the non-shocked economy.


Static, Non-Shocked Economy with Mean-Dynamic
6
5
4
3
2
1
0
-1
0
1
2
3
4
5
6
Figure 1



Static, Shock = -0.5, Two Mean-Dynamic Paths
6
4
2
0
-2
0
0.5
1
1.5
2
2.5
3
Non-Shocked Economy
6
4
2
0
-2
0
0.5
1
1.5
2
2.5
3

Figure 2


Static Model, Shock = 0, b = 0.08, n = 400
40
35
30
25
20
15
10
5
0
0
1
2
3
4
5
6
7
8
9
10
Figure 3

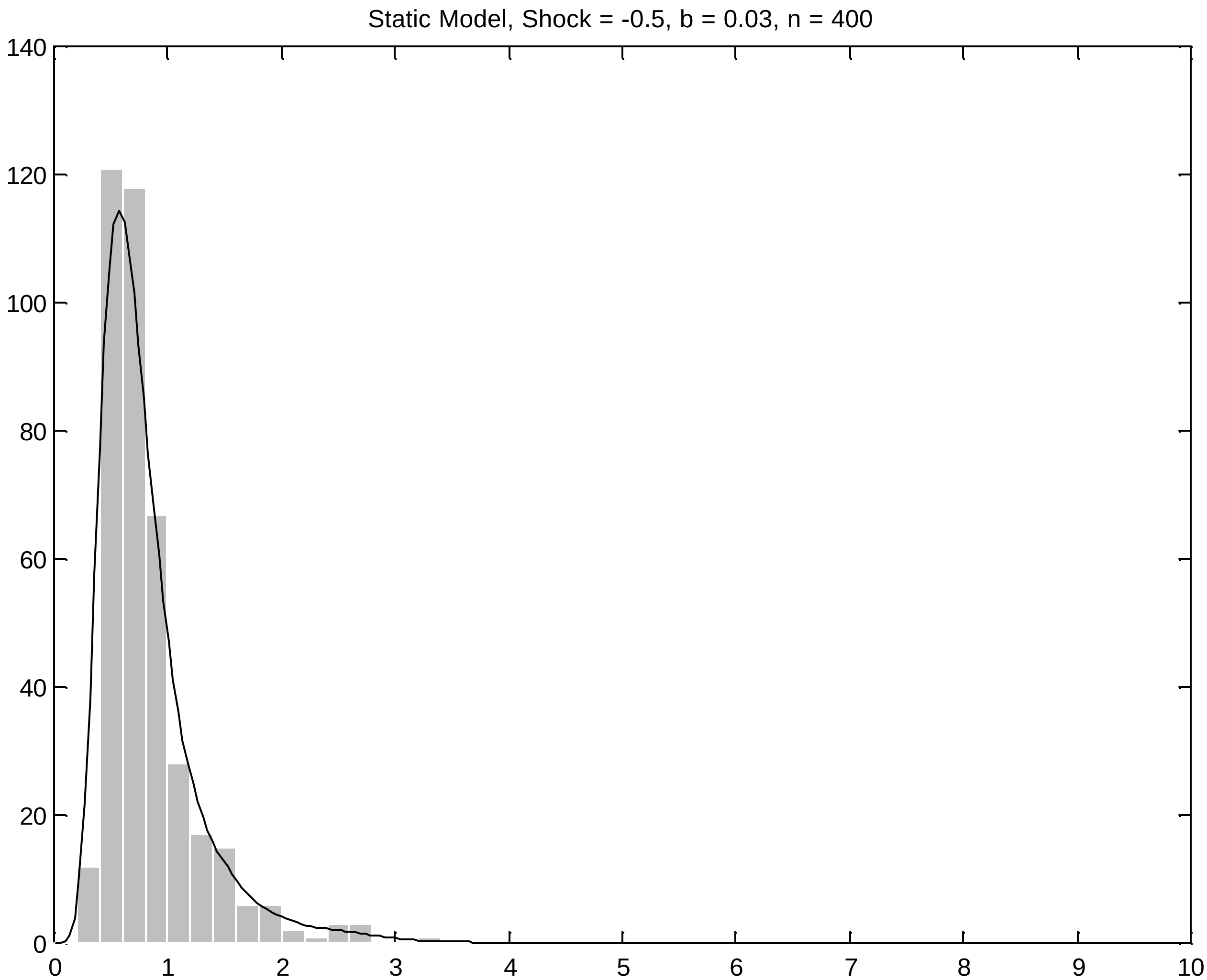
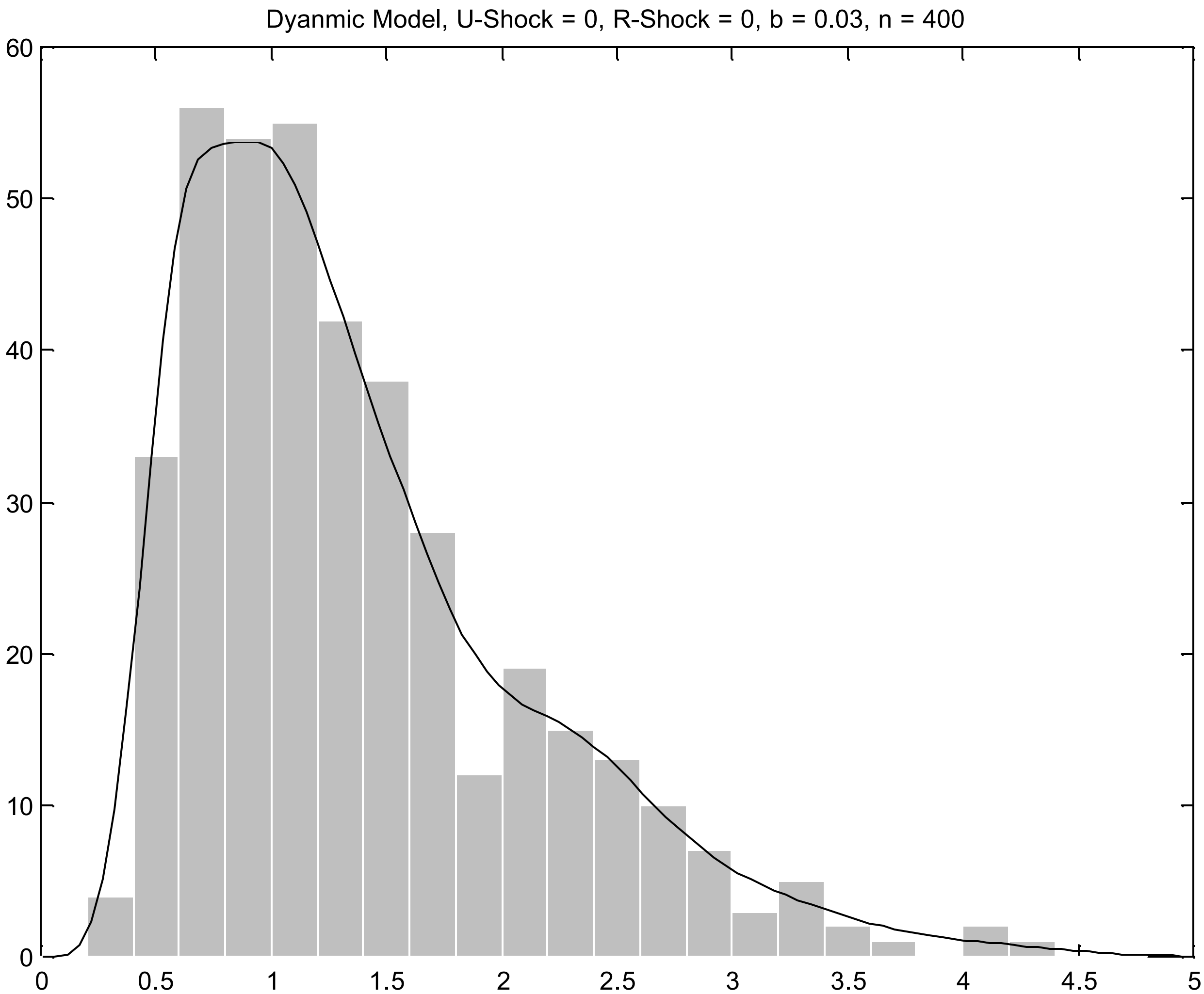
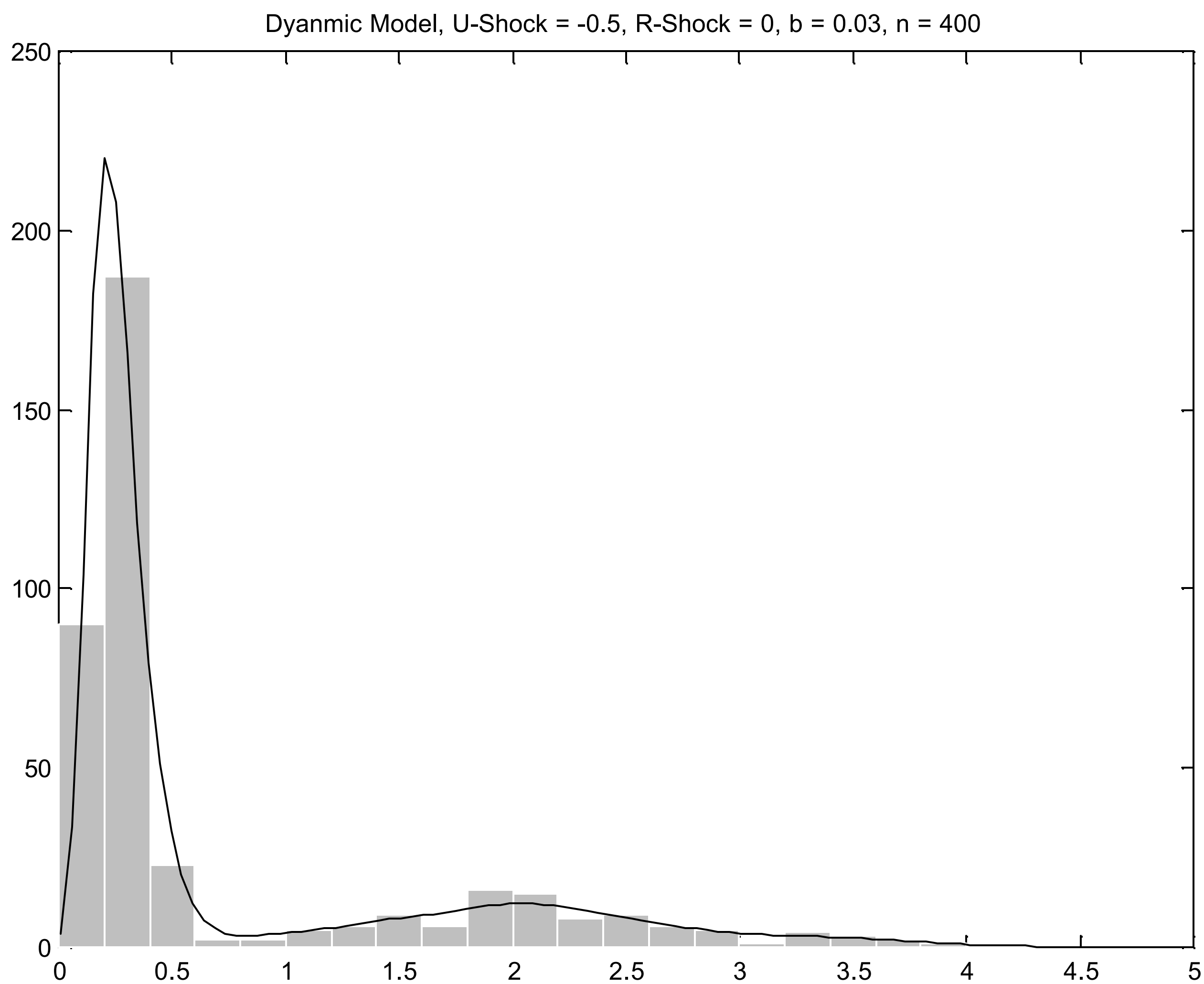

Static Model, Shock = -0.5, b = 0.03, n = 400
140
120
100
80
60
40
20
0
0
1
2
3
4
5
6
7
8
9
10

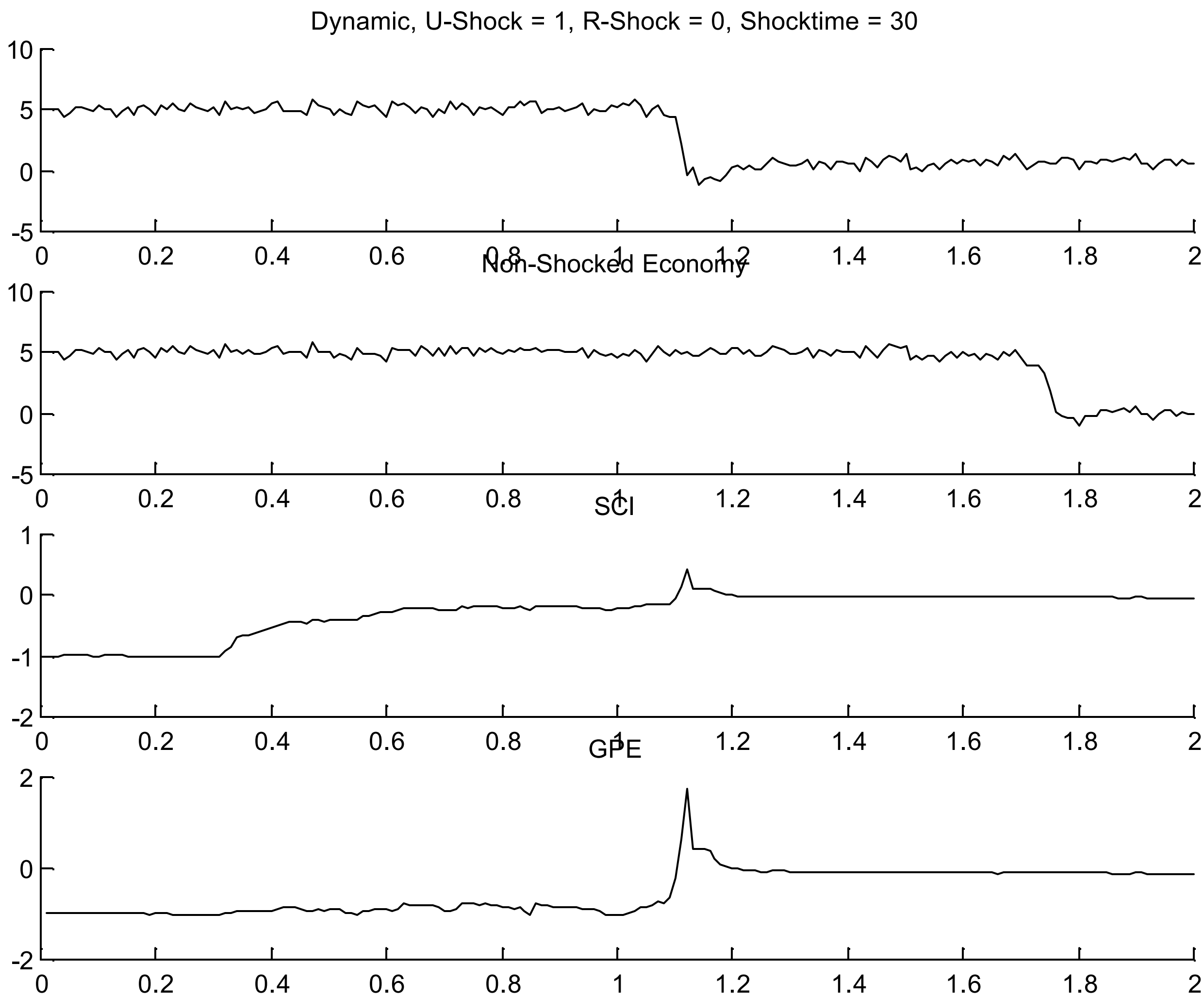
Figure 4


Dyanmic Model, U-Shock = 0, R-Shock = 0, b = 0.03, n = 400
60
50
40
30
20
10
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Figure 5



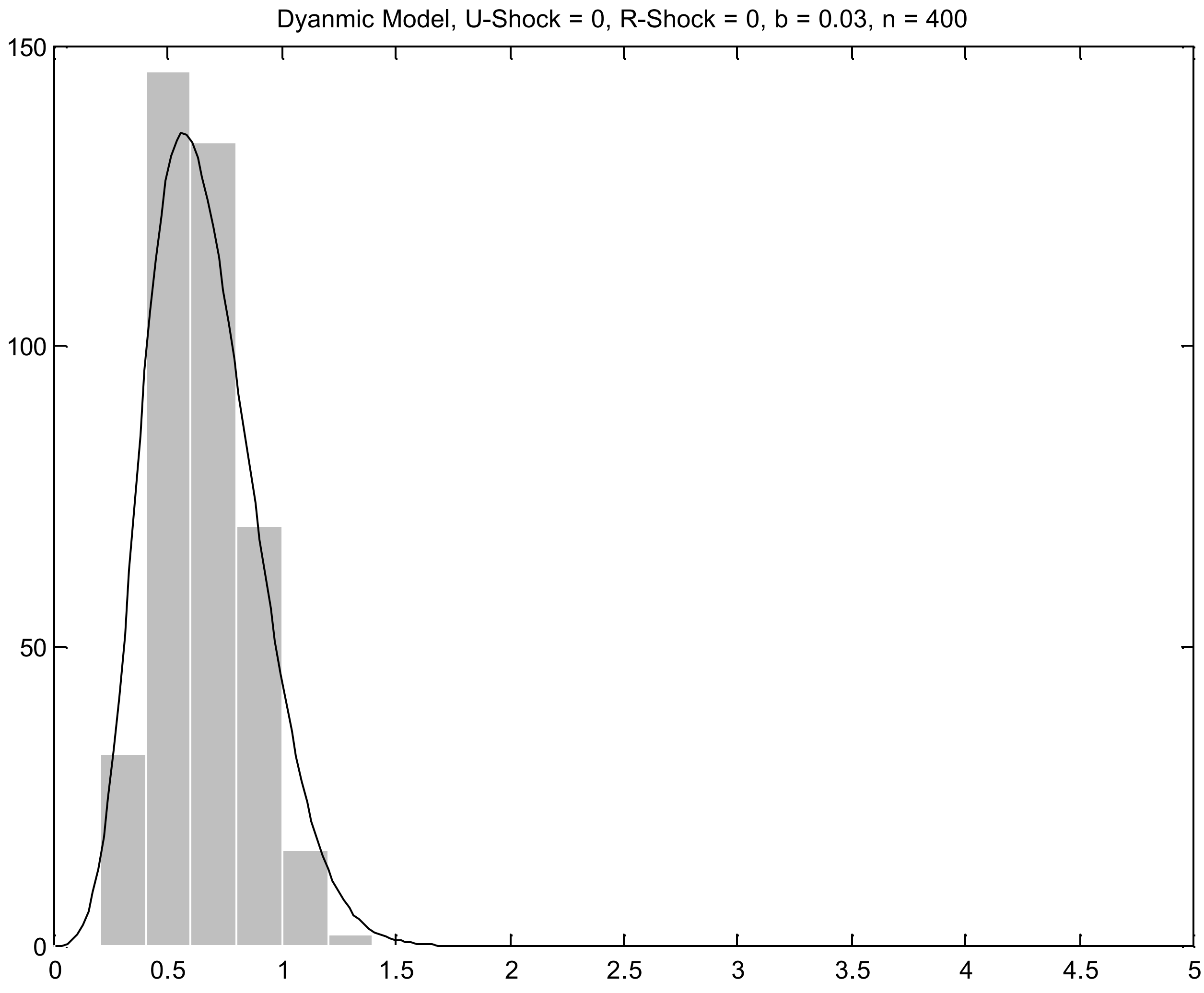
Dyanmic Model, U-Shock = -0.5, R-Shock = 0, b = 0.03, n = 400
250
200
150
100
50
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Figure 6



Dynamic, U-Shock = 1, R-Shock = 0, Shocktime = 30
Non-Shocked Economy
SCI
GPE

Figure 7


Dyanmic Model, U-Shock = 0, R-Shock = 0, b = 0.03, n = 400
150
100
50
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Figure 8

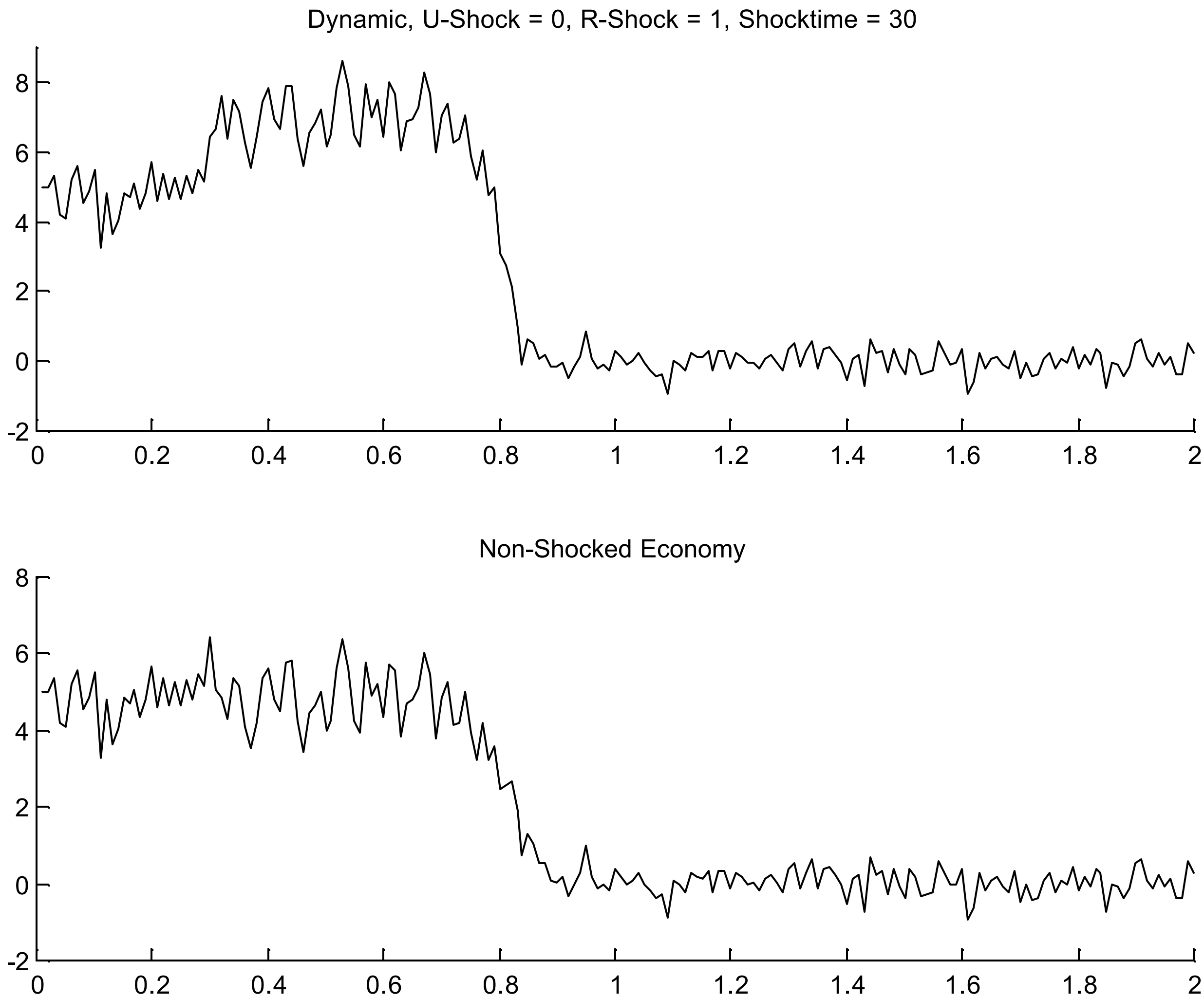

Dynamic, U-Shock = 0, R-Shock = 1, Shocktime = 30
8
6
4
2
0
-2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Non-Shocked Economy
8
6
4
2
0
-2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Figure 9