FADING MEMORY LEARNING IN THE COBWEB MODEL WITH RISK AVERSE HETEROGENEOUS PRODUCERS

CARL CHIARELLA, XUE-ZHONG HE AND PEIYUAN ZHU

School of Finance and Economics
University of Technology, Sydney
PO Box 123 Broadway
NSW 2007, Australia

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ABSTRACT. In this paper we analyse the dynamics of the traditional cobweb model with risk averse heterogeneous agents under bounded rationality. We consider a learning mechanism in which heterogeneous producers seek to learn the distribution of asset prices using a geometric decay processes (GDP)—the expected mean and variance are estimated as a geometric average of past observations—with either finite or infinite fading memory. With constant absolute risk aversion, the dynamics of the model can be characterized with respect to the length of memory window and the memory decay rate of the learning GDP. We show that the decay rate of the GDP of heterogeneous producers plays a complicated role on the pricing dynamics of the nonlinear cobweb model. In general, the decay rate plays a stabilizing role on the local stability of the steady state price when the memory is infinite, but this role becomes less clear when the memory is finite and the heterogeneity has double edged effect on the dynamics in the sense that heterogeneous learning can stabilize an otherwise unstable dynamics in some cases and destablize an otherwise stable dynamics in other cases as well. It is shown that (quasi)periodic solutions and strange (or even chaotic) attractors can be created through Neimark-Hopf bifurcation when the memory is infinite and through flip bifurcation as well when the memory is finite. In addition, it is found that the source of risk is the risk itself in the sense that the behaviour of producers in response to risk can generate market failure.

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1. INTRODUCTION

Consider the well-known cobweb model:

$$\begin{cases} p_t^e = aq_t + b & \text{(supply)}, \\ p_t = \alpha q_t + \mu & \text{(demand)}, \end{cases} \tag{1.1}$$

Here, q_t and p_t are quantities and prices, respectively, at period t, p_t^e is the price expected at time t based on the information at $t-1$, and $a, b, \mu\,(>0)$ and $\alpha < 0$ are constants.

Under the naive expectation scheme $p_t^e = p_{t-1}$, the price either converges to the optimal market equilibrium (when $|\alpha/a| < 1$) or explodes (when $|\alpha/a| > 1$). To obtain more realistic, oscillatory, price time paths, the literature has introduce nonlinearities and time lag into the cobweb model and such nonlinearities can come from either nonlinear supply or demand curve or risk aversion, discussed as follows.

When the producers are homogeneous, it has been shown that non-linearities in the supply or demand curves may lead the cobweb model to exhibit both stable periodic and chaotic behavior (i.e., Artsein (1983), Jensen and Urban (1984), Chiarella (1988), Holmes and Manning (1988), Hommes (1991), Puu (1991) and Day (1992).). These authors consider a variety of backward looking mechanisms for the formation of the expectations p_t^e ranging across the traditional naive expectation $p_t^e = p_{t-1}$, learning expectations (e.g., learning by arithmetic mean $p_t^e = (p_{t-1} + \cdots + p_{t-L})/L$) and adaptive learning expectation $p_t^e = p_{t-1}^e + w(p_{t-1} - p_{t-1}^e)$ with $0 \le w \le 1$.

Nonlinearity can also come from risk and risk aversion (i.e., Boussard and Gerard (1991), Burton (1993) and Boussard (1996)). As pointed in Boussard (1996), with risk averse producers, the traditional linear cobweb model becomes nonlinear. By assuming that the actual price p_t is uncertain so that p_t^e has mean $\bar{p}_t$ and variance $\bar{v}_t$, Boussard (1996) shows that, under the simplest learning scheme $\bar{p}_t = \hat{p}$ and $\bar{v}_t = (p_{t-1} - \hat{p})^2$ with constant $\hat{p}$, the nonlinear model may result in the market generating chaotic price series, and market failure, and therefore the source of risk is the risk itself (p.435, Boussard (1996)). Consequently, the study *"casts a new light on expectations. Not only are expectations pertaining to mean values important for market outcomes. Those pertaining to variability can be just as crucial"* (p.445, Boussard (1996)).

By assuming the bounded rationality, when producers are somewhat uncertain about the dynamics of the economic system in which they are to play out their roles, they need to engage in some learning scheme to update their beliefs. Among various learning schemes, the properties of recursive learning processes under homogeneous expectations have been studied extensively (e.g., Bray (1982, 1983), Evans and Ramey (1992), Balasko and Royer (1996), Evans and Honkapohja (1994, 1995, 1999), Barucci (2001, 2002)). In Bray (1982, 1983) and Evans and Honkapohja (1994, 1995), the agent's expectation is computed as the arithmetic average of all the past observations with full memory (the same weight is employed for each observation). In Barucci (2001, 2002), agent's expectation is computed as a weighted average of all the past observation with no-full memory. The weights of the average are described by a geometric process with a ratio smaller than 1 and therefore, the weights for older observations are smaller than the weights for recent observations. As pointed by Barucci (2001, 2002), these features of fading memory learning mechanism are *appealing because the assumption of a constant weight for past observations are not fully plausible from a behavioral point of view. As a matter of fact, agents do not stop to learn as time goes on and they 'forget' remote observations.* For a class of nonlinear deterministic forward-looking economic model under the fading memory learning, Barucci shows that the decay rate of the memory of the learning process plays a stabilizing role—it enlarges the local stability parameters region of the prefect foresight stationary equilibria.

Apart from Boussard (1996), a great deal of attention in the expectations formation literature has been devoted to schemes for the mean, but very little to schemes for the variance. Chiarella and He (1998, 2000) extend Boussard's framework in a way that takes account of the risk aversion of producers and allows them to estimate both the mean and variance via the arithmetic learning process (ALP) $\bar{p}_t = \frac{1}{L}\sum_{i=1}^{L} p_{t-i}, \quad \bar{v}_t = \frac{1}{L}\sum_{i=1}^{L}[p_{t-i} - \bar{p}_t]^2$ with some integer $L \geq 1$. Chiarella and He (1998, 2000) show that the resulting cobweb dynamics form a complicated nonlinear expectations feedback structure whose dimensionality depends upon the length of the window of past prices (the lag length) used to estimate the moments of the price distributions. It is found that an increase of the window length L can enlarge the parameter region (in terms of $|\alpha/a|$) of the local stability of the steady state and, at the crossover from local stability to local instability, the dynamics exhibits resonance behavior which is indicative of quite complicated dynamical behavior, and even chaos (for the model with constant elasticity supply and demand functions).

Chiarella et al (2003) extend the homogeneous model in Chiarella and He (1998, 2000) to a heterogeneous model and characterise the price dynamics by the window lengths of the heterogeneous ALP. It is found that increase of window lengths plays a stabilizing role when both lags are the same, but this stabilizing effect becomes less clear when both lags are different. This paper extend our discussion further to study the role of the memory decay rate when producers follow GDP. It is found that, when the memory is infinite, the decay rate plays a stabilizing role on the local stability of the steady state price. However, this role becomes less clear when the memory is finite and the heterogeneity has double edged effect on the dynamics in the sense that heterogeneous learning can stabilize an otherwise unstable dynamics in some cases and destablize an otherwise stable dynamics in other cases as well. It is shown that (quasi)periodic solutions and strange (or even chaotic) attractors can be created through Neimark-Hopf bifurcation when the memory is infinite and through flip bifurcation as well when the memory is finite. In addition, it is found that the source of risk is the risk itself in the sense that the behaviour of producers in response to risk can generate market failure.

The paper is organised as follows. A general cobweb model with heterogeneous producers is established in Section 2. The heterogeneous geometric decay (learning) processes (GDP) is introduced, and the existence of steady-state is then discussed in

Section 2 in Section 3. The dynamics of the heterogeneous model is analyzed for both finite and infinite memory in Sections 4 and 5, respectively.

2. Cobweb Model with Heterogeneous Producers

This section is intended to establish a cobweb model when producers are heterogeneous in their risk and expectation formulation on both the mean and variance. In the case of linear supply and demand functions, the model may be written as

$$\begin{cases} \text{Supply:} & p^e_{i,t} = a_i q_{i,t} + b_i, \quad (i = 1, 2); \\ \text{Demand:} & p_t = \alpha q_t + \mu \quad (\alpha < 0), \end{cases} \tag{2.1}$$

where q_t is the aggregate supply , $q_{i,t}$ and $p^e_{i,t}$ are the quantity and price expected of producer i at time t based on the information set at $t-1$, and p_t is the price, and $a_i, b_i, \mu\,(> 0)$ and $\alpha < 0$ are constants.

Our approach to the formation of expectations will be somewhat different in that we assume that the actual price p_t is uncertain so that the heterogeneous producers treat $p^e_{i,t}$ as a random variable drawn from a normal distribution whose mean and variance they are seeking to learn[1].

2.1. **Market Clearing Price and Heterogeneous Model.** Let $\bar{p}_{i,t}$ and $\bar{v}_{i,t}$ be, respectively, subjective mean and variance of price $p^e_{i,t}$ of producer i formed at time t based on the information set at $t-1$, and q_t be quantity at time t. With constant absolute risk aversion A_i, the marginal revenue certainty equivalent of producer i is[2]

$$\tilde{p}_{i,t} = \bar{p}_{i,t} - 2A_i \bar{v}_{i,t} q_{i,t}.. \tag{2.2}$$

Suppose a linear marginal cost, as in (2.1), so that the supply equation, under marginal revenue certainty equivalent becomes

$$\tilde{p}_{i,t} = a_i q_{i,t} + b_i \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$a q_{i,t} + b_i = \bar{p}_{i,t} - 2A_i \bar{v}_{i,t} q_{i,t}$$

and hence the supply for producer i is given by

$$q_{i,t} = \frac{\bar{p}_{i,t} - b_i}{a_i + 2A_i \bar{v}_{i,t}}. \tag{2.4}$$

[1]It would of course be preferable (and more in keeping with models of asset price dynamics in continuous time finance) to treat $p^e_{i,t}$ as log-normally distributed. However this would then move us out of the mean-variance framework so we leave an analysis of this approach to future research.

[2]With constant absolute risk aversion A_i, we assume the certainty equivalent of the receipt $r = pq$ is $R(q_t) = \bar{p}_{i,t} q_t - A_i \bar{v}_{i,t} q_t^2$. Then maximisation of this function with respect to q_t leads to the marginal revenue certainty equivalent $\tilde{p}_t = \frac{\partial R}{\partial q_t} = \bar{p}_{i,t} - 2A_i \bar{v}_{i,t} q_t$. We recall that this objective function is consistent with producers having the utility of receipts function $U_i(r) = -e^{-A_i r}$.

Denote by n_i the proportion of type i producers[3], then the market clearing price is determined by

$$p_t = \mu + \alpha \sum_i n_i \frac{\bar{p}_{i,t} - b_i}{a_i + 2A_i \bar{v}_{i,t}}. \tag{2.5}$$

In fact, it follows from (2.1) and (2.4) that the aggregated supply is given by

$$q_t = \sum n_i q_{i,t} = \frac{p_t - \mu}{\alpha}$$

and hence

$$\frac{p_t - \mu}{\alpha} = \sum n_i \frac{\bar{p}_{i,t} - b_i}{a_i + 2A_i \bar{v}_{i,t}},$$

from which (2.5) follows.

2.2. **Homogeneous Cobweb Model.** As a special case of the heterogeneous model (2.5), assume that producers are homogeneous, that is, $a_i = a$, $b_i = b$, $\bar{p}_{i,t} = \bar{p}_t$, $A_i = A$, $\bar{v}_{i,t} = v_t$, then the corresponding homogeneous model has the form

$$p_t = \mu + \alpha \frac{\bar{p}_t - b}{a + 2Av_t}, \tag{2.6}$$

and its dynamics is considered in Chiarella and He (1998).

2.3. **A Cobweb Model with Two Types of Heterogeneous Producers Following GDP.** In the following discussion, the simplest heterogeneous model when there are two types of producers is considered. Then the population of heterogeneous producers can be measured by a single parameter. Let $n_1 = (1+w)/2, n_2 = (1-w)/2$. Then (2.5) can be rewritten in the following form

$$p_t = \mu + \frac{\alpha}{2}(1+w)\frac{\bar{p}_{1,t} - b_1}{a_1 + 2A_1 \bar{v}_{1,t}} + \frac{\alpha}{2}(1-w)\frac{\bar{p}_{2,t} - b_2}{a_2 + 2A_2 \bar{v}_{2,t}}. \tag{2.7}$$

The heterogeneous model (2.7) is incomplete unless producers' expectations are specified. In this paper, the *geometric decay processes (GDP)* with either finite and infinite memory is assumed. More precise definition is introduced in the following section.

3. Heterogeneous Beliefs—Geometric Decay Process (GDP)

This section introduces the geometric decay process (GDP) with both finite and infinite memory and the dynamics of the GDP of the heterogeneous agent model is then analyzed in the following sections.

[3]In general, the proportion n_i is a function of time t, that is, $n_{i,t}$, which can be measured by certain fitness function and discrete choice probability, as in Brock and Hommes (1997). Because of the complexity of the dynamics, we consider only the case with fixed propitiation and leave the changing proportion problem to our future work.

3.1. **GDP with Finite Memory.** For type i producers, assume that the price follows a geometric probability distribution with decay rate of δ_i over a window length of L_i, that is,

$$\begin{cases} \bar{p}_{i,t} \equiv m_{i,t-1} = & B_i \sum_{j=1}^{L_i} \delta_i^{j-1} p_{t-j}, \\ \bar{v}_{i,t} \equiv v_{i,t-1} = & B_i \sum_{j=1}^{L_i} \delta_i^{j-1} [p_{t-j} - m_{i,t-1})]^j, \end{cases} \tag{3.1}$$

where $B_i = 1/(1 + \delta_i + \delta_i^2 + \cdots + \delta_i^{L_i - 1})$, L_i are integers, and $\delta_i \in [0, 1]$ are constants for $i = 1, 2$.

Two special cases of the geometric decay process (GDP) are of particular interested:

(i) When $\delta_i = 1$, the GDP (3.1) is reduced to the standard arithmetic learning process (ALP), discussed in Chiarella et al (2003).

(ii) When $\delta_i = 0$, the expectation of the mean follows the naive expectation $\bar{p}_{i,t} = p_{t-1}$ and $\bar{v}_{i,t} = 0$.

3.2. **GDP with Infinite Memory.** As the window length $L_i \to \infty$, it is shown (see Appendix A.1) that, as a limiting process of geometric decay process with finite memory, the GDP with infinite memory satisfy

$$\begin{cases} m_{i,t} = & \delta_i m_{i,t-1} + (1 - \delta_i) p_t \\ v_{i,t} = & \delta_i v_{i,t-1} + \delta_i (1 - \delta_i)(p_t - m_{i,t-1})^2. \end{cases} \tag{3.2}$$

3.3. **Existence of the Unique Steady State Price.** Denote by p^* the state steady price of the GDP model with finite memory. Then, for the GDP with finite memory (3.1), it is found from (2.7) that p^* satisfies

$$p^* = \frac{\mu - \frac{\alpha}{2}[(1+w)\frac{b_1}{a_1} + (1-w)\frac{b_2}{a_2}]}{1 - \frac{\alpha}{2}[(1+w)\frac{1}{a_1} + (1-w)\frac{1}{a_2}]}. \tag{3.3}$$

For the GDP model with infinite memory, the state steady $(p_t, m_{i,t}, v_{i,t}) = (p^*, p^*, 0)$. Note that p^* is the same under GDP with both finite and infinite memory.

In the following sections, dynamics of the heterogeneous model (2.7) are studied when agents update their estimations on both mean and variance by using the GDP with both finite memory (3.1) and infinite memory (3.2).

4. Dynamics of the Heterogeneous Model with Finite Memory GDP

This section focuses on the dynamics of (2.7) when producers follow the GDP with finite memory and different window lengths L_i. Without loss of generality, we assume $L_1 \le L_2$. Denote $L = \max\{L_1, L_2\} = L_2$. Because of the dependence of the subjective mean $\bar{p}_t$ and variance $\bar{v}_t$ on price lagged L periods, equation (2.7) is a difference equation of order L (see system (A.7) in Appendix A.2).

The local stability of the unique steady state $p_t = p^*$ is determined by the eigenvalues of the corresponding characteristic equation (equation (A.9) in Appendix A.2), which is difficult to analyze in general, in particular when L is large. In the rest of this section, we examine the case when $L \le 3$.

Denote

$$\begin{cases} \beta_1 &= -\frac{\alpha}{2a_1}(1+w) \\ \beta_2 &= -\frac{\alpha}{2a_2}(1-w). \end{cases} \tag{4.1}$$

Then $\beta_1 > 0, \beta_2 > 0$. As indicated from the following results, the local stability of the steady state depends on the parameters, including those from supply and demand functions a_1, a_2, α, the proportion difference of two types of producers w, the window lengths L_1 and L_2 used by the heterogeneous producers, and the decay rates δ_1, δ_2. The discussion here is focused on two different aspects. On the one hand, for a fixed window length combination of (L_1, L_2), we consider how the demand parameter α and the proportion difference w of producers affect the local stability of the steady state and bifurcation. On the other hand, for a set of fixed parameters, we examine how these results on the local stability and bifurcation are affected by different combination of the window lengths. Regarding the first aspect, it is found that both the local stability region and bifurcation boundary are geometrically easy to construct by using parameters β_1 and β_2, instead of w and α. However, the one-one relation (4.1) between (w, α) and (β_1, β_2) makes it possible to transform the results between different set of parameters, and in addition, to preserve the geometric relation of the local stability regions between the two sets of parameter.[4] In the following discussion, we consider the case $L_1 = L_2 = L$ first and then $L_1 \neq L_2$. Because of the geometric advantage, the results are formulated in terms of (β_1, β_2), although some of the stability regions are plotted using (w, α) as well.

4.1. **Case 1:** $L_1 = L_2 = L$**.** Consider first the case when both types of producer use the same window length, that is $L_1 = L_2 = L$, but different decay rates (δ_1, δ_2).

4.1.1. *Local Stability and Bifurcation Analysis.* The proofs of the following Propositions 4.1-4.3 can be found in Appendix A.3.

The simplest case of $L = 1$ can be treated as special case of GDP when the decay rate $\delta_i = 0$, that is, agents use the traditional naive expectation, taking the latest price as their expected price for the next period.

Proposition 4.1. *For $L_1 = L_2 = 1$, the local stability region D_{11} in terms of (β_1, β_2) is given by*

$$D_{11}(\beta_1, \beta_2) = \{(\beta_1, \beta_2) : 0 \le \beta_1 + \beta_2 < 1\}.$$

Furthermore, a flip bifurcation occurs along the boundary $\beta_1 + \beta_2 = 1$.

Proposition 4.1 indicates that, when agents use the naive expectation, the steady state becomes unstable through a flip bifurcation, leading to a two-period cycle of two prices, one is above and one is below the steady state price.

[4]Note that the determinant of the Jacobian of the transformation (4.1) does not change the sign, implying the reservation of the transformation.

Proposition 4.2. *For $L_1 = L_2 = 2$, the local stability region $D_{22}(\beta_1, \beta_2)$ of the state steady is defined by*

$$D_{22} = \{(\beta_1, \beta_2) : \Delta_1 < 1, \Delta_2 < 1\}$$

where

$$\Delta_1 = \frac{\delta_1}{1+\delta_1}\beta_1 + \frac{\delta_2}{1+\delta_2}\beta_2,$$
$$\Delta_2 = \frac{1-\delta_1}{1+\delta_1}\beta_1 + \frac{1-\delta_2}{1+\delta_2}\beta_2.$$

Furthermore,

- *a flip bifurcation occurs along the boundary $\Delta_2 = 1$ where two eigenvalues satisfy $\lambda_1 = -1, \lambda_2 \in (-1, 1)$;*
- *a Neimark-Hopf bifurcation occurs along the boundary $\Delta_1 = 1$ where the two eigenvalues are given by $\lambda_{1,2} = e^{\pm 2\pi\theta i}$, here θ is determined by*

$$\rho \equiv 2\cos(2\pi\theta) = -\left[\frac{\beta_1}{1+\delta_1} + \frac{\beta_2}{1+\delta_2}\right]. \tag{4.2}$$

Comparing the stability conditions in Propositions 4.1 and 4.2, one can see that the parameter (in terms of (β_1, β_2)) region on the local stability of the steady state is enlarged as L increases from $L = 1$ to $L = 2$. This means that agents can learn the steady state price over a wide region of parameters as they follow the GDP with $L = 2$. However, as one can see from the following discussion, these learning process, in particular the decay rates $\delta_i (i = 1, 2)$, can generate far more complicated dynamics when the steady state price becomes unstable. To understand the effect of parameters $\beta_i, \delta_i \, (i = 1, 2)$ on the stability of the state steady and types of bifurcation, we now undertake a more detailed analysis by considering various cases in terms of parameters (δ_1, δ_2).

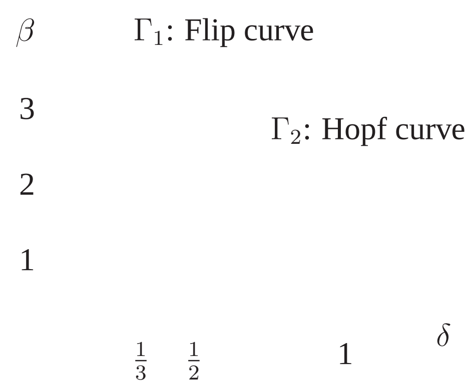


FIGURE 1. Stability region and bifurcation boundaries for $L_1 = L_2 = 2$, $\delta_1 = \delta_2 = \delta$ and $\beta = \beta_1 + \beta_2$.

- *The case* $\delta_1 = \delta_2 = \delta$: In this case, it follows from Proposition 4.2 that the stability region of the state steady can be characterized by two parameters β and δ:

$$D_{22} = \{(\beta_1, \beta_2) : 0 \leq \beta \equiv \beta_1 + \beta_2 < \bar{\beta}\},$$

where

$$\bar{\beta} = \begin{cases} \frac{1+\delta}{1-\delta} & \delta \leq \frac{1}{2} \\ \frac{1+\delta}{\delta} & \delta > \frac{1}{2}. \end{cases}$$

In this case, a flip bifurcation occurs along the boundary

$$\Gamma_1 : \beta = (1+\delta)/(1-\delta), \qquad \delta \in [0, 1/2],$$

and a Neimark-Hopf bifurcation occurs along the boundary

$$\Gamma_2 : \beta = (1+\delta)/\delta, \qquad \delta \in (1/2, 1], \qquad \rho = -1/\delta \in (-2, -1].$$

Note that functions $f(x) = \frac{1+x}{1-x}, g(x) = \frac{1+x}{x}$ satisfy $f' > 0, f'' > 0, g' < 0, g'' > 0$. The stability region D_{22} is plotted in Figure 1. One can see that the different decay rate δ has different effect on the stability:

(i) for $\delta \in [0, \frac{1}{2}]$, the stability region D_{22} in terms of the parameter β is enlarged as δ increases;

(ii) for $\delta \in [\frac{1}{2}, 1]$, the stability region D_{22} in terms of the parameter β is enlarged as δ decreases;

(iii) for $\delta = 0$, we have the smallest parameter β region for the local stability: $0 \leq \beta < 1$; while for $\delta = 1/2$, we have the largest parameter β region for the local stability: $0 \leq \beta < 3$; for $\delta > 1/2$, increase of δ does not enlarge the parameter region on the local stability;

(iv). for small decay rate $\delta \leq 1/2$, the steady state price becomes unstable through flip bifurcation (implying a two-period cycle), while for large decay rate $\delta > 1/2$, the steady state price become unstable through Neimark-Hopf bifurcation, which in turn generates either period cycle or aperiodic orbit.

- *The case* $0 \leq \delta_1, \delta_2 < 1/2$ *and* $\delta_1 \neq \delta_2$. In this case, it follows from Proposition 4.2 that the local stability region is defined by

$$D_{22} = \{(\beta_1, \beta_2) : \Delta_2 < 1\}$$

and the steady state becomes unstable through a flip bifurcation only, as indicated in Figure 2(a). Furthermore, as either δ_1 or δ_2 increases, the local stability region D_{22} of the state steady with respect to parameters (β_1, β_2) is enlarged, as indicated in Figure 3(b).

- *The case* $\delta_1, \delta_2 > 1/2$ *and* $\delta_1 \neq \delta_2$. In this case, it follows from Proposition 4.2 that the local stability region is defined by

$$D_{22} = \{(\beta_1, \beta_2) : \Delta_1 < 1\}$$

β_2

$\frac{1+\delta_2}{1-\delta_2}$

flip curve

$\frac{1+\delta_1}{1-\delta_1}$ β_1

(a) $\delta_1 \neq \delta_2, 0 < \delta_1, \delta_2 < \frac{1}{2}$

β_2

$1 + \frac{1}{\delta_2}$

Hopf curve

$1 + \frac{1}{\delta_1}$ β_1

(b) $\delta_1 \neq \delta_2, \frac{1}{2} < \delta_1, \delta_2 < 1$

β_2

$1 + \frac{1}{\delta_2}$

Hopf curve

A

flip curve

$\frac{1+\delta_1}{1-\delta_1}$ β_1

(c) $\delta_1 \neq \delta_2, \delta_1 < \frac{1}{2}, \delta_2 > \frac{1}{2}$

β_2

$\frac{1+\delta_1}{1-\delta_1}$

flip curve

A

Hopf curve

$1 + \frac{1}{\delta_1}$ β_1

(d) $\delta_1 \neq \delta_2, \delta_1 > \frac{1}{2}, \delta_2 < \frac{1}{2}$

FIGURE 2. Stability region and bifurcation boundaries for (a) $0 \leq \delta_1, \delta_2 \leq 1/2$; (b) $1/2 < \delta_1, \delta_2 \leq 1$; (c) $0 \leq \delta_1 \leq 1/2 < \delta_2 \leq 1$; and (d) $0 \leq \delta_2 \leq 1/2 < \delta_2 \leq 1$, where $A : (\beta_1, \beta_2) = ((1-2\delta_2)(1+\delta_1)/(\delta_1 - \delta_2), (1-2\delta_1)(1+\delta_2)/(\delta_2 - \delta_1)$.

and the steady state becomes unstable through a Neimark-Hopf bifurcation, as indicated in Figures 2(b) and 3(a). Along the bifurcation boundary, the nature of bifurcation is characterised by θ which satisfies (see Appendix A.2 for the details) $\rho \equiv 2\cos(2\pi\theta) \in (-1/\min(\delta_1, \delta_2), -1/\max(\delta_1, \delta_2))$. Say, for example, for fixed $\delta_2 = 2/3$, the region for the parameter ρ various for different δ_1, as illustrated in Table 1.

δ_1	ρ	δ_1	ρ
1/2	(-2, -3/2)	3/4	(-3/2, -4/3)
2/3	-3/2	1	(-3/2, -1)

TABLE 1. Parameter region for ρ with fixed $\delta_2 = 2/3$ and different δ_1.

Different from the previous case, as either δ_1 or δ_2 increases, the local stability region of the parameters (β_1, β_2) becomes smaller, as indicated in Figures 2(c), (d) and 3(b).

- *The case either* $0 < \delta_1 < 1/2, \delta_2 > 1/2$ *or* $0 < \delta_2 < 1/2, \delta_1 > 1/2$. In this case, the stability region is bounded by two bifurcation boundaries, as indicated in Figures 2(c), (d) and 3. One is the flip bifurcation boundary defined by $\Delta_2 = 1$, the other is the Neimark-Hopf bifurcation boundary defined by $\Delta_1 = 1$, along which the types of bifurcation are characterised by θ which satisfies (see Appendix A.2 for the details) $\rho \equiv 2\cos(2\pi\theta) \in (-2, -1/\max(\delta_1, \delta_2))$. It is interesting to see that, unlike the previous case, the parameter ρ is determined only by either δ_1 (when $\delta_1 > 1/2$) or δ_2 (when $\delta_2 > 1/2$). Also, the parameter region for (β_1, β_2) on the local stability is enlarged as either δ_1 increases and δ_2 decreases or δ_2 increases and δ_1 decreases.

The previous Propositions 4.1 and 4.2 seems to indicate that as L increases from 1 to 2, on the one hand, the stability region is enlarge and, on the other hand, instability leads to a more complicated price dynamics. One may expect a similar dynamics would occur if we increase L from 2 to 3. However, the following Proposition 4.3 indicates that this may not be the case.

Proposition 4.3. *For $L_1 = L_2 = 3$, the local stability region $D_{33}(\beta_1, \beta_2)$ of the state steady is defined by*

$$D_{33} = \{(\beta_1, \beta_2) : \Delta_3 < 1\}$$

where

$$\Delta_3 = \frac{1 - \delta_1 + \delta_1^2}{1 + \delta_1 + \delta_1^2}\beta_1 + \frac{1 - \delta_2 + \delta_2^2}{1 + \delta_2 + \delta_2^2}\beta_2.$$

Furthermore, the steady state price becomes unstable through a flip bifurcation boundary defined by $\Delta_3 = 1$.

It is interesting to see that, similar to the case $L = 1$, but different from the case $L = 2$, the steady state becomes unstable only through flip bifurcation when $L = 3$. Moreover, the parameter region on the local stability is enlarged as the decay rates δ_i increase. The stability regions are plotted in Figure 4(a) for $\delta_1 = \delta_2 = \delta, \beta = \beta_1 + \beta_2$ and Figure 4(b) for $\delta_1 \neq \delta_2$ and fixed $\delta_2 = 1/2$.

A general comparison among $L = 1, 2$ and 3 may not be easy for various δ_1 and δ_2. However, such comparison when $\delta_1 = \delta_2 = \delta$ can lead to some insight regarding the role of the decay rate on the price dynamics. In such case, the stability condition for $L = 3$ is given by

$$\beta \equiv \beta_1 + \beta_2 < \frac{1 + \delta + \delta^2}{1 - \delta + \delta^2} \equiv H(\delta).$$

Note that $H(0) = 1, H(1) = 3, H' > 0, H'' > 0$. Stability regions for $L = 1, 2$ and 3 are plotted in Figure 4(c) . One can see that

β_2

Neimark-Hopf bifurcation surface

3

2

1

Flip bifurcation surface

$\frac{1}{2}$ 1 δ_1

1

2

3

(a)

β_1

β_2

Flip bifurcation surface

2

1

1 δ_1

1

2

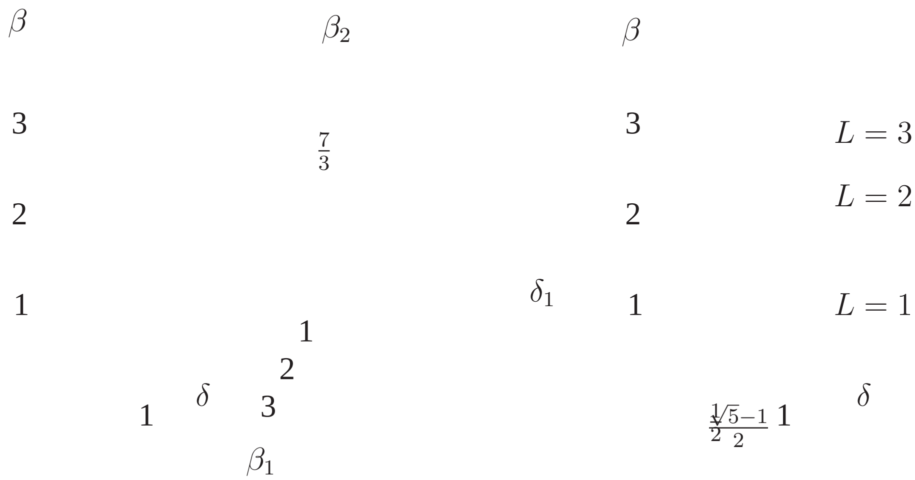
3

Neimark-Hopf bifurcation surface

(b)

β_1

FIGURE 3. Stability region and bifurcation boundary surfaces for (a) $\delta_2 = 2/3$, and (b) $\delta_2 = 1/3$.



(a) (b) (c)

FIGURE 4. Stability region and bifurcation boundary (or surfaces) for $L = 3$ and (a) $\delta_1 = \delta_2 = \delta, \beta = \beta_1 + \beta_2$; (b) $\delta_2 = 1/2$, $\delta_1 \in [0, 1]$. (c) Comparison of the stability regions for $L = 1, 2, 3$.

- for $\delta \in [0, 1/2]$, the parameter β region on the local stability of the steady state is enlarged as δ increase, $L = 2$ leads to the largest stability region, and the steady state becomes unstable through a flip bifurcation;
- for $\delta \in (1/2, 1]$, $L = 2$ gives a larger stability region for $\delta \leq (\sqrt{5} - 1)/2$, while $L = 3$ gives a larger stability region for $\delta > (\sqrt{5} - 1)/2$. In addition, the steady state becomes unstable through a Neimark-Hopf bifurcation for $L = 2$, but a flip bifurcation for $L = 3$.

4.1.2. *Dynamics of the Nonlinear System—Numerical Analysis.* Guided by the above local analysis, numerical simulations are used to demonstrate the dynamics of the nonlinear system (2.7) and (3.1).

For $L = 1$, the GDP is reduced to the naive expectation and numerical simulations show the prices are either converge to the steady state price (when $\beta = \beta_1 + \beta_2 < 1$) or explode (when $\beta = \beta_1 + \beta_2 > 1$). The flip bifurcation does not lead to price oscillation and fluctuation.

For $L = 2$, the stability regions and bifurcation boundaries in terms of parameters (α, w) of the nonlinear system (2.7) are plotted in Figure 5.

- For $\delta_1 = \delta_2 = \delta$, the local stability region is bounded by a flip bifurcation boundary for $0 \leq \delta = 0.25, 0.5 \leq 1/2$ and a Neimark-Hopf bifurcation boundary for $\delta = 0.75, 1 > 1/2$ with $\rho \in [-2, -1]$, respectively, as indicated by Figure 5(a).
- For $\delta_1 \neq \delta_2$ and a fixed $\delta_1 = 0.15$, the local stability region is bounded by a flip bifurcation boundary for $0 \leq \delta_2 = 0.15, 0.5 \leq 1/2$ and both flip and

Neimark-Hopf bifurcation boundaries for $\delta_2 = 0.75 > 1/2$, as indicated in Figure 5(b).

(a) (b)

FIGURE 5. Local stability regions and bifurcation boundaries for $L = 2$ and (a) $\delta_1 = \delta_2 = 1/4, 1/2, 3/4, 1$; (b) $\delta_1 = 0.15$, $\delta_2 = 0.25, 0.5, 0.75$ with parameters $a_1 = 0.8 < a_2 = 1, A_1 = A_2 = 0.05, \beta = 11, b_1 = b_2 = 0$.

To illustrate the dynamics of the memory decay parameter, a bifurcation diagram for parameter δ_2 is plotted in Figure 6 with parameters $\alpha = -2.5, w = -0.6, \delta_1 = 0.15, a_1 = 0.8, a_2 = 1, A_1 = A_2 = 0.005, \beta = 11, b_1 = b_2 = 0$. In particular, for $\delta_2 = 0.2$ and 0.88, the phase plots and the corresponding time series are illustrated in Figure 7. For $\delta_2 = 0.2$, the prices converge to a two-period cycle, characterized by the flip bifurcation, while for $\delta_2 = 0.88$, the prices converge to a closed orbit in their phase plot, which is characterized by the Neimark-Hopf bifurcation.

It is interesting to see that the local stability condition and bifurcation in Proposition 4.1-4.3 are independent of the risk aversion coefficients A_i of the heterogeneous agents. This is because that they are associated with the variance, a higher order term of the linearised system of the nonlinear system at the steady state. In the above simulations in Figures 6 and 7, both the risk aversion coefficients are small, and hence the risk aversion and variance have no significant influence on the price dynamics induced from local stability analysis. When agents are more risk averse and willing to learn both mean and variance, the price dynamics are expected to be stabilized in the sense that irregular price patterns, such as quasi-periodic cycles, with higher variability may become regular, such as cycles, with lower variability. This can be verified (not reported here) for the case corresponding to the right panel in Figure 7, in which the steady state price becomes unstable through a Neimark-Hopf bifurcation and prices converge to aperiodic pattern characterized by the closed orbit on the phase plot for small risk aversion coefficients $A_1 = A_2 = 0.005$. As either A_1 or A_2 increases,

FIGURE 6. Bifurcation diagrams of the nonlinear system for δ_2 with parameters $\alpha = -2.5, w = -0.6, \delta_1 = 0.15, a_1 = 0.8, a_2 = 1, A_1 = A_2 = 0.005, \beta = 11, b_1 = b_2 = 0.$.

(a) (b)

FIGURE 7. Phase plot and time series of the nonlinear system for (a) $\delta_2 = 0.2$ and (b) $\delta_2 = 0.88$ with parameters $\alpha = -2.5, w = -0.6, \delta_1 = 0.15, a_1 = 0.8, a_2 = 1, A_1 = A_2 = 0.005, \beta = 11, b_1 = b_2 = 0.$.

the closed orbit becomes smaller (say for $A_1 = A_2 = A = 0.01$). However, as A_i increases further (say $A = 0.05$), prices converge to either aperiodic cycles (characterised by closed orbits for the phase plots) with lower variability for initial values near the steady state price or 3-period cycles with higher variability for initial values not near the steady state price. Similar price dynamics are also observed when

$\delta_1, \delta_2 > 1/2$. This suggests that, when the steady state price becomes unstable through a Neimark-Hopf bifurcation, an increase in the risk aversion can stabilise otherwise unstable price patterns initially and leads to even simple price dynamics. However, this is not necessarily true when the steady state price becomes unstable through a flip bifurcation.

FIGURE 8. Phase plot and time series of the nonlinear system for $\alpha = -2.5, w = -0.6, \delta_1 = 0.15, \delta_2 = 0.02, a_1 = 0.8, a_2 = 1, A_1 = A_2 = 0.5, \beta = 11, b_1 = b_2 = 0.$.

For a set of parameters:

$$\delta_1 = 0.15, \delta_2 = 0.02, \alpha = -2.5, \beta = 11, b_1 = b_2 = 0, w = -0.6, a_1 = 0.8, a_2 = 1,$$

local stability analysis implies that the steady state price becomes unstable through a flip bifurcation when δ_2 is small. This can be verified for A_i small (say $A_i = 0.005$ or 0.05). As A_i increases, the prices converge to period-4 cycle for $A_i = 0.2$, period-8 cycle for $A_i = 0.35$, period-16 cycle for $A_i = 0.36$, and a strange attractor for $A_i = 0.5$. This strange attractor and the corresponding chaotic time series generated through such flip bifurcation for $A_i = 0.5$ are plotted in Figure 8.

For $L = 3$, numerical simulations (not reported here) show that parameter α region on the stability of the steady state price is enlarged as δ_i increases. The steady state price become unstable through a flip bifurcation only, as indicated by Proposition 4.3. Figure 9 illustrates the phase plot of price dynamics when the steady state price is unstable. For $\alpha = -4$, prices converge to a two-period cycle (as indicated by the flip bifurcation), as α decreases further, the attractors become two coexisting closed orbits

FIGURE 9. Phase plot of the nonlinear system for $L_1 = L_2 = 3$, $\alpha = -4, -5, -6, -7$ and $w = -0.6, \delta_1 = 0.8, \delta_2 = 0.5, a_1 = 0.8, a_2 = 1, A_1 = A_2 = 0.005, \beta = 11, b_1 = b_2 = 0.$.

for $\alpha = -5$ and -6. However, for $\alpha = -7$, prices converge to a 10-period cycle. Furthermore, there seems no chaotic attractor generated from the flip bifurcation, unlike the case of $L = 2$.

4.2. **Case 2:** $L_1 \neq L_2$**.**

Consider now the case when both types of producer use the different window length $L_1 \neq L_2$ and decay rates (δ_1, δ_2).

4.2.1. *Local Stability and Bifurcation Analysis.* When $\delta_1 = 0$, the GDP with $(L_1, L_2) = (2, 2)$ and $(3, 3)$ are reduced to the GDP with $(L_1, L_2) = (1, 2)$ and $(1, 3)$, respectively. Therefore, one obtains the following Corollaries 1-2 from Propositions 4.2-4.3 by taking $\delta_1 = 0$.

Corollary 1. *For $L_1 = 1, L_2 = 2$, the stability region $D_{12}(\beta_1, \beta_2)$ of the state steady is defined by*

$$D_{12} = \{(\beta_1, \beta_2) : \Delta_4 < 1\}$$

for $\delta_2 \in [0, 1/2]$ and

$$D_{12} = \{(\beta_1, \beta_2) : \Delta_4 < 1, \Delta_5 < 1\}$$

for $\delta_2 \in (1/2, 1]$, where

$$\Delta_4 = \beta_1 + \frac{1-\delta_2}{1+\delta_2}\beta_2; \qquad \Delta_5 = \frac{\delta_2}{1+\delta_2}\beta_2.$$

In addition,

- *a flip bifurcation occurs along the boundary $\Delta_4 = 1$ for $\delta_2 \in [0, 1/2]$;*

- *both flip and Neimark-Hopf bifurcations occur along the boundary* $\Delta_4 = 1$ *and* $\Delta_5 = 1$*, respectively, for* $\delta_2 \in (1/2, 1]$*. Furthermore, the nature of the Neimark-Hopf bifurcation is determined by*

$$\rho \equiv 2\cos(2\pi\theta) = -[\beta_1 + 1/\delta_2].$$

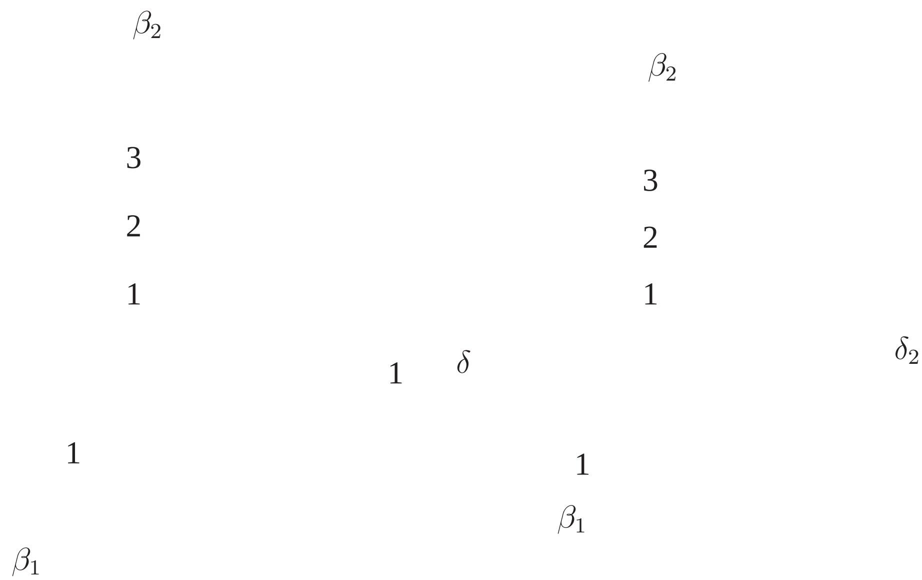


FIGURE 10. Stability region and bifurcation boundaries for (a) $(L_1, L_2) = (1, 2)$, and (b) $(L_1, L_2) = (1, 3)$.

The stability region and the bifurcation boundaries in parameters $(\delta_2, \beta_1, \beta_2)$ space are plotted in Figure 10(a). One can see that the stability region is bounded by a flip bifurcation surface for $\delta_2 \le 1/2$ and both flip and Neimark-Hopf bifurcation surfaces for $\delta_2 > 1/2$.

By applying Proposition 4.3, we obtain the following result for $(L_1, L_2) = (1, 3)$. The stability region and the flip bifurcation surface are plotted in Figure 10(b).

Corollary 2. *For* $L_1 = 1, L_2 = 3$*, the stability region* $D_{13}(\beta_1, \beta_2)$ *of the state steady is defined by*

$$D_{13} = \{(\beta_1, \beta_2) : \Delta_6 < 1\},$$

where

$$\Delta_6 = \beta_1 + \frac{1 - \delta_2 + \delta_2^2}{1 + \delta_2 + \delta_2^2}\beta_2.$$

In addition, the stability region is bounded only by a flip bifurcation boundary defined by $\Delta_6 = 1$.

For $L_1 = 1$, comparing the stability regions between $L_2 = 2$ and $L_2 = 3$, one can verify that $D_{13} \subset D_{12}$ for $\delta_2 \in [0, (\sqrt{5}-1)/2]$. However, for $\delta_2 \in ((\sqrt{5}-1)/2-1, 1]$, $D_{12} \subset D_{13}$ when $\beta_1 \leq \beta_1^*$ and $D_{13} \subset D_{12}$ when $\beta_1 \geq \beta_1^*$, where $\beta_1^* = 1 - (1 + \delta_2^3)/[\delta_2(1 + \delta_2 + \delta_2^2)]$.

For $(L_1, L_2) = (2, 3)$, the following result can be obtained (see Appendix A.4 for the proof).

Proposition 4.4. *For $L_1 = 2, L_2 = 3$, the stability region $D_{23}(\beta_1, \beta_2)$ of the state steady is defined by*

$$D_{23} = \{(\beta_1, \beta_2) : \Delta_7 < 1\}$$

for $\delta_1 \in [0, 1/2]$ and

$$D_{23} = \{(\beta_1, \beta_2) : \Delta_7 < 1, \Delta_8 < 1\}$$

for $\delta_1 \in (1/2, 1]$, where

$$\Delta_7 = \frac{1-\delta_1}{1+\delta_1}\beta_1 + \frac{1-\delta_2+\delta_2^2}{1+\delta_2+\delta_2^2}\beta_2;$$

$$\Delta_8 = \frac{\delta_1}{1+\delta_1}\beta_1 + \frac{\delta_2}{1+\delta_2+\delta_2^2}\beta_2 - \frac{\delta_2\beta_2}{1+\delta_2+\delta_2^2}\left(\frac{\beta_1}{1+\delta_1} + \frac{(1-\delta_2^2)\beta_2}{1+\delta_2+\delta_2^2}\right).$$

Furthermore,

- *a flip bifurcation occurs along the boundary $\Delta_7 = 1$ for $\delta_1 \in [0, 1/2]$;*
- *both flip and Neimark-Hopf bifurcations occur along the boundary $\Delta_7 = 1$ and $\Delta_8 = 1$, respectively, for $\delta_1 \in (1/2, 1]$.*

Because of the nonlinearity of β_i in Δ_8, it is not easy to get a complete geometric relation for $L_1 = 2, L_2 = 3$ and related discussion can be conduct by using numerical simulation in the following subsection.

4.2.2. *Dynamics of the Nonlinear System—Numerical Analysis.* For $(L_1, L_2) = (2, 3)$, we choose a set of parameters $\delta_1 = 0.15, \delta_2 = 0.3, \beta = 11, b_1 = b_2 = 0, w = -0.6, a_1 = 0.8, a_2 = 1$. Since $\delta_1 < 1/2$, the steady state become unstable through a flip bifurcation. It is found that the price dynamics generated through bifurcation parameter α is different from that through the risk aversion coefficients.

- For fixed risk aversion coefficients $A_1 = A_2 = 0.005$, the price dynamics generated through the bifurcation parameter α is similar to the case of $(L_1, L_2) = (1, 3)$. That is, as α decreases, the steady state price becomes unstable and prices converge to 2-period cycle, and then to aperiodic cycles (characterised by two coexisting closed orbits), and then to simple periodic cycles again. In addition, the variability of the prices is also increasing as α decreases.
- For fixed $\alpha = -4$, changing of the risk aversion coefficients can generate a very rich dynamics. For fixed $A_1 = 0.05$ and changing A_2, prices converge to various types of attractors, such as 4-, 8-cycles, strange attractors induced

by period-doubling bifurcation, period-5, 10 cycles and strange attractors, as indicated in Table 2 and Figure 11. For $A_1 = 1$ fixed, as A_2 increases, prices converge to 5-period cycles (say, for $A_2 = 0.6$), strange attractors (say, for $A_2 = 0.7$), three coexisting closed orbits (for $A_2 = 0.79$), and 3-period cycle (for $A_2 = 0.81$), as indicated in Figure 12.

A_2	Attractor	A_2	Attractor
0.005	2-cycle	0.13	5-cycle
0.05	4-cycle	0.35	10-cycle
0.07	8-cycle	0.42	20-cycle
0.08	SA(4)	0.47	SA(5)
0.09	SA(2)	0.6	SA(5)
0.12	SA(1)	1	SA(1)

TABLE 2. Attractors generated by the risk aversion coefficients $A_1 = 0.005$ and various A_2, where SA(m) stands for strange attractor with m pieces on the phase plane.

Instead of $\delta_1 = 0.15 < 1/2$, we can select $\delta_1 = 0.6 > 1/2$. In this case, the steady state price can become unstable through either a flip or Hopf bifurcation. A similar price pattern and bifurcation routine to complicated price dynamics can be observed for changing the risk aversion coefficients, except non-regular, even chaotic, price dynamics can be generated through the bifurcation parameter α.

5. DYNAMICS OF HETEROGENEOUS BELIEFS — GEOMETRIC DECAY PROCESS WITH INFINITE MEMORY

From the discussion in the previous section, we can see that the lags involved in the GDP can have different effect on the stability of the steady state price and price dynamics. In this section, we consider a limiting case when the lags tend to infinite. Let δ_i be the decay rate of agent i's memory. Then it follows from (3.2) that the conditional mean $m_{i,t}$ and variance $v_{i,t}$ are given by

$$\begin{cases} m_{1,t} = & \delta_1 m_{1,t-1} + (1-\delta_1)p_{t-1} \\ m_{2,t} = & \delta_2 m_{2,t-1} + (1-\delta_2)p_{t-1} \\ v_{1,t} = & \delta_1 v_{1,t-1} + \delta_1(1-\delta_1)(p_t - m_{1,t-1})^2 \\ v_{2,t} = & \delta_2 v_{2,t-1} + \delta_2(1-\delta_2)(p_t - m_{1,t-1})^2. \end{cases} \tag{5.1}$$

Let

$$x_t = m_{1,t}, y_t = m_{2,t}, z_t = v_{1,t}, u_t = v_{2,t}.$$

FIGURE 11. Phase plot of the nonlinear system for $(L_1, L_2) = (2, 3)$, $A_2 = 0.08(a), 0.12(b), 0.47(c), 1(d)$ and $\alpha = -4, w = -0.6, \delta_1 = 0.8, \delta_2 = 0.5, a_1 = 0.8, a_2 = 1, A_1 = 0.05, \beta = 11, b_1 = b_2 = 0.$.

Then, under the GDP with infinite memory (5.1), the nonlinear system (2.7) is equivalent to the following 5-dimensional system

$$\begin{cases} p_t &= f(p, x, y, z, u)_{t-1} \\ x_t &= \delta_1 x_{t-1} + (1 - \delta_1) p_{t-1} \\ y_t &= \delta_2 y_{t-1} + (1 - \delta_2) p_{t-1} \\ z_t &= \delta_1 z_{t-1} + \delta_1 (1 - \delta_1)(p_t - x_{t-1})^2 \\ u_t &= \delta_2 u_{t-1} + \delta_2 (1 - \delta_2)(p_t - y_{t-1})^2, \end{cases} \tag{5.2}$$

where

$$f(p, x, y, z, u) = \beta + \frac{\alpha}{2}[(1 + w)\frac{x - b_1}{a_1 + 2A_1 z} + (1 - w)\frac{y - b_2}{a_2 + 2A_2 z}]$$

FIGURE 12. Phase plot of the nonlinear system for $(L_1, L_2) = (2, 3)$, $A_2 = 0.7(a), 0.79, 0.81(b)$ and $\alpha = -4, w = -0.6, \delta_1 = 0.8, \delta_2 = 0.5, a_1 = 0.8, a_2 = 1, A_1 = 1, \beta = 11, b_1 = b_2 = 0.$.

We first obtain the following result on the local stability and bifurcation and its proof can be found in Appendix A.5.

Proposition 5.1. *The fixed equilibrium x^* is LAS if*

$$\begin{aligned}&\left[\delta_1\beta_2(1-\delta_2)+\delta_2\beta_1(1-\delta_1)-\tfrac{\delta_1+\delta_2}{2}\right]^2\\&\qquad+\beta_2(1-\delta_2)+\beta_1(1-\delta_1)<1+\tfrac{(\delta_1-\delta_2)^2}{4}.\end{aligned} \tag{5.3}$$

Furthermore, the steady state becomes unstable through a Neimark-Hopf bifurcation and the nature of the Neimark-Hopf bifurcation is determined by

$$\rho = \delta_1[1-\beta_2(1-\delta_2)] + \delta_2[1-\beta_1(1-\delta_1)]. \tag{5.4}$$

In particular, when $\delta_1 = \delta_2 = \delta$, the steady state is stable if

$$\beta \equiv \beta_1 + \beta_2 < \frac{1}{1-\delta}.$$

and the steady state becomes unstable through a Neimark-Hopf bifurcation with $\rho = \delta \in [0, 1)$.

It is interesting to see that the steady state become unstable through a Neimark-Hopf bifurcation only. It may not be easy to see the effect of the decay rates on the stability region from condition (5.3), but the condition (5.4) when $\delta_1 = \delta_2 = \delta$ indicates that the parameter region for $\beta = \beta_1 + \beta_2$ on the local stability is enlarged as δ increases, as shown in Figure 13(a). In addition, the parameter region on the stability becomes unbounded as $\delta \to 1$. This general feature is still hold when $\delta_1 \neq \delta_2$ and this can be verified by numerical plot of the bifurcation surface, which can be indicated by Figure for fixed $\delta_1 = 0.5$. Hence the stability region is enlarged as the decay rates increase.

β β $L = +\infty$

3 3 $L = 3$

2 2 $L = 2$

1 1 $L = 1$

δ $\frac{1}{2}$ 1 δ

$\delta_1 = \delta_2 = \delta$

(a) (b)

FIGURE 13. (a) Stability region and bifurcation boundary for GDP with $L = \infty$ and $\delta_1 = \delta_2 = \delta$; (b) Comparison of stability regions for $L_1 = L_2 = L = 1, 2, 3, \infty$ and $\delta_1 = \delta_2 = \delta$.

A comparison when $L_1 = L_2 = L$ and $\delta_1 = \delta_2 = \delta$ is plotted in Figure 13(b) for $L = 1, 2, 3$ and $L = \infty$. One can see that, for δ not close to 1, the stability region may not be enlarged as L increased for L to be both finite and infinite. However, this is indeed the case as δ is close to 1. Therefore, loosely speaking, high decay rate with long memory can improve the stability of the steady state price.

Numerical simulations can be used to show various price dynamics when the steady state price becomes unstable and it is found that the price dynamics is more dependent on the decay rates, rather than the risk aversion coefficients. For a set of parameters: $\beta = 11, w = 0, a_1 = 0.8, a_2 = 1, b_1 = b_2 = 0$, we have the following observations.

- For both decay rates close to 1, say $\delta_1 = \delta_2 = 0.9$, the steady state price becomes unstable when α is small, say $\alpha = -8$. As α decreases further, prices converge aperiodically, characterised by closed orbits in the phase plot, with high variability. Also, for fixed α, a sufficient high δ_i (close to 1) can lead an otherwise unstable price dynamics to converge to the steady state price, as indicated by the above local stability analysis.
- For fixed $\alpha = -10, \delta_1 = 0.2, \delta_2 = 0.9$ and $A_1 = 0.05$, prices converge to some strange attractors for a wide range of A_2 (say $A_2 \in (0.05, 2)$), as shown in Figure 14(b) for $A_1 = 0.05$. However, when we fixed $A_2 = 0.05$ and increase A_1 from 0.05 up to 2, it is found that prices in the phase plane converge to strange attractors for A_1 small (say, $(A_1 = 0.05, 0.8)$), and then to a 5-period cycle for $A_1 = 1.2$, and then to a strange attractor for $A_1 = 1.5$. This

FIGURE 14. Phase plot of the nonlinear system for GDP with infinite memory and (a) $\alpha = -20, \delta_1 = 0.6$; (b) $\alpha = -10, \delta_1 = 0.2$ and $A_1 = A_2 = 0.05, w = 0, \delta_1 = 0.8, \delta_2 = 0.5, a_1 = 0.8, a_2 = 1, \beta = 11, b_1 = b_2 = 0.$.

indicates that when agents have infinite memory, the risk aversion coefficient has no significant influence on the price dynamics when agents have high decay rate (and in particular, when agents have almost full memory over the whole history of price), however such influence can be significant when agents have a low decay rate.

- For the GDP with finite memory case discussed in the previous section, some of the regular or strange attractors are generated through bifurcation with certain period cycles. However, for the GDP with infinite memory, such attractor may have no connection with such periodic-cycle-induced bifurcation, as shown in Figure 14(a).

6. CONCLUSIONS

In this paper we have introduced a GDP learning mechanism into the traditional cobweb model with risk averse heterogeneous agents by allowing producers to learn both mean and variance with different geometric decay rate. For a class of nonlinear forward-looking models with homogeneous agents, Barucci (2001, 2002) show that, when the memory is infinite, the memory decay rate plays a stabilizing role in the sense that increasing the decay rate of the learning process the parameters stability region of a stationary rational expectation equilibrium becomes larger and eliminate cycles and chaotic attractors created through flip bifurcation, but not Hopf bifurcation. We have shown in this paper that the memory decay rate plays a similar stabilizing role and complicated price dynamics can be created through Neimark-Hopf bifurcation, not flip bifurcation, when memory is infinite and agents are heterogeneous. However, when memory is finite, we show that the decay rate of the GDP of heterogeneous producers plays a complicated role on the pricing dynamics. When both the lag lengths are odd, increasing of the decay rate enlarges the parameters region of the stability of the steady state and complicated price dynamics can only be created through flip bifurcation. However when both the lag lengths are not odd, there exists a critical value (between 0 and 1) such that, when the decay rate is below the critical value, the decay rate plays the stabilizing role and, for the decay rate is above the critical value, the decay rate plays a destabilizing role in the sense that the parameters stability region becomes smaller as the decay rate increases. In addition, (quasi)periodic cycles and strange attractors can be created through flip bifurcations when the decay rate is below the critical value and Neimark-Hopf bifurcations when the decay rate is above the critical value. It is also found that the source of risk is the risk itself in the sense that the behaviour of producers in response to risk can generate market failure.

The heterogeneous GDP considered in this paper are some of the simplest learning processes and the analysis has shown how they yield very rich dynamics in terms of the stability, bifurcation and routes to complicated dynamics. In practice, agents revise their expectations by adapting the decay rate in accordance to observations. How the GDP learning affects the dynamics in general is a question left for future work.

APPENDIX

A.1. **Mean and Variance of GDP with Infinite Memory.** Let m_t and v_t be the mean and variance of the GDP with lag length L, that is

$$\begin{cases} m_{t-1} = & B[p_{t-1} + \delta p_{t-2} + \cdots + \delta^{L-1} p_{t-L}], \\ v_{t-1} = & B[(p_{t-1} - m_{t-1})^2 + \delta(p_{t-2} - m_{t-1})^2 \\ & + \delta^2 (p_{t-3} - m_{t-1})^2 + \cdots + \delta^{L-1}(p_{t-L} - m_{t-1})^2], \end{cases} \tag{A.1}$$

where

$$B = \frac{1-\delta}{1-\delta^L} \quad \text{for } \delta \in [0,1) \quad \text{and } B = \frac{1}{L} \quad \text{for } \delta = 1.$$

The mean process m_t can be rearranged as follows:

$$m_t = B[p_t - \delta^L p_{t-L}] + \delta m_{t-1}.$$

Then for $\delta \in [0,1)$, as $L \to \infty$, the limiting mean process is given by

$$m_t = (1-\delta)p_t + \delta m_{t-1},$$

which can be written as follows

$$m_t - m_{t-1} = (1-\delta)(p_t - m_{t-1}) \tag{A.2}$$

or

$$m_t - p_t = \delta(m_{t-1} - p_t). \tag{A.3}$$

For the variance process, from

$$v_t = B[(p_t - m_t)^2 + \delta(p_{t-1} - m_t)^2 + \cdots + \delta^{L-1}(p_{t-(L-1)} - m_t)^2].$$

we have

$$\begin{aligned} v_t - \delta v_{t-1} &= B[(p_t - m_t)^2 \\ &\quad + \delta[(p_{t-1} - m_t)^2 - (p_{t-1} - m_{t-1})^2] \\ &\quad + \delta^2[(p_{t-2} - m_t)^2 - (p_{t-2} - m_{t-1})^2] \\ &\quad + \cdots \\ &\quad + \delta^{L-1}[(p_{t-(L-1)} - m_t)^2 - (p_{t-(L-1)} - m_{t-1})^2] \\ &\quad - \delta^L (p_{t-L} - m_{t-1})^2, \end{aligned}$$

which can be rewritten as follows:

$$\begin{aligned} v_t - \delta v_{t-1} &= B(p_t - m_t)^2 - B\delta^L (p_{t-L} - m_{t-1})^2 \\ &\quad + B\{\delta[(p_{t-1} - m_t) + (p_{t-1} - m_{t-1})][m_{t-1} - m_t] \\ &\quad + \delta^2[(p_{t-2} - m_t) + (p_{t-2} - m_{t-1})][m_{t-1} - m_t] \\ &\quad + \cdots \\ &\quad + \delta^{L-1}[(p_{t-(L-1)} - m_t) + (p_{t-(L-1)} - m_{t-1})][m_{t-1} - m_t]\} \\ &= B(p_t - m_t)^2 - B\delta^L (p_{t-L} - m_{t-1})^2 \\ &\quad + (m_{t-1} - m_t)\{B[\delta(p_{t-1} - m_t) + \delta^2(p_{t-2} - m_t) \\ &\quad + \cdots + \delta^{L-1}(p_{t-(L-1)} - m_t)] \\ &\quad + B[\delta(p_{t-1} - m_{t-1}) + \delta^2(p_{t-2} - m_{t-1}) + \cdots + \delta^{L-1}(p_{t-(L-1)} - m_t)]\} \\ &= B(p_t - m_t)^2 - B\delta^L (p_{t-L} - m_t)^2 \\ &\quad + (m_{t-1} - m_t)[-B(p_t - m_t) - B\delta^L (p_{t-L} - m_t)] \end{aligned}$$

Note that, for $\delta \in [0, 1)$, as $L \to \infty$,

$$B = \frac{1-\delta}{1-\delta^L} \to 1 - \delta$$

and, using (A.3),

$$\begin{aligned} p_{t-L} - m_t &= \delta(p_{t-L} - m_{t-1}) = \delta^2(p_{t-L} - m_{t-2}) = \cdots \\ &= \delta^L(p_{t-L} - m_{t-L}) \to 0. \end{aligned}$$

Therefore the limiting variance process is given by

$$\begin{aligned} v_t - \delta v_{t-1} &= (1-\delta)(p_t - m_t)^2 + (m_{t-1} - m_t)[-(1-\delta)(p_t - m_t)] \\ &= (1-\delta)(p_t - m_t)[(p_t - m_t) + (m_t - m_{t-1})] \\ &= (1-\delta)(p_t - m_t)(p_t - m_{t-1}), \end{aligned}$$

that is,

$$v_t = \delta v_{t-1} + (1-\delta)(p_t - m_t)(p_t - m_{t-1}). \tag{A.4}$$

Based on the above argument, for $\delta \in [0,1)$, the limiting process (as $L \to \infty$) of the mean and variance are given by

$$\begin{cases} m_t = & \delta m_{t-1} + (1-\delta)p_t \\ v_t = & \delta v_{t-1} + (1-\delta)(p_t - m_t)(p_t - m_{t-1}) \\ = & \delta v_{t-1} + \delta(1-\delta)(p_t - m_{t-1})^2. \end{cases} \tag{A.5}$$

A.2. **Characteristic Equation of the Heterogeneous GDP Model with Finite Memory.** When the memory is finite, the heterogeneous GDP can be written as follows:

$$\begin{cases} \bar{p}_{i,t} = & \sum_{j=1}^{L_i} w_{ij} p_{t-j} \\ \bar{v}_{i,t} = & \sum_{j=1}^{L_i} [\bar{p}_{i,t} - p_{t-j}]^2 \end{cases} \tag{A.6}$$

in which, $w_{ij} = B_i \delta^{j-1}$ ($i = 1, 2$ and $j = 1, \cdots, L_i$). Let

$$\begin{cases} x_{1,t} & = p_t \\ x_{2,t} & = p_{t-1} \\ x_{3,t} & = p_{t-2} \\ \vdots & \\ x_{L,t} & = p_{t-(L-1)}, \end{cases}$$

where $L = \max\{L_1, L_2\}$. Then, (2.7) with finite memory GDP is equivalent to the following L-dimensional difference system

$$\begin{cases} x_{1,t+1} & = f(x_t) \\ x_{2,t+1} & = x_{1,t} \\ & \vdots \\ x_{L,t+1} & = x_{L-1,t} \end{cases} \tag{A.7}$$

where

$$\begin{cases} f(\mathbf{x}_t) & = \beta + \frac{\alpha}{2}(1+w)\frac{\bar{x}_{1,t}-b_1}{a_1+2A_1\bar{v}_1} + \frac{\alpha}{2}(1-w)\frac{\bar{x}_{2,t}-b_2}{a_2+2A_2\bar{v}_2} \\ \mathbf{x}_t & = (x_{1,t}, x_{2,t}, \cdots, x_{L,t}) \\ \bar{x}_{i,t} & = \sum_{j=1}^{L_i} w_{ij} x_{j,t} \\ \bar{v}_{i,t} & = \sum_{j=1}^{L_i} w_{ij} [\bar{x}_{i,t} - x_{j,t}]^2. \end{cases}$$

At the steady state p^*, $\bar{x}_1 = \bar{x}_2 = p^*$ and $\bar{v}_1 = \bar{v}_2 = 0$. Without loss generality, it is assumed that $L_1 \le L_2$ and then $L = L_2$. Evaluating function $f(\mathbf{x}_t)$ at the steady state, one obtain that

$$\begin{aligned} \frac{\partial f}{\partial x_j} &= \frac{\alpha}{2}[(1+w)\frac{1}{a_1}w_{1j} + (1-w)\frac{1}{a_2}w_{2j}] \\ &= -[w_{1j}\beta_1 + w_{2j}\beta_2] \end{aligned}$$

for $j = 1, \cdots, L_1$ and

$$\frac{\partial f}{\partial x_j} = -w_{2j}\beta_2$$

for $j = L_1 + 1, \cdots, L$. Therefore the corresponding characteristic equation is given by

$$\Gamma(\lambda) \equiv \lambda^L + \sum_{j=1}^{L_1} [w_{1j}\beta_1 + w_{2j}\beta_2]\lambda^{L-j} + \sum_{j=L_1+1}^{L} w_{2j}\beta_2\lambda^{L-j}. \tag{A.8}$$

In particular, for the GDP, it follows from $w_{ij} = B_i\delta^{j-1}$, $L_1 \le L_2$ and (A.8) that

$$\Gamma(\lambda) \equiv \lambda^L + \sum_{j=1}^{L_1} \left[\beta_1 B_1 \delta_i^{j-1} + \beta_2 B_2 \delta_2^{j-1}\right]\lambda^{L-j} + \sum_{j=L_1+1}^{L} \beta_2 B_2 \delta_2^{j-1}\lambda^{L-j} = 0. \tag{A.9}$$

A.3. **Stability and Bifurcation Analysis When** $L_1 = L_2 = L$**.** When $L_1 = L_2 = L$, one can see from (A.9) that the corresponding characteristic equation is given by

$$\Gamma_L(\lambda) \equiv \lambda^L + \sum_{j=1}^{L} \left[\beta_1 B_1 \delta_1^{j-1} + \beta_2 B_2 \delta_2^{j-1}\right]\lambda^{L-j} = 0. \tag{A.10}$$

Case $L = 1$—Consider first the case when $L = 1$. Then

$$\Gamma_1(\lambda) \equiv \lambda + [\beta_1 + \beta_2] = 0.$$

Hence, $|\lambda| < 1$ holds if and only if $\beta \equiv \beta_1 + \beta_2 < 1$. Furthermore, $\lambda = -1$ when $\beta = 1$. which leads to a flip bifurcation.

Case $L = 2$—When $L = 2$, the characteristic equation has the form

$$\Gamma_2(\lambda) \equiv \lambda^2 + [\beta_1 B_1 + \beta_2 B_2]\lambda + [\beta_1 B_1 \delta_1 + \beta_2 B_2 \delta_2] = 0,$$

where $B_i = 1/[1 + \delta_i]$ $(i = 1, 2)$. It follows from Jury's test that $|\lambda_i| < 1$ if and only if

(i). $\Gamma_2(1) = 1 + \beta_1 + \beta_2 > 0$;

(ii). $\Gamma_2(-1) = 1 - [\beta_1 B_1 + \beta_2 B_2] + [\beta_1 B_1 \delta_1 + \beta_2 B_2 \delta_2] > 0$, which can be rewritten as

$$\Delta_2 \equiv \frac{1 - \delta_1}{1 + \delta_1}\beta_1 + \frac{1 - \delta_2}{1 + \delta_2}\beta_2 = 1. \tag{A.11}$$

(iii). $\beta_1 B_1 \delta_1 + \beta_2 B_2 \delta_2 < 1$, which can be rewritten as

$$\Delta_1 \equiv \frac{\delta_1}{1 + \delta_1}\beta_1 + \frac{\delta_2}{1 + \delta_2}\beta_2 < 1. \tag{A.12}$$

Therefore, $|\lambda_1| < 1$ if and only if (A.11) and (A.12) hold. Note that $\Gamma_2(-1) = 0$ implies that a flip bifurcation occurs when $\Delta_2 = 1$. Also, when $\lambda_{1,2} = e^{\pm 2\pi\theta i}$, we have $\lambda_1\lambda_2 = \beta_1 B_1 \delta_1 + \beta_2 B_2 \delta_2 = \Delta_1 = 1$ and $\lambda_1 + \lambda_2 = -[\beta_1 B_1 + \beta_2 B_2] = 2\cos(2\pi\theta) \equiv \rho$, which implies that $\Delta_1 = 1$ leads to a Neimark-Hopf bifurcation. In addition, the nature of the bifurcation is characterised by the parameter θ, which is determined by (4.2).

When the local stability region is bounded by Neimark-Hopf bifurcation, the nature of the bifurcation is characterised by values of ρ which have different region for different combination of (δ_1, δ_2).

- When $1/2 \leq \delta_1, \delta_2 \leq 1$, the stability region is bounded only by the Neimark-Hopf bifurcation boundary $\Delta_1 = 1$. Then, $\rho = -1/\delta_2$ for $(\beta_1, \beta_2) = (0, [1+\delta_2]/\delta_2)$ and $\rho = -1/\delta_1$ for $(\beta_1, \beta_2) = ([1+\delta_1]/\delta_1, 0)$. Hence

$$\rho \equiv 2\cos(2\pi\theta) \in \Big(-\frac{1}{\min(\delta_1, \delta_2)}, \qquad -\frac{1}{\max(\delta_1, \delta_2)} \Big).$$

- When $0 \leq \delta_1 \leq 1/2, 1/2 \leq \delta_2 \leq 1$, the stability region is bounded by both flip and Neimark-Hopf bifurcation boundaries. The Neimark-Hopf bifurcation boundary corresponds to the line segment between $A : (\beta_1, \beta_2) = (0, [1+\delta_2]/\delta_2)$ and B which is the interaction point between $\Delta_1 = 1$ and $\Delta_2 = 1$, leading to $\rho = -2$. Therefore,

$$\rho \equiv 2\cos(2\pi\theta) \in \Big(-2, \qquad -\frac{1}{\max(\delta_1, \delta_2)} \Big).$$

Case $L = 3$**—**When $L = 3$, the characteristic equation has the form

$$\Gamma_3(\lambda) \equiv \lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0,$$

where

$$\begin{aligned} c_1 &= [\beta_1 B_1 + \beta_2 B_2], \\ c_2 &= [\beta_1 B_1 \delta_1 + \beta_2 B_2 \delta_2], \\ c_3 &= [\beta_1 B_1 \delta_1^2 + \beta_2 B_2 \delta_2^2], \\ B_i &= 1/[1 + \delta_i + \delta_i^2] \qquad (i = 1, 2). \end{aligned}$$

It follows from Jury's test that $|\lambda_i| < 1$ if and only if

(i). $\Gamma_3(1) = 1 + \beta_1 + \beta_2 > 0$;

(ii). $(-1)^3\Gamma_3(-1) > 0$, which is equivalent to

$$\Delta_3 \equiv \frac{1 - \delta_1 + \delta_1^2}{1 + \delta_1 + \delta_1^2}\beta_1 + \frac{1 - \delta_2 + \delta_2^2}{1 + \delta_2 + \delta_2^2}\beta_2 < 1. \tag{A.13}$$

(iii). $c_2 + c_3(c_3 - c_1) < 1$, which is equivalent to

$$\delta_1\gamma_1 + \delta_2\gamma_2 + (\delta_1^2\gamma_1 + \delta_2^2\gamma_2)[(\delta_1^2 - 1)\gamma_1 + (\delta_2^2 - 1)\gamma_2] < 1, \tag{A.14}$$

where $\gamma_i = \frac{\beta_i}{1+\delta_i+\delta_i^2}$.

(iv). $c_2 \equiv \delta_1\gamma_1 + \delta_2\gamma_2 < 3$.

It follows from $\beta_i > 0, \delta_i \in [0, 1]$ and $\delta_i < 1 - \delta_i + \delta_i^2$ that condition (i) is satisfied and condition (ii) implies conditions (iii) and (iv). Hence the only condition for the local stability is $\Delta_3 < 1$. In addition, $\lambda = -1$ when $\Delta_3 = 1$, implying that the stability region is bounded by the flip bifurcation boundary defined by $\Delta_3 = 1$.

A.4. **Stability and Bifurcation Analysis For** $(L_1, L_2) = (2, 3)$**.** For $L_1 = 2, L_2 = 3$, the characteristic equation is given by

$$\Gamma_{2,3}(\lambda) \equiv \lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0,$$

where

$$c_1 = [\gamma_1 + \gamma_2], \qquad c_2 = \gamma_1\delta_1 + \gamma_2\delta_2, \qquad c_3 = \gamma_2\delta_2^2,$$
$$\gamma_1 = \beta_1/[1 + \delta_1], \qquad \gamma_2 = \beta_2/[1 + \delta_2 + \delta_2^2].$$

It follows from Jury's test that $|\lambda_i| < 1$ if and only if

(i). $\Gamma_{2,3}(1) = 1 + \beta_1 + \beta_2 > 0$;

(ii). $(-1)^3\Gamma_{2,3}(-1) > 0$, which is equivalent to

$$\Delta_7 \equiv \frac{1-\delta_1}{1+\delta_1}\beta_1 + \frac{1-\delta_2+\delta_2^2}{1+\delta_2+\delta_2^2}\beta_2 < 1. \tag{A.15}$$

(iii). $c_2 + c_3(c_3 - c_1) < 1$, which is equivalent to

$$\Delta_8 \equiv \frac{\delta_1}{1+\delta_1}\beta_1 + \frac{\delta_2}{1+\delta_2+\delta_2^2}\beta_2 - \frac{\delta_2\beta_2}{1+\delta_2+\delta_2^2}\left(\frac{\beta_1}{1+\delta_1} + \frac{(1-\delta_2^2)\beta_2}{1+\delta_2+\delta_2^2}\right) < 1. \tag{A.16}$$

(iv). $c_2 \equiv \delta_1\gamma_1 + \delta_2\gamma_2 < 3$.

Note that $\beta_i > 0, \delta_i \in [0, 1]$ and $\delta_2 < 1 - \delta_2 + \delta_2^2$, one can see that $\Delta_7 < 1$ implies condition (iv). In addition $\lambda = -1$ when $\Delta_7 = 1$ is satisfied and $\Delta_7 < 1$ implies $\Delta_8 < 1$ for $\delta_2 \leq 1/2$.

A.5. **Proof of Proposition 5.1.**

$$\begin{cases} p_t &= f_1(p, x, y, z, u)_{t-1} \\ x_t &= \delta_1 x_{t-1} + (1-\delta_1)p_{t-1} = f_2 \\ y_t &= \delta_2 y_{t-1} + (1-\delta_2)p_{t-1} = f_3 \\ z_t &= \delta_1 z_{t-1} + \delta_1(1-\delta_1)(p_t - x_{t-1})^2 = f_4 \\ u_t &= \delta_2 u_{t-1} + \delta_2(1-\delta_2)(p_t - y_{t-1})^2 = f_5 \end{cases}$$

$$f_1 = \beta + \frac{\alpha}{2}[(1+w)\frac{x-b_1}{a_1+2A_1z} + (1-w)\frac{y-b_2}{a_2+2A_2z}]$$

Evaluating at the unique fixed point $(p_t, x_t, y_t, z_t, u_t) = (p^*, p^*, p^*, 0, 0)$:

$$\begin{cases} \frac{\partial f_1}{\partial p} &= 0, \\ \frac{\partial f_1}{\partial x} &= \frac{\alpha}{2}\frac{(1+w)}{a_1} = -\beta_1, \\ \frac{\partial f_1}{\partial y} &= \frac{\alpha}{2}\frac{(1-w)}{a_2} = -\beta_2, \\ \frac{\partial f_1}{\partial z} &= \frac{\alpha}{2}(1+w)\frac{-2A_1(p^*-b_1)}{a_1^2} \equiv \Delta_1^*, \\ \frac{\partial f_1}{\partial u} &= \frac{\alpha}{2}(1+w)\frac{-2A_2(p^*-b_2)}{a_2^2} \equiv \Delta_2^*, \end{cases}$$

and

$$\begin{cases} \frac{\partial f_2}{\partial p} &= 1-\delta_1, \quad \frac{\partial f_2}{\partial x} = \delta_1, \quad \frac{\partial f_2}{\partial y} = \frac{\partial f_2}{\partial z} = \frac{\partial f}{\partial u} = 0, \\ \frac{\partial f_3}{\partial p} &= 1-\delta_2, \quad \frac{\partial f_3}{\partial y} = \delta_2, \quad \frac{\partial f_3}{\partial x} = \frac{\partial f_3}{\partial z} = \frac{\partial f_3}{\partial u} = 0, \\ \frac{\partial f_4}{\partial p} &= 0 = \frac{\partial f_4}{\partial x} = \frac{\partial f_4}{\partial y} = 0, \quad \frac{\partial f_4}{\partial z} = \delta_1, \quad \frac{\partial f_4}{\partial u} = 0, \\ \frac{\partial f_5}{\partial u} &= \delta_2, \quad \frac{\partial f_5}{\partial p} = \frac{\partial f_5}{\partial x} = \frac{\partial f_5}{\partial y} = \frac{\partial f_5}{\partial z} = 0. \end{cases}$$

Hence

$$J = \begin{bmatrix} 0 & -\beta_1 & -\beta_2 & \Delta_1^* & \Delta_2^* \\ 1-\delta_1 & \delta_1 & 0 & 0 & 0 \\ 1-\delta_2 & 0 & \delta_2 & 0 & 0 \\ 0 & 0 & 0 & \delta_1 & 0 \\ 0 & 0 & 0 & 0 & \delta_2 \end{bmatrix}.$$

The characteristic equation is given by

$$\begin{aligned} \Gamma(\lambda) \equiv |\lambda I - J| &= \begin{vmatrix} \lambda & \beta_1 & \beta_2 & -\Delta_1^* & -\Delta_2^* \\ -(1-\delta_1) & \lambda-\delta_1 & 0 & 0 & 0 \\ -(1-\delta_2) & 0 & \lambda-\delta_2 & 0 & 0 \\ 0 & 0 & 0 & \lambda-\delta_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda-\delta_2 \end{vmatrix} \\ &= (\lambda-\delta_1)(\lambda-\delta_2) \begin{vmatrix} \lambda & \beta_1 & \beta_2 \\ -(1-\delta_1) & \lambda-\delta_1 & 0 \\ -(1-\delta_2) & 0 & \lambda-\delta_2 \end{vmatrix} \\ &= (\lambda-\delta_1)(\lambda-\delta_2)h(\lambda), \end{aligned}$$

where

$$h(\lambda) = \begin{vmatrix} \lambda & \beta_1 & \beta_2 \\ -(1-\delta_1) & \lambda-\delta_1 & 0 \\ -(1-\delta_2) & 0 & \lambda-\delta_2 \end{vmatrix} = \lambda^3 + c_1\lambda^2 + c_2\lambda + c_3$$

$$\begin{cases} c_1 &= -(\delta_1+\delta_2), \\ c_2 &= \delta_1\delta_2 + \beta_2(1-\delta_2) + \beta_1(1-\delta_1), \\ c_3 &= -\delta_1\beta_2(1-\delta_2) - \delta_2\beta_1(1-\delta_1). \end{cases}$$

Since $\delta_1, \delta_2 \in (0,1)$ applying Jury's test to $h(\lambda) = 0$, one can see that $|\lambda_i| < 1$ if $\pi_i > 0$, where

$$\begin{cases} \pi_1 &= 1 + c_1 + c_2 + c_3, \\ \pi_2 &= 1 - c_1 + c_2 - c_3, \\ \pi_3 &= 1 - c_2 + c_3(c_1 - c_3), \\ c_2 &< 3. \end{cases}$$

Note that

$$\begin{aligned}\pi_1 > 0 \quad &\Leftrightarrow \quad (1-\delta_1)(1-\delta_2)[1+\beta_1+\beta_2] > 0\\ \pi_2 > 0 \quad &\Leftrightarrow \quad -\left[\frac{1-\delta_1}{1+\delta}\beta_1 + \frac{1-\delta_2}{1+\delta_2}\beta_2\right] < 1\\ \pi_3 > 0 \quad &\Leftrightarrow \quad \left[\delta_1\beta_2(1-\delta_2) + \delta_2\beta_1(1-\delta_1) + \frac{\delta_1+\delta_2}{2}\right]^2\\ &\qquad +\beta_2(1-\delta_2) + \beta_1(1-\delta_1) < 1 + \frac{(\delta_1-\delta_2)^2}{4}\end{aligned}$$

and $c_2 < 3$ is implied by $\pi_3 > 0$. Therefore, the only condition we need for the local stability is $\pi_3 > 0$. Furthermore, from $h(1) = \pi_1, (-1)^3 h(-1) = \pi_2$, there is no saddle-node and flip bifurcation and the only boundary of the stability region is given by Neimark-Hopf bifurcation boundary, defined by $\pi_3 = 0$. Along the bifurcation boundary, let

$$\lambda_{1,2} = e^{\pm 2\pi\theta i}, \qquad \lambda_3 = r \in (-1,1).$$

Then it follows from

$$\begin{aligned}&[\lambda_1+\lambda_2+\lambda_2] = -[\rho + r] = -[\delta_1+\delta_2],\\ &\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 1 + r\rho\\ &\qquad = \delta_1\delta_2 + \beta_1(1-\delta_1) + \beta_2(1-\delta_2),\\ &\lambda_1\lambda_2\lambda_3 = -r = -[\delta_1\beta_2(1-\delta_2 + \delta_2\beta_1(1-\delta_1)]\end{aligned}$$

that

$$\rho = \delta_1[1-\beta_2(1-\delta_2)] + \delta_2[1-\beta_1(1-\delta_1)].$$

In particular, for $\delta_1 = \delta_2 = \delta$, the stability condition becomes

$$[1-\beta(1-\delta)][\delta^2\beta(1-\delta) + (1-\delta^2)] > 0,$$

which is equivalent to

$$\beta < \frac{1}{1-\delta}, \qquad \text{where} \qquad \beta = \beta_1 + \beta_2.$$

Along the bifurcation boundary, $\beta(1-\delta) = 1$, and hence $\rho = \delta$.

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