Learning to Forecast and Cyclical Behavior of Output and Inflation

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Abstract

This paper considers a sticky price model with a cash-in-advance constraint where agents forecast inflation rates with the help of econometric models. Agents use least squares learning to estimate two competing models of which one is consistent with rational expectations once learning is complete. When past performance governs the choice of forecast model, agents may prefer to use the inconsistent forecast model, which generates an equilibrium where forecasts are only constrained rational. While average output and inflation result the same as under rational expectations, higher moments differ substantially: output and inflation show persistence, inflation responds sluggishly to nominal disturbances, and the dynamic correlations of output and inflation match U.S. data surprisingly well.

Keywords: Learning, Business Cycles, Rational Expectations, Inefficient Forecasts, Output and Inflation Persistence

JEL-Class.:E31, E32, E37

1 Introduction

One of the main objectives of macroeconomic modeling is to understand the joint behavior of aggregate output and inflation at the business cycle frequency. Rational expectations models with nominal rigidities, workhorses of current macroeconomics, seem to have rather weak internal propagation mechanisms and therefore face substantial difficulties in matching the persistence inherent in output and inflation data. Especially, matching the reactions of output and inflation in response to nominal shocks has proven cumbersome (e.g. Chari et al. (2000) and Nelson (1998)).[1]

The aim of this paper is to analyze what role deviations from perfect forecast rationality might play in strengthening the internal propagation mechanisms and, therefore, the models' ability to match the data.

Presented is a simple business cycle model with monopolistic competition where prices are preset for one period and agents hold money to satisfy a cash-in-advance constraint. The paper deviates from rational expectations by imposing that agents can choose between either of two forecasting models to predict future inflation rates and that they must learn the parameters of the forecast functions.[2]

While one of the available forecast models can correspond to a rational expectations equilibrium once learning is complete, the other available model is 'inconsistent' with rational expectations in the sense that it delivers misspecified inflation forecasts whenever it is employed for forecasting.

Although the forecasting restriction itself does not preclude that agents acquire rational expectations (in the limit), I find that agents may learn to use the inconsistent forecast model, which gives rise to an equilibrium where forecasts are only constrained rational. Use of the inconsistent model can be optimal because it induces a law of motion for inflation that causes *both* available forecasting models to be underparameterized. Agents then prefer the inconsistent model whenever it provides a better approximation to the law of motion it generates.[3]

[1] This is not to say that matching the empirical impulse responses is impossible. However, it seems to require a battery of auxiliary assumptions, such as adjustment costs, habit formation, etc., e.g. Christiano et al. (2001).

[2] The restriction to two such forecast models is crucial for the results that follow and detailed justifications for it are developed in the paper.

[3] The term 'better' should be understood in the sense of producing a lower mean-squared forecast error, which represents a quadratic approximation to a utility-based preference relation.

I find that the model's propagation mechanism with constrained rational expectations differs strongly from that under rational expectations. Moreover, with constrained rational expectations the model is able to match important features of U.S. output and inflation data using white noise nominal demand shocks as the unique driving forces. Output and inflation then display persistent deviations from steady state, output deviations tend to be followed by persistent inflation deviations in the same direction, and inflation is an indicator of future output losses. While these features all show up in U.S. data none of them is obtained when expectations are fully rational. Output and inflation are then white noise as the underlying shock process.

In the equilibrium with constrained rational forecasts firms initially underreact to demand shocks. This is the case because inflation expectations fail to pick up immediately in response to a shock.[4] As a result, output displays persistence and inflation reacts sluggishly in response to demand shocks.

Once inflation has picked up, however, firms' inflation expectations increase. Since expectations are on average unbiased, inflation expectations now overreact and generate an amount of inflation that creates a demand slump in future periods. This implies that inflation is an indicator of future output losses.

While the idea of allowing choice in a limited class of forecast models has been developed before, this has not been done in a business cycle context. Most closely related is Evans and Honkapohja (1993) who consider an overlapping generations model that can generate endogenous cycles when agents choose between forecasting rules with different (constant) gain parameters. Brock and Hommes (1997) consider a cobweb model where agents choose between rational and naive predictors and show that cyclical or even chaotic behavior may emerge. Evans and Ramey (1992) consider a model where agents can engage in costly expectation revisions and show that this may give rise to long run non-neutrality and hysteresis effects.

Also the learning approach has rarely been taken to macroeconomic data as the existing literature remains largely theoretical.[5] Exceptions are the early hyperinflation study by Cagan (1956), the paper by Marcet and Nicolini (1996), who analyze hyperinflations in South America in a model closely related to the one in this paper, and Sargent (1999) who

[4]In this equilibrium expecations of future inflation depend on current inflation, which is preset.

[5]Evans and Honkapohja (2001) and Packalen (2000), for example, study the learnability of rational expectations equilibria in real business cycle models. Evans and Honkapohja (2002) and Bullard and Mitra (2002) study learnability in new Keynesian Models.

explains the history of American inflation with policy makers who learn about the Phillips curve trade-off.[6]

A number of empirical contributions have investigated the effects of deviations from forecast rationality. However, these usually do not provide microfoundations for why deviations from forecast rationality occur. Roberts (1997), for example, reinterprets the model of Fuhrer and Moore (1995) as a sticky price model and shows that boundedly rational inflation forecasts generate empirically plausible inflation behavior. Similarly, Ball (2000) shows that introducing first-order autoregressive inflation expectations significantly improves the empirical performance of otherwise standard sticky price models.

The paper is organized as follows. Section 2 describes important features of U.S. output and inflation that the model seeks to match. The business cycle model is presented in section 3 and its rational expectations solutions are outlined in section 4. Section 5 introduces learning agents with forecasting constraints. Conditions under which such agents might acquire rational or constrained rational expectations are determined in section 6. This section also compares the implied equilibrium dynamics with features of U.S. data. Section 7 then discusses the degree of forecast rationality and the generality of the mechanism generating equilibria with constrained rational forecasts. Section 8 provides an outlook on work that lies ahead. Technical details can be found in the appendix.

2 U.S. Output and Inflation: Some Facts

This section presents key features of the behavior of U.S. output and inflation that I seek to match in this paper.

The subsequent analysis is based on log quarterly U.S. GNP data (not seasonally adjusted, from Q1:1959 to Q3:1999) at constant and current prices with quarterly inflation defined as the implicit GNP-deflator and transformed into yearly rates.[7] As in Stock and Watson (1999) business cycle components have been obtained by using a band-pass filter on log-output and inflation.[8] The filtered series are shown in figure 1.

[6]Applications of learning models to financial markets include Timmermann (1993), Kasa (2002), and Bossaerts (2002). Further applications of learning models to financial markets and macroeconomics can be found in the collection introduced by Arifovic and Bullard (2001).

[7]The data is made available by Datastream International and has been compiled using U.S. Department of Commerce and Federal Reserve Bank data.

[8]The filter takes out fluctuations with a frequency below 2 and above 32 quarters to get rid of seasonal and trend components. Filtering with 4-32 quarters or using an HP-filter with a smoothing parameter of 1600 leads to very similar results.

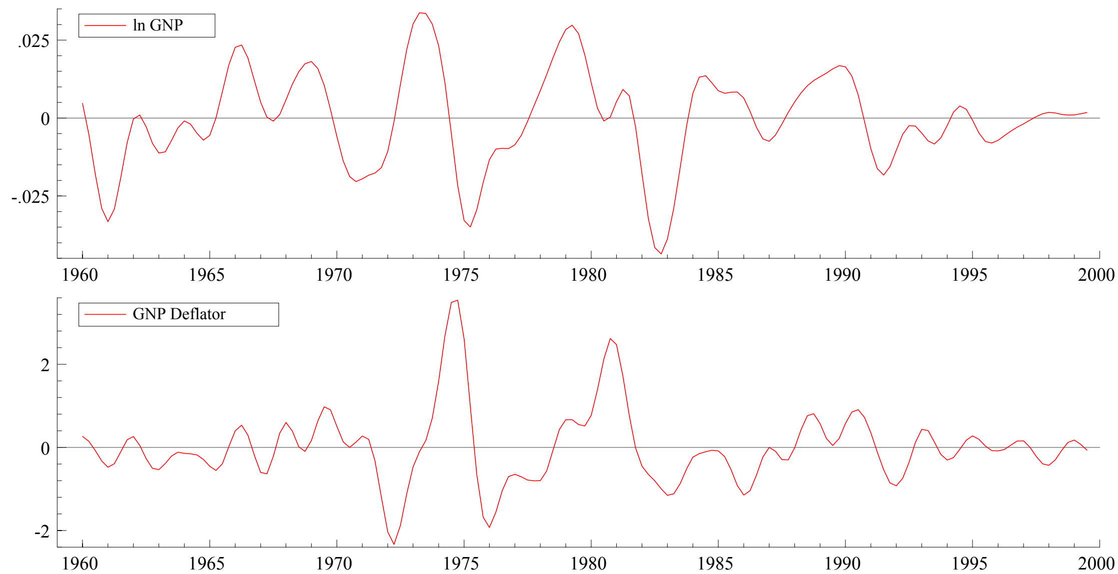

Figure 1: Detrended U.S. Data

Figure 2 depicts the auto- and crosscorrelations for the detrended data. The panels on the main diagonal of the figure show the autocorrelation for output and inflation, respectively. Output and inflation are positively autocorrelated for about 1 year, which shows that there is considerable persistence in these variables. Thereafter, the autocorrelations turn negative showing that average output (inflation) tends to be followed by below average output (inflation) circa 1 to 4 years down the road. These findings are consistent with the ones reported in Stock and Watson (1999).

The lower-left graph in figure 2 reveals that inflation is positively correlated with lagged output for about $2\frac{1}{2}$ years, with the maximum correlation being attained at around 1 year. This suggests that inflation responds sluggishly to output deviations. Correspondingly, the upper-right graph shows that output is negatively correlated with lagged inflation for about the first 3 years, with the maximum effect at around $1\frac{1}{2}$ years: inflation is an indicator of future output losses. Taylor (1999) has called these features the 'reverse dynamic cross correlation' of output and inflation.

Qualitatively similar results as the ones depicted in figure 2 have been reported by Fuhrer and Moore (1995) who estimated autocorrelation functions for output and inflation using a VAR that included a short term nominal interest rate.

While an analysis of the dynamic correlations is informative about the co-movements

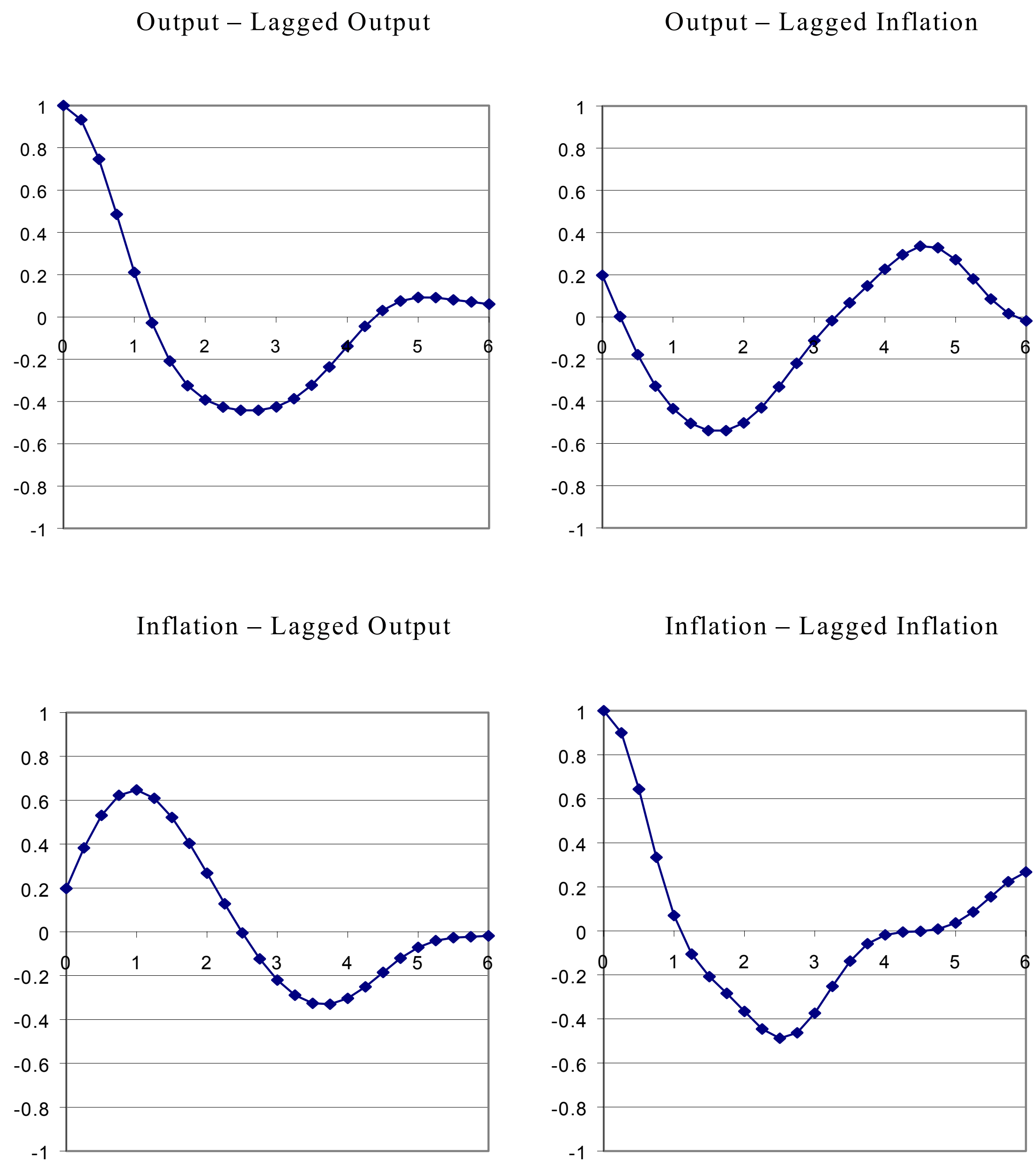

Figure 2: Auto- and crosscorrelations

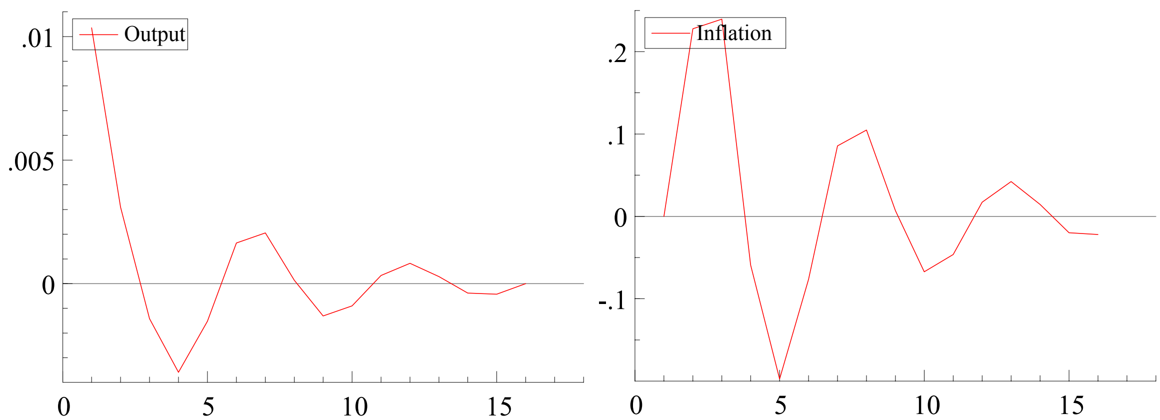

Figure 3: Impulse Responses to a Demand Shock, VAR(2)

present in the data, it remains uninformative about the causes underlying these movements. Impulse response functions provide answers about potential causal links but require identifying assumptions.

Figure 3 depicts the impulse response to a nominal demand shock obtained from fitting a vector autoregression (VAR) with 2 lags to yearly output and inflation data.[9] Identification of the demand shocks is based on the assumption that prices are preset and cannot react contemporaneously to the shock, as is the case with the model presented in the latter part of the paper. Such an identification assumption seems justified on the grounds that the benchmark results in Christiano, Eichenbaum, and Evans (1999) indicate that the GDP deflator reacts only after about $1\frac{1}{2}$ years to a monetary policy shock while output reacts well within 1 year.

Figure 3 shows the impulse response for a positive demand shock of a magnitude of one standard deviation. Output remains about 0.3 standard deviations above average in the year after the shock, illustrating that there is considerable output persistence. Inflation increases by 0.4 standard deviations in both years after the shock which indicates that the price response is rather sluggish.

In summary, the preceding analysis suggests that business cycle models should match the following features: demand shocks should generate a persistent increase in output and a

[9] Yearly data has been obtained by taking averages of the quarterly detrended values and facilitates the comparison with the theoretical model that will be presented in section 3.

sluggish and persistent increase in inflation. Moreover, inflation should display persistence and should be followed by a persistent decrease in future output.

The simple model discussed in the remaining part of this paper can replicate all of these facts using white noise nominal demand disturbances as driving forces.

3 A Simple Business Cycle Model

This section outlines a highly stylized business cycle model with utility maximizing consumers, profit maximizing entrepreneurs, and a government that distributes money through lump sum transfers.

There are two deviations from a frictionless environment, which insure that agents hold money in equilibrium and guarantee that monetary policy has real effects under rational expectations. Firstly, firms must commit to prices one period in advance, however can reset them each period. Secondly, consumption is subject to a cash-in-advance constraint forcing agents to use money to finance current consumption.

There is a unit mass of entrepreneurs who own monopolistically competitive firms. Each entrepreneur i produces an intermediate consumption good q^i that is an imperfect substitute in the construction of an aggregate consumption good c:

$$c = \left(\int_{i \in [0,1]} \left(q^i\right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \qquad \text{with } 1 > \sigma \geq 0$$

where σ indexes the degree of imperfect competition (Dixit and Stiglitz (1977)). This setup gives rise constant elasticity demand functions and implies that each entrepreneur i maximizes expected profits by choosing its price P_t^i as a fixed mark-up over expected production costs:

$$P_t^i = \frac{1}{1-\sigma} E_{t-1}[P_t w_t]$$

The previous equation assumes that the production technology is linear in labor such that nominal marginal costs are given by the product of the price index of the final consumption good P_t and the real wage w_t.

Dividing the previous equation by P_{t-1} and assuming that entrepreneurs have identical expectations delivers

$$\Pi_t = \frac{1}{1-\sigma} E_{t-1}[\Pi_t w_t] \tag{1}$$

where $\Pi_t = \frac{P_t}{P_{t-1}}$ is the inflation factor from period $t-1$ to period t.[10]

There is also a unit mass of workers. Each worker j chooses consumption c_t^j and labor supply n_t^j to maximize:

$$\begin{aligned} \max_{\{c_t^j, n_t^j\}} E_t \textstyle\sum_{t=0}^{\infty} \beta^t \left(u(c_t^j) - v(n_t^j)\right) \quad & \text{s.t.} \\ & c_t^j \leq \tfrac{m_{t-1}^j}{\Pi_t} + \tau_t \\ & m_t^j = \tfrac{m_{t-1}^j}{\Pi_t} - c_t^j + n_t^j w_t + \tau_t \end{aligned}$$

where m_t^j denotes the worker's real money holdings at the end of period t and τ_t the real value of the government cash transfer, which might be negative. The first constraint forces workers to use cash to pay for consumption goods (with leisure being the credit good) while the second constraint is the workers' flow budget constraint.[11]

When $u,\ v \in C^2,\ u' > 0, u'' < 0,\ v' > 0,\ v'' \geq 0,\ -\frac{u''(c) \cdot c}{u'(c)} < 1$ for all $c \geq 0$, and the cash-in-advance constraint is binding, utility maximization implies a labor supply function of the form:

$$n_t = n(w_t, E_t[\Pi_{t+1}])$$

Inverting this labor supply function with respect to the first argument delivers an expression for the real wage:

$$w_t = w(y_t, E_t[\Pi_{t+1}]) \qquad \textit{with} \quad \frac{\partial w}{\partial y} > 0, \frac{\partial w}{\partial E_t[\Pi_{t+1}]} > 0 \tag{2}$$

where the linearity of the production function has been used to substitute n_t by y_t. Given the specified utility functions, the real wage increases in the demand for labor and in the expected inflation tax.

The government issues money trough lump sum transfers to agents. This causes real money balances to evolve according to

$$m_t = \frac{m_{t-1}}{\Pi_t} + \tau_t$$

where τ_t is a mean zero white noise shock with small bounded support and is the only source of randomness in the model. When prices are preset and the cash-in-advance constraint is binding, output is demand determined in the short run and the previous equation is a

[10]Note that Π_t must be treated as part of the time t-information set along a (disequilibrium) learning path.

[11]We assume that firm's profits are consumed by entrepreneurs to avoid that labor supply depends on expected profits.

specification of the demand side of the economy.[12] This implies that output can be written as

$$y_t = \frac{y_{t-1}}{\Pi_t} + \tau_t \tag{3}$$

Using (1), (2), and (3) one obtains the temporary equilibrium equations that describe current output and inflation as a function of past output and expectations about future inflation rates:

$$\Pi_t = \frac{1}{1-\sigma} E_{t-1}[\Pi_t w(\frac{y_{t-1}}{\Pi_t} + \tau_t, \Pi_{t+1})] \tag{4}$$

$$y_t = \frac{(1-\sigma) y_{t-1}}{E_{t-1}[\Pi_t w(\frac{y_{t-1}}{\Pi_t} + \tau_t, \Pi_{t+1})]} + \tau_t \tag{5}$$

The remaining part of the paper will consider the linearizations of these equations around the deterministic steady state equilibrium and will make different assumptions about how agents forecast the inflation rates appearing in (4) and (5).

4 Rational Expectations Equilibria

The deterministic steady state of the model is given by:[13]

$$\begin{aligned} \overline{\Pi} &= 1 \\ \overline{y} &= n(1-\sigma, 1) \end{aligned}$$

Linearizing (4) and (5) around the steady state yields the following linear approximation for the stochastic system:[14]

$$\begin{aligned} \begin{pmatrix} \Pi_t \\ y_t \end{pmatrix} &= \begin{pmatrix} -1 \\ 2\overline{y} \end{pmatrix} + \begin{pmatrix} 1-\frac{1}{\varepsilon} & 0 \\ -\overline{y}(1-\frac{1}{\varepsilon}) & 0 \end{pmatrix} E_{t-1} \begin{pmatrix} \Pi_t \\ y_t \end{pmatrix} \\ &+ \begin{pmatrix} 1 & 0 \\ -\overline{y} & 0 \end{pmatrix} E_{t-1} \begin{pmatrix} \Pi_{t+1} \\ y_{t+1} \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{\overline{y}\varepsilon} \\ 0 & 1-\frac{1}{\varepsilon} \end{pmatrix} \begin{pmatrix} \Pi_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \tau_t \end{pmatrix} \end{aligned} \tag{6}$$

[12] Along a deterministic equilibrium path where the expected inflation factor is above the discount factor, the cash-in-advance constraint strictly binds if period zero money balances are not too high. Also, in the stochastic case with small support for the shocks surprise deflation and shocks to cash holdings will be small such that agents do not wish to postpone consumption by saving money.

[13] There exists another (economically uninteresting) steady state where money is worthless.

[14] The linearization uses the fact that $\frac{\partial w / \partial E\Pi_{t+1}}{w} = 1$ in a steady state where $\Pi = 1$, which follows from workers' first order conditions .

where ε denotes the real wage elasticity of labor supply at the deterministic steady state. Given the steady state values, ε remains the only free parameter of the model.

In appendix 9.2 it is shown that the rational expectations solutions to (6) have a minimum state variable representation as a two-dimensional AR(1) process

$$\begin{pmatrix} \Pi_t \\ y_t \end{pmatrix} = a + B \begin{pmatrix} \Pi_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \tau_t \end{pmatrix} \tag{7a}$$

There are two rational expectations solutions of this form. One solution is stationary and given by

$$\begin{pmatrix} \Pi_t \\ y_t \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{y} \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{\overline{y}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Pi_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \tau_t \end{pmatrix} \tag{8}$$

Output in this equilibrium is white noise and inflation is lagging output deviations by one period. The other solution is non-stationary:

$$\begin{pmatrix} \Pi_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1+\varepsilon \\ -\frac{\overline{y}}{\varepsilon} \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{\varepsilon\overline{y}} \\ 0 & 1+\frac{1}{\varepsilon} \end{pmatrix} \begin{pmatrix} \Pi_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \tau_t \end{pmatrix} \tag{9}$$

As one can easily see, this equilibrium path is diverging from the steady state (almost surely). The diverging paths have either increasing inflation rates and decreasing output levels or decreasing inflation rates and increasing output levels.[15]

5 Learning to Forecast Inflation Rates

While the previous section considered rational inflation forecasts, this section assumes that forecasts must be generated by econometric models and that agents consider only a limited set of such models.

5.1 Forecast Models, Model Estimation, and Model Selection

This section describes the set of econometric models that agents can use for forecasting and how agents estimate and select between competing models. The consequences of choosing different sets of forecasting models are addressed in section 7.2.

Assume that agents use simple regression models of the form

$$\Pi_t = \alpha_{t-1} + \beta_{t-1} x_{t-1} + \varepsilon_t \tag{10}$$

[15] The path with increasing output levels exists only in a local sense, see Adam (2002a) for details.

to forecast inflation rates, where x is an explanatory variable that is believed to predict future inflation. The parameters $(\alpha_{t-1}, \beta_{t-1})$ denote agents' (least squares) estimates of the relationship between x_{t-1} and Π_t. These estimates are based on information up to time $t-1$ and are constantly reestimated.

Admittedly, the above assumption is ad hoc. However, evidence obtained from the experimental laboratory has given strong support to the notion that agents' use simple models of the form (10) for forecasting, see Adam (2002b). Moreover, as will become clear below the above restriction is sufficiently general to generate rational expectations.

With the economy being described by two state variables, i.e. real output and inflation, there are only two forecasting models of the form (10):[16]

$$\begin{aligned} \text{Model Y}: \ \Pi_t &= \alpha_Y + \beta_Y y_{t-1} \\ \text{Model } \Pi: \ \Pi_t &= \alpha_\Pi + \beta_\Pi \Pi_{t-1} \end{aligned}$$

Model Y conditions inflation forecasts on past output and Model Π conditions on past inflation. Note that Model Y can generate the expectations of the rational expectations equilibria (8) and (9). Model Π, however, will never generate rational expectations since inflation is independent of lagged inflation in a rational expectations equilibrium.

Given that our agents do not possess a priori knowledge about the structure of the economy's rational expectations solution, it seems a reasonable starting point to assume that agents consider models that can and models that cannot deliver rational expectations.

To choose between and to parameterize the forecast models it is assumed that agents use the mean squared error criterion, i.e. agents use least squares to estimate parameters α and β of the two forecast models and then choose the model with the lowest past mean squared forecast error to predict future inflation. This can be justified on the grounds that mean squared errors constitute a second order approximation to the correct utility based choice criterion.[17]

What economic interpretation can be given to the above restriction on the class of forecast models?

Firstly, one may interpret the restriction as an exogenous restriction imposed on agents via the available prediction technology. The restriction then captures agents' computational knowledge or, more specifically, their knowledge about forecast procedures.

[16]For simplicity the time subscripts for α and β will be dropped from now on.

[17]This holds because least squares produce uncondionally unbiased estimates such that the first order terms drop off.

Secondly, the restriction may be interpreted as the result of an optimal choice of a class of models that trades-off the forecasting performance with the cost of considering smaller or larger classes of forecast models. The class is then an artefact of existing calculation costs. As argued in section 7, consideration of larger classes does not pay even when additional calculation costs are arbitrarily small, provided the variance of money shocks is sufficiently low.

Thirdly, a restricted class of forecast models can be seen as a temporary phenomenon due to agents who perform a specification search for suitable forecast models and start out by considering a certain class of models. Unsatisfactory prediction performance may then lead to changes in the considered class.

In principle all three economic interpretations are consistent with the setup of the model. A discussion about the forecast errors associated with the restriction above will be given in section 7.

5.2 Equilibrium with Learning Agents

Given the setup from the previous section, the economy will evolve as explained in the following.

Each period agents estimate both Model Y and Π by least squares and choose the model with the lowest past mean squared forecast error to forecast inflation. All agents then maximize utility under the assumption that the future evolution of the economy is described by the forecast produced by the selected forecast model. This generates a new inflation rate and output level according to equation (6), where the operator $E[\cdot]$ might now denote the potentially non-rational expectations generated by the chosen forecast model. Using the new data point, agents adapt their least squares estimates and their model choices, and the process repeats itself.

An equilibrium is a situation where the new inflation rate and output level confirm the previous estimates and the previous model choice. Formally

Definition 1 *A Model Equilibrium consists of least squares estimates* (α_Y^*, β_Y^*) *and* $(\alpha_\Pi^*, \beta_\Pi^*)$ *for Model Y and* Π*, respectively, and all agents using either Model Y or Model* Π *to forecast inflation rates such that*

i. *Agents choose the model with minimum mean squared forecast error.*

ii. *Given the forecast behavior, the economic outcomes resulting from (6) reconfirm the least squares estimates* (α_Y^*, β_Y^*) *and* $(\alpha_\Pi^*, \beta_\Pi^*)$.

An equilibrium where agents use Model M ($M = Y, \Pi$) to forecast inflation will be called a Model M Equilibrium.[18] The definition implies that in equilibrium agents forecast optimally subject to the constraints that have been imposed upon them. In this sense agents forecasts will be rational in any potential Model Equilibrium.

The equilibrium concept above is most closely related to the *restricted perceptions equilibrium* in Evans and Honkapohja (2001) and the *rational expectations equilibria with econometric models* of Anderson and Sonnenschein (1985). However, both equilibrium concepts are defined in terms of the regressors that enter the agents' (single) forecast function, while the present notion requires to specify the restrictions in terms of the class of forecast models.

Also related are the *consistent-expectations equilibrium* of Hommes and Sorger (1997) and the *self-confirming equilibrium* of Fudenberg and Levine (1993) recently used by Sargent (1999) in a macroeconomic context. Both types of equilibria are also equilibria in the sense of definition 1. However, the converse is not true because Model Equilibria allow for autocorrelation in the forecasts errors.

6 Calculating Model Equilibria

This section derives conditions under which Model Y and Model Π Equilibria exist and compares their properties with U.S. data.

6.1 Model Y Equilibria

Suppose that agents use Model Y to forecast inflation rates. Substituting the forecasts of Model Y for the expectations in the temporary equilibrium equations (6) delivers that the actual law of motion for inflation will be given by

$$\Pi_t = a(\alpha_Y, \beta_Y) + b(\alpha_Y, \beta_Y) y_{t-1} \tag{11}$$

where the coefficients a and b depend on agents' least squares estimates α_Y and β_Y.

Equation (11) reveals that the actual law of motion for inflation coincides with the structural assumption of Model Y, which implies that in a Model Y Equilibrium

$$a(\alpha_Y, b_Y) = \alpha_Y \qquad \text{and} \qquad b(\alpha_Y, \beta_Y) = \beta_Y$$

[18]Note that the definition excludes equilibria where some share of agents uses Model Y to forecast and the remaining agents use Model Π. In the present model such equilibria are not robust to small perturbations in the share of agents using the respective models.

Otherwise the parameter estimates would not remain stable. Consequently, a Model Y Equilibrium is a rational expectations equilibrium.

The converse is not necessarily true. At a Model Equilibrium the least squares estimates of the forecast models must get reconfirmed, which excludes rational expectations equilibria that are unstable under least squares learning. As shown in appendix 9.3, the stationary rational expectations equilibrium (8) is the unique Model Y Equilibrium since the non-stationary equilibrium (9) is unstable under least squares learning.[19]

Output and inflation in Model Y Equilibrium are thus given by

$$\Pi_t = \frac{1}{\overline{y}} y_{t-1} \tag{12}$$

$$y_t = \overline{y} + \tau_t \tag{13}$$

A one percent money shock increases output temporarily by the same amount. Equation (12) reveals that it also causes inflation expectations to increase by one percent.[20] As a result, entrepreneurs increase prices by one percent (see equation (1) and note that the expected wage is equal to $1 - \sigma$) which implies that the excess money stock will not persist into the next period.

Consequently, with rational expectations output and inflation are white noise processes with inflation lagging output by one period due to the price stickiness. The rational expectations equilibrium, therefore, performs poorly in matching the features of U.S. output and inflation data documented in section 2.

6.2 Model Π Equilibria

This section considers equilibria where agents use Model Π to predict inflation rates. As argued before, agents expectations are then only constrained rational.

6.2.1 Preliminaries

Suppose agents use Model Π to forecast inflation rates. Since least squares estimates deliver forecasts that are (on average) unbiased, average inflation in a Model Π Equilibrium will coincide with average expected inflation. Given that such a relation holds only at a (rational

[19] At the latter equilibrium agents adapt their least squares estimates in response to shocks in a way such that their adapted expectations generate new inflation rates and output levels that cause these estimates to diverge even further.

[20] Agents' Model Y estimate is asymptotically identical to equation (12).

expectations) steady state, average output and average inflation in a stationary Model Π Equilibrium will be same as in a Model Y Equilibrium.[21]

This causes constraint rational expectations equilibria to alter only the second moments of the equilibrium time series while leaving the first moments unchanged. Moreover, on a more technical level, it implies that one can linearize the model around the same steady state values as one usually does when calculating the linearized rational expectations solutions.

Substituting the predictions of Model Π for the inflation expectations in (6) delivers an equation that describes current inflation and output as a function of the past values of these variables, the Model Π parameters (α_Π, β_Π), and the labor supply elasticity ε:[22]

$$\begin{pmatrix} \Pi_t \\ y_t \end{pmatrix} = \begin{pmatrix} -1 + \alpha_\Pi(2 + \beta_\Pi - \frac{1}{\varepsilon}) \\ \left(2 - \alpha_\Pi(2 + \beta_\Pi - \frac{1}{\varepsilon})\right)\overline{y} \end{pmatrix} + \begin{pmatrix} \left(1 + \beta_\Pi - \frac{1}{\varepsilon}\right)\beta_\Pi & \frac{1}{\varepsilon}\frac{1}{\overline{y}} \\ -\left(1 + \beta_\Pi - \frac{1}{\varepsilon}\right)\beta_\Pi\overline{y} & 1 - \frac{1}{\varepsilon} \end{pmatrix} \begin{pmatrix} \Pi_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \tau_t \end{pmatrix} \qquad (14)$$

The preceding equation reveals that in a potential Model Π Equilibrium inflation depends (generically) on past inflation *and* past output. The law of motion for inflation, thus, lies outside the class of forecast models that agents consider. With *all* forecast models being misspecified, Model Π may deliver a better fit to (14) than Model Y.

6.2.2 Existence of Model Π Equilibria

This section discusses under which conditions Model Π Equilibria exist.

In a potential Model Π Equilibrium the least squares estimate β_Π must be identical to the correlation coefficient

$$\frac{cov(\Pi_t, \Pi_{t-1})}{var(\Pi_t)}$$

of the actual law of motion (14). Since β_Π enters (14), calculating a Model Π equilibrium involves solving a fixed point problem, as when calculating a standard rational expectations equilibrium.

Appendix 9.4 shows how to solve for the fixed point β^*_Π and figure 4 graphs the solution as a function of the labor supply elasticity ε.

When $\varepsilon = 1$, a one percent demand shock causes a one percent increase in expected costs and, thus, in inflation. This amount of inflation devaluates excess money just back to

[21] This does not need to hold for non-stationary equilibria.

[22] The relevant features of this process are unaffected by the average level of output $\overline{y}$.

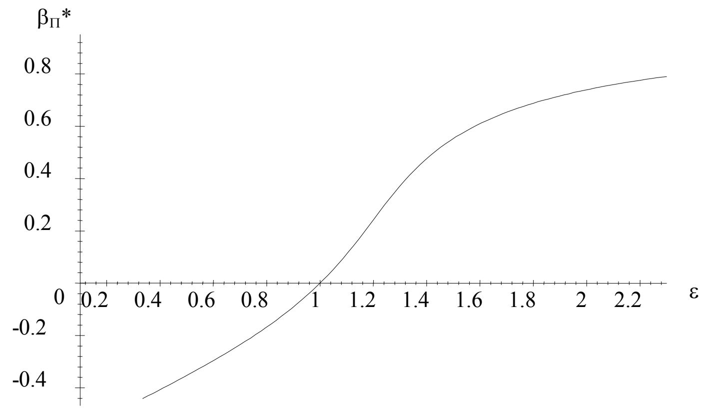

Figure 4: β^*_Π as a function of ε

its equilibrium level. As a result, there is no persistence in excess demands, inflation is white noise, and β^*_Π is equal to zero.

As labor supply becomes more elastic, a one percent demand shock generates a less than proportionate labor cost and inflation increase. Excess money balances are then not devaluated in a single period but persist into future periods where they cause again above average costs and inflation. Inflation rates are then positively autocorrelated which explains the positive slope in figure 4.

Substituting the solution depicted in figure (4) into (14) yields a *candidate process* for a Model Π equilibrium.[23] Importantly, the properties of this process depend only on the elasticity ε. Numerical calculations show that the process is stationary for $0.35 \leq \varepsilon \leq 2.15$ which will be the range of values considered from now on.[24]

For this *candidate process* to be a Model Π Equilibrium one has to verify that Model Π is predicting inflation better than Model Y. Clearly, for values of ε around 1 this cannot be expected, since past inflation has almost no predictive power. However, for larger and smaller values inflation becomes an increasingly better predictor since inflation rates are autocorrelated, as shown in figure 4.

Figure (5) depicts the mean squared forecast errors of Model Y and Model Π for various values of ε assuming that agents use Model Π to forecast. The graph reveals that for a sufficiently elastic labor supply ($\varepsilon_{n,w} \geq 1.75$) Model Π will generate better predictions than

[23]One also has to set $\alpha^*_\Pi = 1 - \beta^*_\Pi$, which follows from the unbiasedness of least squares forecasts.

[24]These and the subsequent boundaries are only approximate.

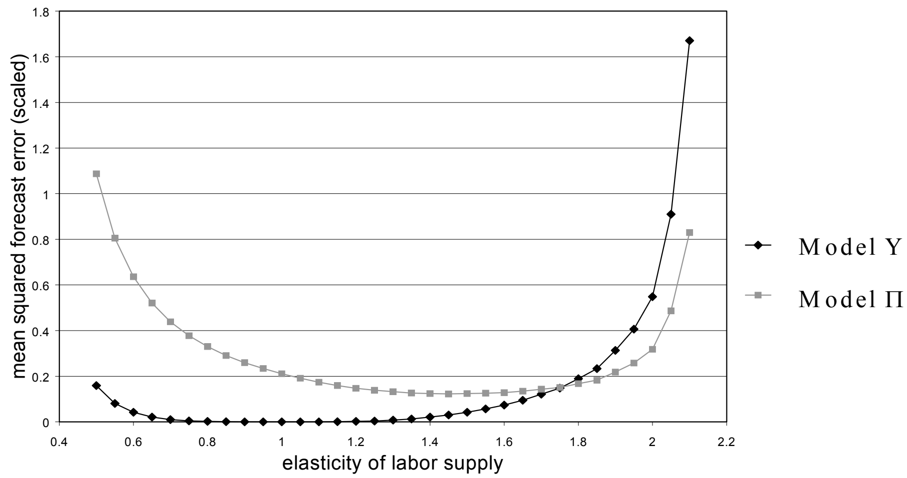

Figure 5: Mean squared forecast error, Model Y vs. Model Π

Model Y. Potential Model Π Equilibria therefore exist for $1.75 \leq \varepsilon_{n,w} \leq 2.25$. Appendix 9.3 shows that the potential Model Π Equilibria are also stable under least squares learning, which establishes that Model Π Equilibria exist for these values of the labor supply elasticity.

Admittedly, the required supply elasticity lies on the high end of plausible values. However, similar elasticity levels are not uncommon in the literature. Christiano et al.(1997), for example, report satisfactory performance of a limited participation model for a labor supply elasticity of 2. Moreover, labor is the unique variable input factor in the present model. Therefore, the elasticity should rather be interpreted as an elasticity of total marginal costs of production, which includes other variable factors such as capital (in terms of utilization rates), raw materials, and energy. Moreover, one should be concerned with working hours which include overtime and are therefore more elastic.

6.2.3 Output and Inflation in Model Π Equilibrium

This section presents the output and inflation dynamics in Model Π Equilibrium and compares these with the behavior of U.S. data. For illustrative purposes the section assumes a labor supply elasticity of $\varepsilon = 1.8$, which is at the lower end of values for which Model Π Equilibria exist. The effects of larger elasticity values are discussed at the end of the section.

Figure 6 depicts the auto- and crosscorrelations of output and inflation in Model Π Equilibrium for 6 periods. This corresponds to the 6 years of U.S. data shown in figure 2 if

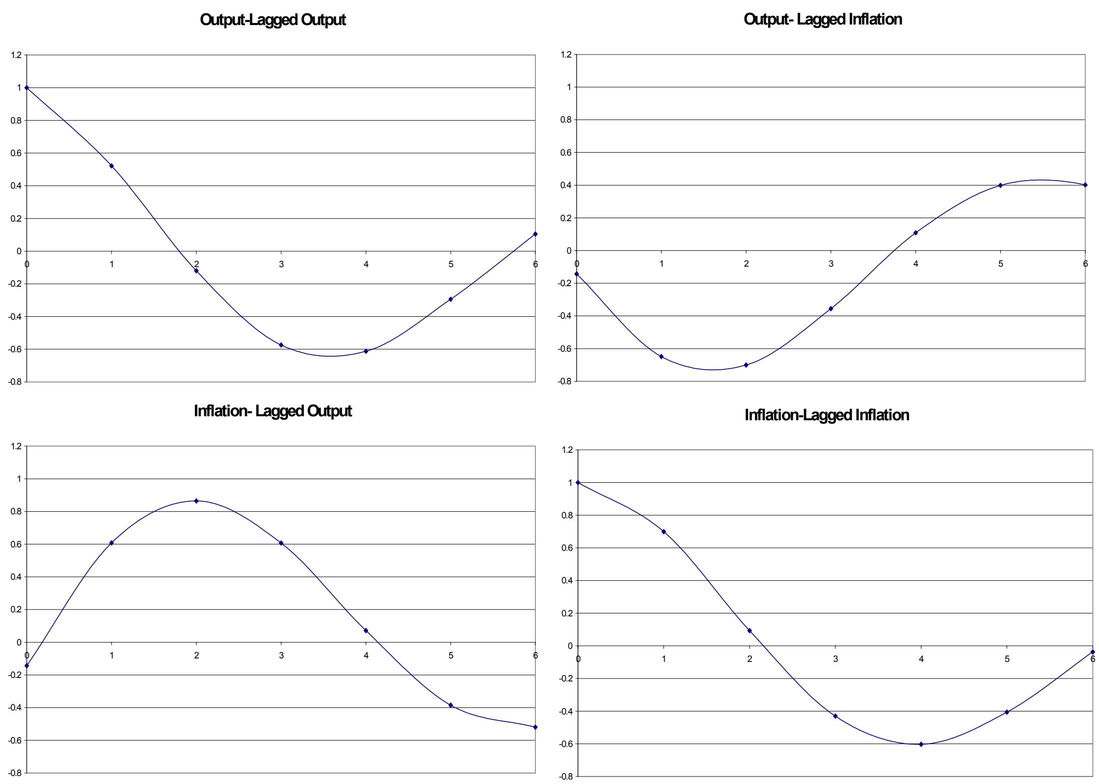

Figure 6: Auto- and crosscorrelations in Model II Equilibrium

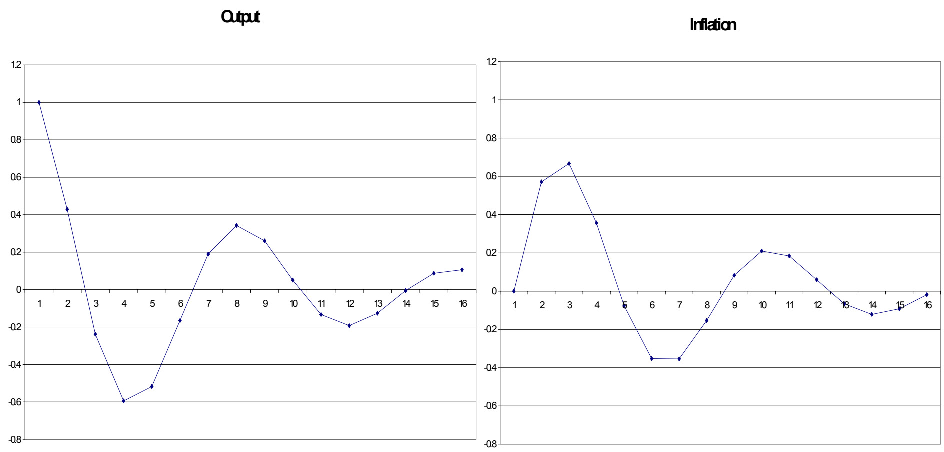

Figure 7: Impulse Responses to a Demand Shock in Model Π Equilibrium

a model period is interpreted as a year. Such an interpretation seems reasonable given the degree of price stickiness.[25] The auto- and crosscorrelations in figure 6 match those of figure 2 remarkably well. Output and inflation are persistent. They are positively correlated for short lags and negatively for longer lags. Output is a positive leading indicator for inflation and inflation is a leading indicator of future output losses. None of these features shows up in a rational expectations equilibrium.

Figure 7 shows the impulse responses to a demand shock in a Model Π Equilibrium. The shocks hits the economy in period 1. Prices react sluggishly in period 2 such that output remains above steady state. In period 3 inflation increases even further and generates a demand slump. Output and inflation then slowly return to their equilibrium values. Note that these responses match the estimated responses for U.S. data shown in figure 3.

The weak and sluggish reaction of inflation in response to a demand shock can be explained as follows. In general, inflation increases because firms expect either inflation or real wages to increase. When a demand shock hits the economy current prices are preset and, given that Model Π forecasts condition on current inflation, inflation expectations are equally preset. Thus, cost expectations must initially drive inflation.

[25] The figure depicts the correlations for the unfiltered data. Transforming this 'yearly' data into 4 equal quarterly values and applying the bandpass filter that has been used for detrending the data leads to a very similar graph. Unfiltered data are shown because they allow for a clear interpretation in terms of the model's mechanisms.

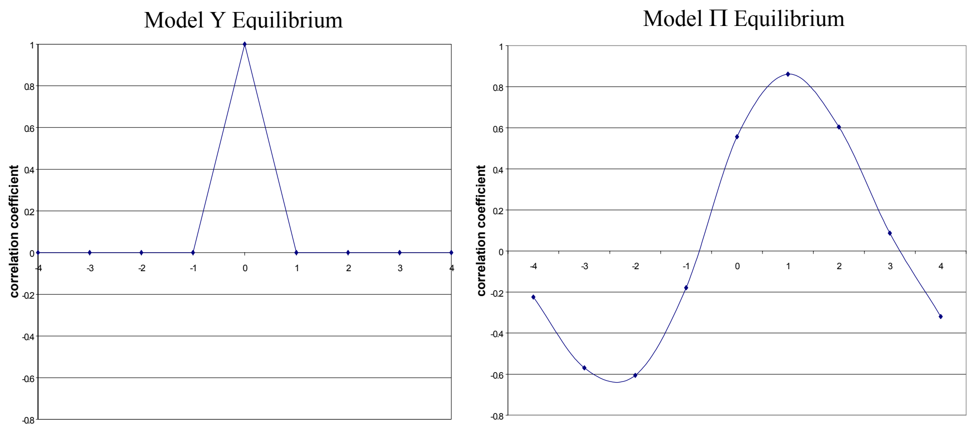

Figure 8: Correlation between output (t) and wage(t+x)

With future inflation expected to be unchanged, current excess demand is expected to persist into the next period, which will increase expected future real wages. Since labor supply is relatively elastic, costs are expected to increase only slightly and thereby cause a rather weak inflation reaction.

Yet, once inflation has picked up, inflation expectations will start to pick up and then start to drive actual inflation. This explains why inflation in period 3 is higher than in period 2, see the inflation panel in figure 7.

The previous results show that Model Π Equilibrium is able to capture all features of U.S. output and inflation that have been presented in section 2. In addition, Model Π Equilibria also generate an empirically plausible behavior for real wages, as will be shown below.

Galí and Gertler (1999) have stressed that in the U.S. real wages are lagging over the output cycle. The panel on the right of figure 8 displays the correlation coefficient between the real wage and output in Model Π Equilibrium. Although output is positively correlated with *current* wages (due to an upward sloping labor supply function), output shows a higher correlation with the wage in the *next* model period. Since inflation and inflation expectations are lagging over the cycle, workers expect a higher inflation tax after the peak of the cycle. This shifts the labor supply curve upwards and generates higher real wages despite a decreasing output level.

The panel on the left of figure 8 displays the same correlation in a Model Y Equilibrium.

Since output and inflation are white noise, output is only correlated with current wages but neither with leading nor lagging values.

I now briefly discuss the effects a more elastic labor supply for the impulse responses shown in figure 7. Clearly, the larger ε the lower the expected cost increase. The initial inflation response in period 2, therefore, will be more sluggish and output will be higher. In period 3, when inflation expectations are picking up, there are two opposing effects. On the one hand inflation shows higher persistence, see figure 4, therefore inflation expectations tend to pick up more for any given inflation increase . At the same time, the initial inflation increase has been less pronounced. As it turns out, the net effect is positive and inflation in period 3 is higher and output lower than shown in figure 7.

Higher elasticity values, thus, generate a more sluggish initial inflation reaction but a stronger reaction thereafter. This results in a shorter length of the cycle and causes the correlations shown in figure 6 to cross the axis at an earlier date.

7 Discussion

This section discusses two important issues concerning the plausibility of Model Π Equilibria: Firstly, how severe is the deviation form perfect forecast rationality and what is the utility loss associated with it? Secondly, how general is the result that forecasting restrictions might give rise to equilibria where optimal forecasts are only constrained rational?

7.1 Forecast Inefficiency in Model Π Equilibrium

A diagnostic test for the rationality of forecasts in Model Π Equilibrium can be based on the correlation structure of forecast errors.

The panel on the left-hand side of figure 9 displays the autocorrelations for the one-step-ahead forecast error in Model Π Equilibrium for a labor supply elasticity of $\varepsilon = 1.8$.[26] Not surprisingly, the figure shows that forecast errors are correlated indicating that forecasts are inefficient.

How long would it take to discover the inefficiency? The panel on the right-hand side of figure 9 depicts the likelihood with which Box-Pierce tests accept the null hypothesis that forecast errors are white noise in Model Π Equilibrium.[27] While with 20 to 30 data points

[26] This is the value used in the previous section. Higher elasticities lead to very similar error structures. The auto-correlations have been obtained numerically from 10000 simulated data points.

[27] The shown rejection probabilities are for the asymptotic 1% critical value of the test and have been obtained by Monte Carlo simulations of 10000 series.

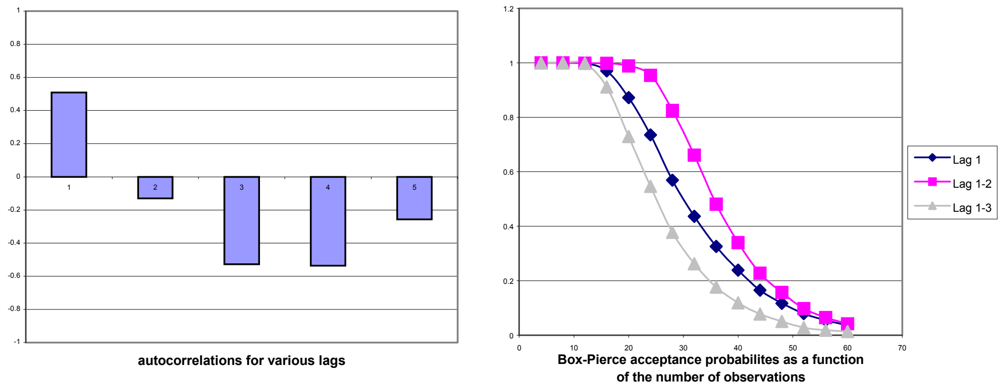

Figure 9: Forecast errors in Model II Equilibrium

the hypothesis is almost always accepted, it is almost certainly rejected when 50 data points become available.

Table 1 reports the expected number of observations for which the tests deliver the first rejection.[28] Depending on the precise test used one can expect the class of forecast models to be in place for around 33 observations, i.e. more than three decades if a model period is interpreted as a year, as before.

Table 1: Specification Tests

Box-Pierce Test	**Lag 1**	**Lag 1-2**	**Lag 1-3**
Expected duration until rejection	33.8	39.2	27.3

Although the correlation structure of forecast errors is informative about whether forecasts are rational, it is not useful to assess the utility costs associated with inefficient forecasting. Utility costs, however, seem to be the economically more relevant criterion to evaluate the degree of non-rationality.

The first line of equation (14) reveals that the best possible forecast model in Model II Equilibrium predicts with zero error.[29] The size of Model II's forecast error is thus a quadratic approximation to the utility loss associated with the use of simple regression models. Interestingly, the size of Model II′s forecast errors depends on the variance of the

[28] Agents are assumed to perform a Box-Pierce test with a 1% critical value each time a new data point is obtained. The table reports the expected number of observations at which the first rejection is obtained based on 10000 Monte Carlo simulations.

[29] This is the case because supply shocks are absent.

demand shocks. When the variance of demand shocks approaches zero, the forecast errors of Model Π disappear: Since Model Π forecasts are unbiased, forecasts will become rational in the limit when shocks are absent. This is a consequence of the more general observation that perfect foresight equilibria can be the limit of constrained rational equilibria of stochastic economies with vanishing noise, see Adam (2002a).

The previous observation implies that the utility loss that agents incur by using Model Π decreases with the variance of the demand shock and can be made arbitrarily small. Moreover, due to an argument made by Akerlof and Yellen (1985) the agent's utility loss is of an order smaller than the size of the forecast errors. Further results from Cochrane (1989) indicate that utility losses associated with non-optimal consumption behavior are in the order of a few cents per month given the variability of U.S. data.

All this suggests that the utility loss associated with use of Model Π is relatively small despite the statistical properties of forecast errors.

7.2 Generality of Results

As argued before, the essential ingredient allowing for equilibria with inefficient equilibrium expectations is that the class of forecast models is open in the sense that use of some forecast model causes the actual law of motion of the forecasted variable to lie outside the considered class. How likely is it that some arbitrary class of forecast models is open?

For the present model there exists an obvious way to obtain a closed class. One simply has to add to the two forecast models considered thus far a model containing both lagged output and lagged inflation. The induced laws of motion will then depend at most on lagged output and inflation, see equation (6).

The previous point, however, is not entirely convincing. Once agents use forecast equations with two variables they might as well consider other two variable systems, including an equation where inflation is assumed to depend on lagged and twice lagged inflation. Use of the latter forecast model will lead to an actual law of motion where inflation depends on three variables (two lags of inflation and one lag of output, see equation (6)). The class of forecast models will then be open again. A similar logic applies when allowing for three or more variables in the forecast equations.

In general openness in linear models remains always a possibility as long as the maximum number of lags is finite.[30] The class of forecast models is only guaranteed to be closed

[30]However, openness is only a necessary condition for inefficient expectations equilibria to exists. Once the class of models increases, the class of competing models increases and the candiate model projecting the

if the number of lags can be arbitrarily large: linear forecasts in a linear model lead to linear actual laws of motion, which are then contained in the class by definition.

Closedness is likely to be even more difficult to obtain when allowing for non-linear forecasts and/or non-linear models. Moreover, models deviating from the representative agent assumption and allowing agents to have heterogeneous forecasting capabilities may also increase the likelihood of openness in the sense above.

In the light of this discussion the openness property of the class of simple regression models considered in this paper seems to be a virtue rather than a deficiency since openness is likely to be obtained in many situations involving forecasting constraints.

8 Outlook on Work Ahead

Some questions related to the findings in this paper deserve further attention.

Firstly, it seems desirable to evaluate the plausibility of the imposed forecasting restrictions not only via the implied model predictions but also in a more direct way. Inflation survey data or experimental evidence, as generated in Adam (2002b), will help to shed light on the plausibility of equilibria with constrained rational expectations.

Secondly, output and consumption in Model Π Equilibrium display a higher variance than in Model Y Equilibrium. This suggests that Model Π Equilibria are Pareto dominated by Model Y Equilibria for low enough discount rates.[31] This raises the question whether suitable policies could improve upon the situation and prevent the economy from converging to a Model Π Equilibrium.

Finally, the construction of models where agents differ in their forecasting constraints would be of interest. Does heterogeneity of forecasts generated along these lines makes it even more difficult for forecasters to detect the true underlying economic relationships?

9 Appendix

9.1 Vector Autoregression

A VAR with two lags and a constant was estimated by OLS regression for yearly output and inflation data. The data consisted of the yearly averages of the bandpass-filtered quarterly

actual law of motion outside the considered class is less likely to be the best approximation to that actual law.

[31] However, entrepreneurs and workers profit differently from output fluctuations.

inflation and output data depicted in figure 1. The estimation results are :

	Π_t	*Std.Error*	y_t	*Std.Error*
const.	0.020562	0.089495	0.00033680	0.0016824
Π_{t-1}	0.056692	0.16023	-0.0087039	0.0030122
Π_{t-2}	-0.39029	0.15483	-0.0042423	0.0029107
y_{t-1}	22.005	8.7542	0.29932	0.16457
y_{t-2}	15.306	9.2518	-0.035335	0.17393
σ	0.55062	-	0.010351	-
R^2	0.75406	-	0.69924	-

The actual and the fitted values are shown in figure 10. Figure 11 depicts the autocorrelation of the regression errors.

9.2 Calculation of the Rational Expectations Equilibria

Consider a stochastic linear expectational difference equation of the form

$$z_t = k + B_0 E_{t-1}[z_t] + B_1 E_{t-1}[z_{t+1}] + D z_{t-1} + u_t \tag{15}$$

with $z_t, u_t, k \in R^n$, $B_0, B_1, D \in R^{n \times n}$, and $B_1 \neq 0$, $D \neq 0$. The minimum state variable solutions of (15) take the form

$$z_t = a + B z_{t-1} + u_t$$

provided there exists a real solution to the matrix quadratic equation

$$B_1 B^2 - (B_0 - I)B + D = 0 \tag{16}$$

see chapter 10.2 in Evans and Honkapohja (2001). Then a is given by

$$(I - B_0 - B_1(I + B))a - k = 0 \tag{17}$$

The AR-solutions can be calculated by solving the matrix equations (16) and (17) for a and B. Due to the sparsity of the matrices B_0, B_1, and D in the present model, see equation (6), this is straightforward and delivers the solutions (8) and (9).

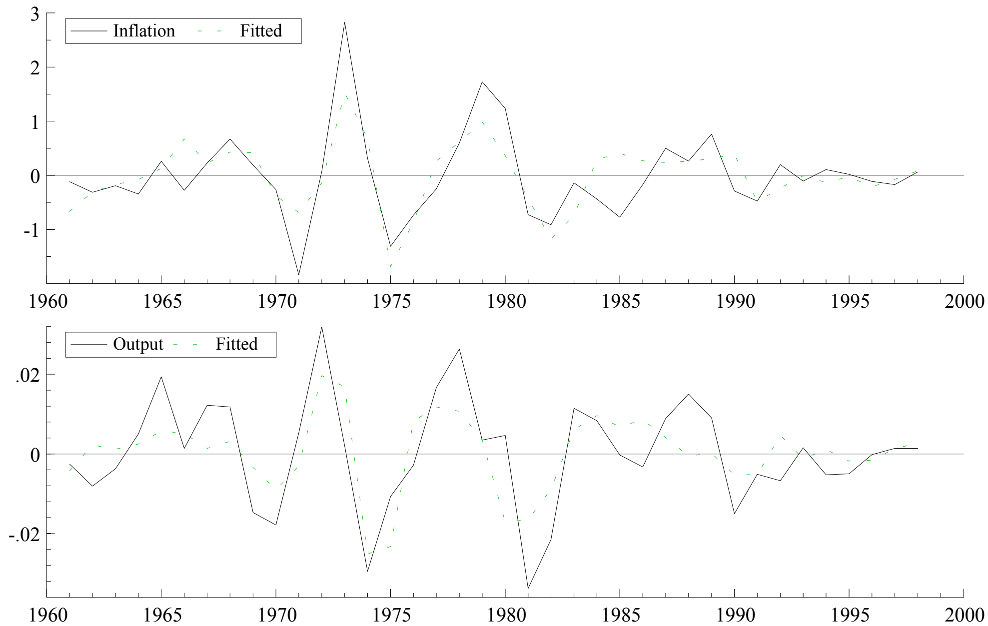

Figure 10: VAR: Actual and fitted values

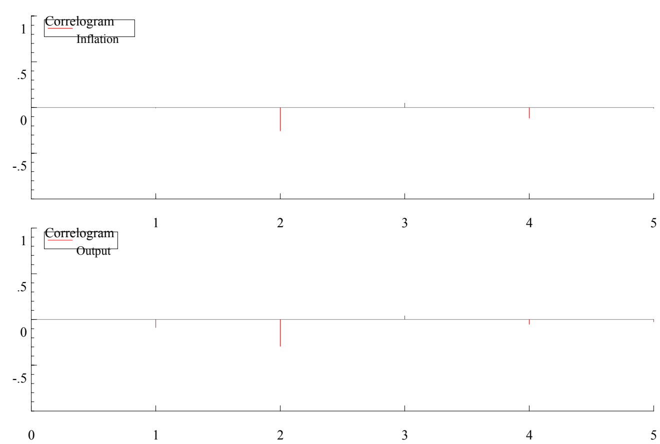

Figure 11: VAR: Autocorrelation of residuals

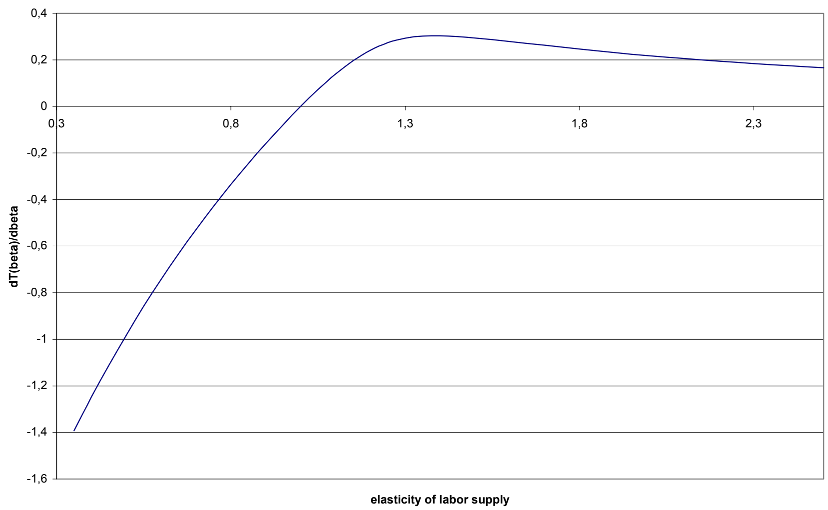

Figure 12: $\partial T(\beta)/\partial(\beta)$ as a function of ε

9.3 Stability of Rational Expectations under Least Squares Learning

The stability of the rational expectations equilibria (8) and (9) under least-squares learning is governed by the so-called E-stability criterion, see Evans and Honkapohja (1998). Using

$$\begin{aligned} E_{t-1}[\Pi_t] &= \alpha + \beta y_{t-1} \\ E_{t-1}[\Pi_{t+1}] &= \alpha + \beta E_{t-1}[y_t] \\ &= \alpha + \beta\left(\overline{y} + \frac{1}{\overline{\Pi}} y_{t-1} - \frac{\overline{y}}{\overline{\Pi}^2}(\alpha + \beta y_{t-1})\right) \end{aligned}$$

to substitute the expectations in (6) and using $\overline{\Pi} = 1$, one obtains

$$\Pi_t = T_a(\alpha, \beta) + T_b(\beta) y_{t-1}$$

where

$$\begin{aligned} T_a(\alpha, \beta) &= -1 + (2 - \frac{1}{\varepsilon})\alpha + \beta\overline{y}(1 - \alpha) \\ T_b(\beta) &= \frac{1}{\overline{y}\varepsilon} + (1 - \frac{1}{\varepsilon})\beta - \overline{y}\beta^2 \end{aligned}$$

The associated differential equation is given by

$$\begin{pmatrix} \frac{\partial\alpha}{\partial t} \\ \frac{\partial\beta}{\partial t} \end{pmatrix} = \begin{pmatrix} T_a(\alpha, \beta) \\ T_b(\alpha, \beta) \end{pmatrix} - \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

and E-Stability (E-Instability) is satisfied when the eigenvalues of

$$\begin{pmatrix} \frac{\partial T_a(\alpha,\beta)}{\partial \alpha} & \frac{\partial T_a(\alpha,\beta)}{\partial \beta} \\ \frac{\partial T_b(\alpha,\beta)}{\partial \alpha} & \frac{\partial T_b(\alpha,\beta)}{\partial \beta} \end{pmatrix} \tag{18}$$

are smaller (larger) than one.

For the stationary rational expectations solution (8) the eigenvalues of (18) are given by $\lambda_1 = 1 - \frac{1}{\varepsilon}$ and $\lambda_2 = -\frac{1}{\varepsilon}$ and for the non-stationary solution (9) by $\lambda_1 = 2$ and $\lambda_2 = 2 + \frac{1}{\varepsilon}$. This proves that the stationary rational expectations solution is stable and that the non-stationary rational expectations solution unstable under least squares learning of Model Y.

9.4 Calculating β^*_Π

We now determine β^*_Π which is a function of the covariances of (14). Since the covariances are independent of the constant appearing in (14) we can ignore it and write this equation as

$$z_t = B z_{t-1} + u_t \tag{19}$$

where $z_t = (\Pi_t, y_t)'$ and $u_t = (0, \tau_t)'$. Define also $\Omega = Var(u_t)$, $\Sigma = Var(z_t)$, $\Gamma = Cov(z_t, z_{t-1})$, and $B = (b_{i,j})$.

Taking variances on both sides of (19) yields

$$\Sigma = B \Sigma B' + \Omega$$

which implies

$$\begin{aligned} vec(\Sigma) &= (B \otimes B) vec(\Sigma) + vec(\Omega) \\ &= (I - B \otimes B)^{-1} vec(\Omega) \end{aligned} \tag{20}$$

Multiplying (19) by z_{t-1} and taking expectations one obtains the covariance with lagged variables:

$$\Gamma = B \Sigma \tag{21}$$

Using equations (20) and (21) and remembering that

$$\Omega = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_\tau^2 \end{pmatrix}$$

where σ_τ^2 is the variance of the money shock, the matrices Σ and Γ are easily calculated. Using the expression for the variance of Π_t from Σ and the expression for the covariance of Π_t and Π_{t-1} from Γ one obtains

$$\begin{aligned}\beta_\Pi &= \frac{cov(\Pi_t, \Pi_{t-1})}{var(\Pi_t)} \\ &= \frac{b_{11} + b_{22}}{1 + (b_{11}b_{22} - b_{12}b_{21})} \\ &= \frac{(1 + \beta_\Pi - \frac{1}{\varepsilon})\beta_\Pi + 1 - \frac{1}{\varepsilon}}{(1 + \beta_\Pi - \frac{1}{\varepsilon})\beta_\Pi + 1}\end{aligned} \tag{22}$$

The unique real solution to this equation is given by

$$\beta_\Pi^* = \sqrt[3]{z} - \frac{1}{9}\frac{3\varepsilon - 1}{\varepsilon^2\sqrt[3]{z}} + \frac{1}{3\varepsilon}$$

$$where$$

$$\begin{aligned}z = &\frac{1}{54}\frac{2 - 9\varepsilon - 27\varepsilon^2 + 27\varepsilon^3}{\varepsilon^3} \\ &+ \frac{1}{18}\frac{\sqrt{3}}{\varepsilon^2}\sqrt{(-5 + 26\varepsilon + 9\varepsilon^2 - 54\varepsilon^3 + 27\varepsilon^4)}\end{aligned}$$

9.5 Stability of Model Π Equilibrium under Least Squares Learning

E-Stability governs the stability under least squares learning, see Evans and Honkapohja (1998). The differential equation determining the stability of Model Π Equilibria is given by

$$\frac{\partial\beta}{\partial\tau} = T(\beta) - \beta \tag{24}$$

where $T(\beta)$ is given by equation (22). Whenever (24) is locally asymptotically stable at the Model Π equilibrium, i.e. when $\frac{\partial T(\beta)}{\partial\beta} < 1$ at the equilibrium value of β, then the Model Π equilibrium is stable under least squares learning. Figure 12 shows $\frac{\partial T(\beta)}{\partial\beta}$ for the relevant parameter space of ε and reveals that β converges to its equilibrium value under least squares learning. From $\alpha = 1 - \beta$ it follows that Model Π equilibria are stable under least squares learning.

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