Intrinsic Heterogeneity in Expectation Formation*

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Abstract

We introduce the concept of a *Misspecification Equilibrium* to dynamic macroeconomics. A Misspecification Equilibrium occurs in a stochastic process when agents forecast optimally given that they must choose from a list of misspecified econometric models. With appropriate restrictions on the asymptotic properties of the exogeneous process and on the feedback of expectations, the Misspecification Equilibrium will exhibit *Intrinsic Heterogeneity.* Intrinsic Heterogeneity is a Misspecification Equilibrium where all misspecified models receive positive weight in the distribution of predictors across agents. Interestingly, the existence of heterogeneity depends on the self-referential property of the model. Our derivation of heterogeneous expectations as the equilibrium outcome of a model is a departure from the previous literature which makes *ad hoc* assumptions about the degree of heterogeneity.

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1 Introduction

Despite its dominance in dynamic macroeconomic models, the Rational Expectations Hypothesis has limitations. The most frequently cited drawback to the rational expectations approach is that it assumes agents know the underlying economic structure. In response to this criticism one popular alternative is to model agents as econometricians (Evans and Honkapohja 2001). This adaptive learning approach typically

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assumes agents have a correctly specified model with unknown parameters. Agents then use a reasonable estimator to obtain their coefficient estimates. In many models these beliefs converge to rational expectations.

In practice, however, econometricians often misspecify their models. Economic forecasters who use VAR's purposely limit the number of variables and the number of lags because of degree of freedom problems. If agents are expected to behave like econometricians then they can also be expected to misspecify their models. (Evans and Honkapohja 2001) consider a model where agents underparameterize and show existence of a Restricted Perceptions Equilibrium (RPE) in which agents form their beliefs optimally given their misspecification.

In this paper we examine expectation formation in an environment where agents must forecast using an underparameterized econometric model. More specifically we confront agents with a list of misspecified econometric models, but, given this restriction, assume that agents forecast optimally. Agents choose between these optimal underparameterized models based on their relative mean success.

We investigate this framework in a linear stochastic framework, developing the analysis in the context of the cobweb model. Because the economic model is self-referential, in the sense that expectation formation affects the law of motion for the endogenous variables, the optimal parameters of each misspecified econometric model depend on the proportions of agents using the different models. We define a new equilibrium concept, called a *Misspecification Equilibrium*, in which these proportions are consistent with optimal forecasting from each econometric model. We show that for some economic model parameters and exogenous driving variables, agents will be distributed heterogeneously between the various predictors, even as we approach the limiting case in which agents choose only between the best performing statistical models. We say a Misspecification Equilibrium with such a property exhibits *Intrinsic Heterogeneity.*

Heterogeneity in expectations has been considered previously in papers by (Townsend 1983) and (Haltiwanger and Waldman 1985) who assume a certain fraction of agents are not rational. In adaptive learning models (Honkapohja and Mitra 2001) allow agents to have different specific learning rules. Even the seminal (Bray and Savin 1986) allows for heterogeneity in priors. These papers all assume an *ad hoc* degree of heterogeneity, and the heterogeneity dissappears in the limit. (Evans, Honkapohja and Marimon 2001) allow for stochastic heterogeneity in learning rules, but again the heterogenous expectations is only transitory.

(Brock and Hommes 1997) were the first to model heterogeneous expectations as an endogeneous outcome.[1] (Brock and Hommes 1997) examine a cobweb model where agents choose a predictor from a set of costly alternatives. They base this choice on

[1](Sethi and Franke 1995) also find heterogeneity as an outcome of evolution in a model of stochastic strategic complementarities.

the most recent realized profits of the alternatives in a cobweb model. If agents are boundedly rational in the sense that their 'intensity of choice' between predictors is finite (that is, they do not fully optimize), then there will be heterogeneity and the degree of heterogeneity will vary in a complex manner.

Brock and Hommes illustrate these results in a particular case of rational versus myopic beliefs. Because agents always react to recent changes in profits their predictor choice will oscillate along with the equilibrium price. Our model is closely related to Brock and Hommes. Like their model, we assume that the map from predictor benefits to predictor choice resembles a multinomial logit. The multinomial logit has proven to be an important approach to modeling economic choices.[2] Additionally, there has been an explosion of recent work in dynamic macroeconomics which uses the multinomial logit. Extensions of the (Brock and Hommes 1997) predictor selection dynamic appear in (Brock and deFountnouvelle 2000), (Brock and Hommes 1998, 2000), (Brock, Hommes, and Wagener 2001), (Branch 2002a, 2002b) and (Hommes 2001). (Brock and Durlauf 2001) extend the framework so agent specific choices depend on the expected choices of others.

There are two important departures in our model. First, agents do not choose between a costly accurate forecast and a costless inaccurate forecast; rather, they are forced to choose between misspecified models. We will show that even if they optimally choose between these misspecified models heterogeneity may arise. Second, we assume agents make their choices based on unconditional mean payoffs rather than on the most recent period's realized payoff. This is appropriate in a stochastic environment since otherwise agents could be misled by single period anomalies. Given that agents base decisions on mean profits it is not at all obvious that heterogeneity would be possible if the 'intensity of choice' is large. Indeed, we will show instances of asymptotically homogeneous expectations do arise.

The main difference in our results is, unlike previous work, we derive heterogeneity as a possible equilibrium outcome of a self-referential model where agents must underparameterize. We assume that agents are fully rational except that they misspecify by omitting at least one relevant variable or lag. We focus on the cobweb for two reasons. First, we want to stay close to (Brock and Hommes 1997) in order to highlight the key differences. Second, the cobweb model is the simplest self-referential model that effectively illustrates the intuition of Intrinsic Heterogeneity.

We obtain conditions under which there is an equilibrium with agents heterogeneously split between the misspecified models even as the 'intensity of choice' becomes arbitrarily large. The intuition for this possibility is as follows. Suppose the cobweb price is driven by a two-dimensional vector of demand shocks. If both components of the demand shock matter for predicting prices, and if the feedback through expectations is sufficiently large, then there will be an incentive to deviate from homogeneity.

[2]See, for example, (Manski and McFadden 1981).

If all agents coordinate on the same model the negative feedback through expectations will make the consensus model less useful for forecasting. In these instances an agent could profit by forecasting with the alternative model. With Intrinsic Heterogeneity the equilibrium is such that beliefs and predictor proportions drive expected profits to be identical.

The plan for this paper is as follows. Section 2 introduces the set-up in a general cobweb model. We show the existence of a Misspecification Equilibrium and the conditions under which there is Intrinsic Heterogeneity. Section 3 demonstrates these results for the special case of a process driven by a two dimensional VAR(1) shock and agents choose between two underparameterized models. Section 4 concludes and describes future work.

2 Model

In this section we consider a self-referential stochastic process that is driven by vector autoregressive exogenous shocks. We assume that agents' expectations are based on misspecified models of the economy. Previous work has assumed, in an equilibrium, a particular structure of agents' misspecification. We allow the misspecification to be endogeneous and consider equilibria that jointly determine the misspecification and the equilibrium path.

We do this by extending the Adaptively Rational Equilibrium Dynamics (A.R.E.D.) of Brock and Hommes (1997) to allow agents to select between underparameterized models. Agents consider the unconditional expected payoff of the various possible underparameterizations and select the one which returns the highest payoff. Once they have selected a model they then form their expectations as the optimal linear projection given their misspecification.

We first show that, for given predictor proportions, there exists a Restricted Perceptions Equilibrium (RPE) in which agents' misspecified beliefs are verified by the actual equilibrium process. We next allow for predictor proportions to be endogeneously determined and show the existence of a Misspecification Equilibrium. Finally, we state a condition under which the model exhibits Intrinsic Heterogeneity.

2.1 Set-up

We consider a cobweb model of the form

$$p_t = -\phi p_t^e + \gamma' z_t + v_t \tag{1}$$

where v_t is white noise. Although there are several well-known economic models that fit the form (1), we focus on the "cobweb" model in order to keep the close connection between our model and (Brock and Hommes 1997). z_t is a vector of observable demand disturbances, which will be further specified below.

We normally expect $\phi > 0$ in the cobweb model, which corresponds to upward sloping supply curves and downward sloping demand curves. Bray and Savin (1986) showed that $\phi > -1$ was the condition for the model to be stable under least squares learning. In this paper we focus on the negative feedback case of $\phi > 0$ and leave $\phi < 0$ for future work.[3]

In the cobweb model firms have a one-period production lag. We assume that firms have quadratic costs given by $FQ_t^* + \frac{1}{2}G(Q_t^*)^2$, where Q_t^* is planned output. In addition we allow for exogeneous productivity shocks realized after production decisions are made so that total quantity is $Q_t = Q_t^* + \kappa_t$. Here κ_t is iid with zero mean. Firms aim to maximize expected profits.[4] Thus,

$$\begin{aligned} \max_{Q_t^*} \; E_{t-1}\pi_t &= E_{t-1}\left[p_t\left(Q_t^* + \kappa_t\right) - FQ_t^* - \frac{1}{2}G(Q_t^*)^2\right] \\ &= Q_t^* E_{t-1}p_t + E_{t-1}(p_t\kappa_t) - FQ_t^* - \frac{1}{2}G(Q_t^*)^2 \end{aligned}$$

Solving this problem leads to the supply relation[5]

$$Q_t^* = G^{-1}p_t^e \tag{2}$$

where $p_t^e = E_{t-1}p_t$. Then actual supply follows $Q_t = G^{-1}p_t^e + \kappa_t$.

Demand is given by

$$Q_t = C - Dp_t + h'\zeta_t \tag{3}$$

where ζ_t is an $m \times 1$ vector of demand shocks that follows a zero-mean stationary VAR(n). The ζ_t process is assumed independent of κ_t. Setting demand equal to actual supply we have the following stochastic equilibrium price process

$$p_t = -(DG)^{-1}p_t^e + D^{-1}h'\zeta_t - D^{-1}\kappa_t, \tag{4}$$

where, for convenience, we have expressed p_t and p_t^e in deviation from the mean form.

It is convenient to rewrite the model in terms of an exogenous VAR(1) process. Defining

$$z_t' = (\zeta_t', \zeta_{t-1}', \cdots, \zeta_{t-n+1}')$$

[3]Equation (1) with $-1 < \phi < 0$ takes the same form as a Lucas-type monetary model. In future work we will pursue the possibility of heterogeneity in that model.

[4]It would be possible to extend the model to incorporate risk by assuming agents respond to variances of profits as well as expected profits. We make the expected profits assumption to keep the model as simple as possible.

[5]We have set, without loss of generality, $F = 0$. We are also assuming that agents treat $E_{t-1}(p_t\kappa_t)$ as a constant independent of the choice of Q_t^*. That this is a reasonable assumption can be verified by (4) below.

we can write z_t in its standard VAR(1) form

$$z_t = Az_{t-1} + \varepsilon_t$$

for appropriately defined A and appropriately defined ε_t, which is exogenous white noise. Here z_t is $mn \times 1$ and A is $mn \times mn$. We denote the covariance matrix of z_t as $\Omega = Ezz'$, and Ω is assumed to be positive definite. Setting

$$\phi = (DG)^{-1}, \gamma' = (D^{-1}h', 0, \ldots, 0) \text{ and } v_t = -D^{-1}\kappa_t$$

we can rewrite (4) in the form (1).

2.2 Model Misspecification

To close the model we need to specify the determination of p_t^e. We assume that there are K econometric models available to form expectations and that model $j = 1, \ldots, K$ uses $k_j < mn$ explanatory variables. The market expectation is given by the weighted sum of the individual expectations

$$p_t^e = \sum_{j=1}^{K} n_j p_{j,t}^e \tag{5}$$

where $p_{j,t}^e = b^{j\prime} x_{t-1}^j$, $x_t^j = u^j z_t$. The $k_j \times m$ matrix u^j is a selector matrix that picks out those elements of z_t used in predictor j and b^j is $k_j \times 1$. Thus, k_j is the number of elements in z_t that predictor j uses. We can rewrite (5) as

$$p_t^e = \sum_{j=1}^{K} n_j b^{j\prime} u^j z_{t-1}$$

This set-up forces agents to underparameterize the variables included in their information set and/or the number of lags of those variables. We believe this is a reasonable approximation of actual expectation formation. Cognitive and computing time constraints (as well as degrees of freedom) restrict the number of variables even the most diligent econometricians use in their models. Our form of misspecification makes agents be (at least a little bit) parsimonious in their expectation formation.

We next specify the determination of the parameters b^j. In a fully specified econometric model, and under rational expectations, all variables z_t would be included and the coefficients used to form p_t^e would be given by the least squares projection of p_t on z_t. Here each predictor is constrained to use a subset x_j of relevant variables, and thus each predictor differs from rational expectations. However, we will insist that the beliefs b^j are formed optimally in the sense that b^j is the least squares projection of p_t on $u^j z_{t-1}$. That is, b^j must satisfy

$$Eu^j z_{t-1}\left(p_t - b^{j\prime} u^j z_{t-1}\right) = 0$$

Even though agents will never be "fully" accurate, they will be as accurate as possible given the variables in their information set.

2.3 Misspecification Equilibrium

Given the belief process (5) the actual law of motion (ALM) for this economy is

$$p_t = \left[\gamma' A - \phi\left(\sum_{j=1}^{K} n_j b^{j\prime} u^j\right)\right] z_{t-1} + \gamma' \varepsilon_t + v_t$$

or

$$p_t = \xi' z_{t-1} + \gamma' \varepsilon_t + v_t, \tag{6}$$

where

$$\xi' = \gamma' A - \phi\left(\sum_{j=1}^{K} n_j b^{j\prime} u^j\right). \tag{7}$$

Here $n = [n_1, \cdots, n_K]'$ and $b = [b_1, \cdots, b_K]$. Given these equations and the parameter orthogonality condition we obtain

$$b^j = \left(u^j \Omega u^{j\prime}\right)^{-1} u^j \Omega \xi. \tag{8}$$

We now introduce the concept of Restricted Perceptions Equilibrium (RPE).[6] An RPE is an equilibrium process for p_t such that the parameters b^j are optimal given the misspecification. Note that, like a rational expectations equilibrium, an RPE is self-referential in that the optimal beliefs depend on the vector of parameters ξ which depend in turn on the vector of beliefs b. Thus, an RPE can be defined as a process (6) such that ξ is a solution to (7) and (8) for fixed n.

Substituting (8) into (7) yields

$$\xi' = \gamma' A - \phi \sum_{j=1}^{K} n_j \xi' \Omega u^{j\prime}\left(u^j \Omega u^{j\prime}\right)^{-1} u^j$$

or

$$\xi = \left[I + \phi \sum_{j=1}^{K} n_j u^{j\prime}\left(u^j \Omega u^{j\prime}\right)^{-1} u^j \Omega\right]^{-1} A' \gamma \tag{9}$$

[6]See (Evans and Honkapohja 2001) for a definition and examples. The concept introduced here extends the concept of RPE to incorporate multiple misspecified models.

For a given n an RPE exists (and is unique), provided the inverse in (9) exists.

In the Misspecification Equilibrium, which we define below, n is determined endogenously. Equation (9) gives a well-defined mapping $\xi = \xi(n)$ provided the indicated inverse exists for all n in the unit simplex. We therefore assume that the following condition holds:

Condition $\triangle$*:* $\triangle \neq 0$ *for all* n *in the unit simplex* $S = \{n \in \mathbb{R}^K : n_i \geq 0 \text{ and } \sum_{i=1}^K n_i = 1\}$*, where*

$$\triangle = \det\left(I + \phi \sum_{j=1}^{K} n_j u^{j\prime}\left(u^j \Omega u^{j\prime}\right)^{-1} u^j \Omega\right).$$

Condition $\triangle$ is a necessary and sufficient condition for the existence of a unique RPE for all $n \in S$.

We have the following result:

Proposition 1 *For* $\phi \geq 0$ *sufficiently small, Condition* $\triangle$ *is satisfied and hence for all* n *there exists a unique RPE given by (6) and (8).*

All proofs are contained in the Appendix. In the next Section we demonstrate that Condition $\triangle$ holds for all $\phi \geq 0$ in the case $K = 2$.

We now embed the RPE concept in an equilibrium concept in which n is endogeneously determined by the mean profits of each predictor. We will call this a *Misspecification Equilibrium.* Note that the profits of each predictor depend on the parameters ξ which in turn depend on n.

In order to discuss the mapping for predictor proportions we need the profits for predictor j, which are given by

$$\begin{aligned} \pi_t^j &= p_t\left(\phi D p_{i,t}^e - D v_t\right) - \frac{1}{2}\phi D\left(p_{i,t}^e\right)^2 \\ &= \left[\xi(n)' z_{t-1} + \gamma' \varepsilon_t + v_t\right]\left[\phi D b^{j\prime} u^j z_{t-1} - D v_t\right] - \frac{1}{2}\phi D\left(b^{j\prime} u^j z_{t-1}\right)^2, \end{aligned}$$

where, again, we have expressed profits in deviation from mean form. Taking unconditional expectations of profits yields

$$E\pi_t^j = \phi D b^{j\prime} u^j \Omega\left(\xi(n) - \frac{1}{2} u^{j\prime} b^j\right) - D E v_t^2.$$

Evaluating expected profits in an RPE (i.e. plugging in (8)) leads to

$$E\pi^j = \phi D \xi(n)' \Omega u^{j\prime}(u^j \Omega u^{j\prime})^{-1} u^j \Omega\left(\xi(n) - \frac{1}{2} u^{j\prime}(u^j \Omega u^{j\prime})^{-1} u^j \Omega \xi(n)\right) - D E v_t^2. \quad (10)$$

Note that $E\pi^j$ is well-defined and finite for all n, provided Condition Δ holds so that $\xi(n)$ is well-defined. It will be convenient to denote the function given by (10) as

$$\tilde{F}_j(n) : S \rightarrow \mathbb{R}, \text{ for } j = 1, \ldots, K,$$

and to define $\tilde{F}(n) : S \rightarrow \mathbb{R}^{\mathbb{K}}$ by $\tilde{F}(n) = (\tilde{F}_1(n), \ldots, \tilde{F}_K(n))'$. Note that $\tilde{F}_j(n)$ and $\tilde{F}(n)$ are continuous on S provided Condition Δ holds.

We now follow (Brock and Hommes 1997) in assuming that the predictor proportions follow a multinomial logit (MNL) law of motion. Brock and Hommes consider the cobweb model without noise where agents choose between rational and naive expectations. Agents adapt their choices based on the most recent relative predictor success.[7] This clearly would not be appropriate in the stochastic framework employed here and we instead assume that agents base their decision on unconditional expected relative pay-offs.[8]

The MNL approach leads to the following mapping, for each predictor i,

$$n_i = \frac{exp\{\alpha E\pi^i\}}{\sum_{j=1}^{K} exp\{\alpha E\pi^j\}}. \tag{11}$$

Note that $n_i > 0$ for α and the $E\pi^j$ finite and $\sum_j n_j = 1$. Again, it will be convenient to denote the map defined by (11) as

$$\tilde{H}_\alpha(E\pi^1, \ldots, E\pi^K) : \mathbb{R}^K \rightarrow S,$$

and clearly $\tilde{H}_\alpha$ is continuous. The parameter α is called the 'intensity of choice,' and parameterizes one dimension of agents' bounded rationality. For $\alpha = +\infty$ we have the 'neoclassical' case. We will be interested in heterogeneity for the neoclassical case of full optimization. We remark that our choice of payoff function $E\pi^j$ allows us to consider the fixed point of a map rather than the sequence of difference equations as in Brock and Hommes.

We now define the mapping

$$\tilde{T}_\alpha : S \rightarrow S \text{ where } \tilde{T}_\alpha = \tilde{H}_\alpha \circ \tilde{F}.$$

Under Condition Δ this map is well-defined and continuous. $\tilde{T}_\alpha$ maps a vector of predictor choices, n, through the belief parameter mapping ξ into a vector of expected profits and then to a new predictor choice n. We are now in a position to present our central equilibrium concept:

Definition *A Misspecification Equilibrium (ME) is a fixed point, n^*, of $\tilde{T}_\alpha$.*

Applying the Brouwer Fixed Point Theorem we immediately have:

[7](Branch 2002) shows that many of the qualitative properties in the model without noise carry over to a model with small demand disturbances.

[8]In future work we plan to consider a framework in which agents dynamically respond to recent realizations of payoffs and in which b^j is estimated recursively.

Theorem 2 *Assume Condition Δ. There exists a Misspecification Equilibrium.*

In general we cannot rule out multiple equilibria. Let

$$N_\alpha = \{n^* | \tilde{T}_\alpha(n^*) = n^*\}.$$

For α finite and $E\pi^j$ finite, it is apparent that all components are positive for every fixed point n^*. Thus, heterogeneity for finite α is simply a by-product of the MNL assumption, which ensures that all predictors are used even if they differ in terms of their performance. However, it is of interest to know if heterogeneity continues to arise if agents are highly sensitive to relative performance, so that asymptotically they only use predictors that are not dominated in performance. This leads to the following concept:

Definition *A model is said to exhibit Intrinsic Heterogeneity if (i) an ME exists for all $\alpha > 0$ and (ii) there exists $\bar{n} < 1$ such that $n_j^* \leq \bar{n}$, $j = 1, \ldots, K$, for all α and all ME $n^* \in N_\alpha$.*

It can be shown that a model with intrinsic heterogeneity arises whenever the following additional condition is satisfied.

***Condition P**:* Let e_i denote the $K \times 1$ coordinate vector with 1 in position i and 0 elsewhere. Condition P is said to be satisfied if for each $i = 1, \ldots, K$ there exists $j \neq i$ such that $\tilde{F}_j(e_i) - \tilde{F}_i(e_i) > 0$.

Theorem 3 *Assume Condition Δ and also Condition P. Then the model exhibits intrinsic heterogeneity.*

The next section will present a simple example to illustrate our concepts. In particular we present cases in which Condition P holds and the model exhibits Intrinsic Heterogeneity.

3 Example: Bivariate Case

To illustrate the properties of a Misspecification Equilibrium we will simplify the model by considering a special case in which detailed results can be obtained. In this section we assume that z_t is a two-dimensional VAR(1) $z_t = Az_{t-1} + \varepsilon_t$, where A is 2×2 and $E\varepsilon_t\varepsilon_t' = \Sigma_\varepsilon$. Each misspecified model will omit one explanatory variable and thus $K = 2$ and $k_j = 1$ for $j = 1, 2$. This is the simplest possible illustration of our framework, and we will see that it can generate cases with Intrinsic Heterogeneity.

With bivariate demand shocks the predictors are now

$$\begin{aligned} p_{1,t}^e &= b^1 u^1 z_{t-1} = b^1 z_{1,t-1} \\ p_{2,t}^e &= b^2 u^2 z_{t-1} = b^2 z_{2,t-1} \end{aligned}$$

Plugging these predictors into the law of motion for price and collecting terms leads to

$$p_t = \xi_1 z_{1,t-1} + \xi_2 z_{2,t-1} + \eta_t \tag{12}$$

$$\begin{bmatrix} 1+n_1\phi & \phi n_1 \rho \\ \phi n_2 \tilde{\rho} & 1+n_2\phi \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = A'\gamma, \tag{13}$$

where

$$\rho = \frac{Ez_{1t}z_{2t}}{Ez_{1t}^2},\ \tilde{\rho} = \frac{Ez_{1t}z_{2t}}{Ez_{2t}^2},$$

and $\eta_t = \gamma'\varepsilon_t + v_t$. From the general results of the preceding section we know that a Misspecification Equilibrium exists for $\phi \geq 0$ sufficiently small. For the bivariate case existence can be shown for all $\phi \geq 0$. Furthermore, we will show that this equilibrium is unique.

3.1 Misspecification Equilibrium

If condition $\triangle$ is satisfied then this guarantees a unique ξ_1, ξ_2 for each $n' = (n_1, n_2)$, and a unique RPE. Since $n_2 = 1 - n_1$, in this section we define the key functions in terms of n_1 rather than n. Thus, in particular, if Condition Δ holds then (13) defines a continuous map $\xi = \xi(n_1)$.

Proposition 4 *In the bivariate model, Condition $\triangle$ is satisfied for all $\phi \geq 0$. Hence there exists a unique RPE for every $n_1 \in [0, 1]$.*

From Theorem 2 it follows that there exists a ME. By developing the details we can obtain additional results. The profit functions are given by

$$\begin{aligned} E\pi^1 &= \frac{1}{2}\phi D\left(\xi_1^2(n_1) - \xi_2^2(n_1)\rho^2\right) Ez_{1t}^2 + \phi D\left(\xi_1(n_1) + \xi_2(n_1)\rho\right)\xi_2(n_1) Ez_1z_2 - D\sigma_v^2 \\ E\pi^2 &= \frac{1}{2}\phi D\left(\xi_2^2(n_1) - \xi_1^2(n_1)\tilde{\rho}^2\right) Ez_{2t}^2 + \phi D\left(\xi_2(n_1) + \xi_1(n_1)\tilde{\rho}\right)\xi_1(n_1) Ez_1z_2 - D\sigma_v^2, \end{aligned}$$

and we define

$$F(n_1) = E\pi^1 - E\pi^2.$$

In order to prove existence of a unique ME, we need to show that the profit difference function $F(\xi(n_1))$ is monotonic.

Lemma 5 *In the bivariate model, the function $F(n_1)$ is monotonically decreasing for all $\phi \geq 0$.*

We remark that it is possible to instead have a positive slope for the profit difference function $F(n_1)$ when $\phi < 0$. In this case it will be possible to have multiple equilibria. Examples with $\phi < 0$ are the focus of future research.

The predictor proportion mapping (11) can be written

$$n_1 = \frac{1}{2}\tanh\left[\frac{\alpha}{2}\left(E\pi^1 - E\pi^2\right)\right] + \frac{1}{2} \equiv H_\alpha(E\pi^1 - E\pi^2),$$

where $H_\alpha : \mathbb{R} \to [0,1]$ is a strictly increasing function. Note that we use F and H_α in place of $\tilde{F}$ and $\tilde{H}_\alpha$ to emphasize that in contrast to the previous section the domain of F and the range of H_α is now $[0,1]$ instead of the unit simplex S. This will simplify some of the arguments below.

Because Condition $\triangle$ is satisfied for all $\phi \geq 0$, there exists a well defined mapping $T_\alpha = H_\alpha \circ F$. $T_\alpha : [0,1] \to [0,1]$, which is continuous. From Lemma 5 it follows that T_α is a continuous, decreasing function for each α. It immediately follows that there is a unique fixed point, i.e., we have:

Theorem 6 *Suppose z_t is a bivariate VAR(1). If $\phi \geq 0$ the model has a unique Misspecification Equilibrium.*

Theorem 6 demonstrates that there is a unique equilibrium in the belief parameters and the proportion of agents using the two misspecified models. It does not tell us how agents are distributed between the predictors. Our main interest is in showing that in an equilibrium with $\alpha = +\infty$ it is possible for there to be heterogeneity. Unlike (Brock and Hommes 1997) who obtain heterogeneity as an automatic implication of assuming that α is finite, we want to show that there exists cases of heterogeneity even in the limit as $\alpha \to \infty$. We now take up this issue.

3.2 Intrinsic Heterogeneity

The previous section established uniqueness of the misspecification equilibrium. We now discuss more specific properties of this equilibrium.

From the equations for expected profit, it can be shown that[9]

$$\begin{aligned} F(1) &\gtrless 0 \text{ iff } \xi_1^2(1) \gtrless \xi_2^2(1)Q, \text{ and} \\ F(0) &\gtrless 0 \text{ iff } \xi_1^2(0) \gtrless \xi_2^2(0)Q \end{aligned}$$

[9] The Appendix contains additional details of these derivations.

where $Q = \frac{Ez_2^2}{Ez_1^2} > 0$. Furthermore, from (13) we have

$$\begin{aligned} \frac{\xi_1^2(0)}{\xi_2^2(0)} &= \frac{(1+\phi)^2 (\gamma_1 a_{11} + \gamma_2 a_{21})^2}{(\gamma_1 a_{12} + \gamma_2 a_{22} - \phi\tilde{\rho}(\gamma_1 a_{11} + \gamma_2 a_{21}))^2} \equiv B_0 \\ \frac{\xi_1^2(1)}{\xi_2^2(1)} &= \frac{(\gamma_1 a_{11} + \gamma_2 a_{21} - \phi\rho(\gamma_1 a_{12} + \gamma_2 a_{22}))^2}{(1+\phi)^2 (\gamma_1 a_{12} + \gamma_2 a_{22})^2} \equiv B_1 \end{aligned}$$

These expressions assume that the denominators of these expressions are non-zero.

There are three possible cases depending on A and Σ_ϵ:

Lemma 7

1. *Condition P: $F(0) > 0$ and $F(1) < 0$. Condition P is satisfied when $B_1 < Q < B_0$.*
2. *Condition P0: $F(0) < 0$ and $F(1) < 0$. Condition P0 is satisfied when $Q > B_0$.*
3. *Condition P1: $F(0) > 0$ and $F(1) > 0$. Condition P1 is satisfied when $Q < B_1$.*

Below we give numerical examples of when each condition may arise.

Under Condition P0, $F(1) < 0$ implies that model 2 is always more profitable. Under Condition P1, model 1 is always more profitable. In these cases we anticipate homogeneous expectations as the 'intensity of choice' $\alpha \to \infty$. However, if Condition P obtains there is an incentive to deviate from the consensus selection. We have the following result.

Proposition 8 *Consider again the model with z_t a bivariate VAR(1) and $\phi \geq 0$. The unique Misspecification Equilibrium n_1^* has one of the following properties:*

1. *Condition P implies that as $\alpha \to \infty$, $n_1^* \to \hat{n}_1 \in (0, 1)$ where $F(\hat{n}_1) = 0$. That is, n_1^* has Intrinsic Heterogeneity.*
2. *Condition P0 implies that as $\alpha \to \infty$, $n_1^* \to 0$.*
3. *Condition P1 implies that as $\alpha \to \infty$, $n_1^* \to 1$.*

Proposition 8 establishes the possibility of Intrinsic Heterogeneity in a Misspecification Equilibrium. We discuss the intuition further below. This result is novel because, for high α, rationality of agents is bounded only through their model parameterizations. Agents fully optimize given their (misspecified) model of the economy. In Brock and Hommes' A.R.E.D. heterogeneity arises because there are calculation costs and, most importantly, because a proportion of agents, with finite α, do not optimize by ignoring profit differences. Only in a steady-state will agents be evenly distributed across predictors.[10] In our model, agents optimize given their misspecification, all predictors are equally "sophisticated" and costless, and Intrinsic Heterogeneity can arise as part of a stochastic equilibrium. Most interestingly, it is the self-referential feature of the model that generates this heterogeneity.

We argue that the assumption of underparameterization is reasonable. The adaptive learning literature has argued in favor of modeling agents as econometricians as a reasonable deviation from the rational expectations assumption. But, econometricians misspecify their econometric models. Computational time and limits on degrees of freedom make it impossible for an econometrician to include all economically relevant variables and lags. Imposing such restrictions on our agents we find that this can lead to heterogeneous forecasting models.

3.3 Connection to the Rational Expectations Equilibrium

Our equilibrium differs from the Restricted Perceptions Equilibrium in (Evans and Honkapohja 2001). There agents also underparameterize in one dimension but the dimension is imposed by the model and all agents are homogeneous in their misspecification. These expectations differ from rational expectations by ignoring relevant information. Since all agents ignore the same information in their perceived law of motion it is clear that in equilibrium the parameters of the model will differ from a Rational Expectations Equilibrium (REE). In a Misspecification Equilibrium with Intrinsic Heterogeneity each agent misspecifies but aggregate expectations are conditioned on all available information. In principle, it is conceivable that a ME could reproduce the REE. In this subsection we show that this is not the case: the parameters of the model in a ME will differ from an REE for all asymptotic properties of z_t.

Recall that the equilibrium process is

$$p_t = -\phi p_t^e + \gamma' A z_{t-1} + \eta_t \tag{14}$$

where γ is (2×1), A is (2×2) with elements a_{ij} for $j = 1, 2$, and $\eta_t = \gamma' \varepsilon_t + v_t$. Under rational expectations

$$p_t^e = E_{t-1} p_t \tag{15}$$

An REE is a stochastic process p_t which satisfies (14) given (15). The cobweb model has one such solution and it is given by

$$p_t = \hat{\xi}_1 z_{1,t-1} + \hat{\xi}_2 z_{2,t-1} + \eta_t$$

[10] This is because in (Brock and Hommes' 1997) set-up all predictors return the same forecast in a steady-state. So if a predictor is costless, then it will return the same steady-state net benefit as all other costless predictors. In our model, the nature of the equilibrium forces each predictor to return the same mean profit for large α.

where

$$\begin{aligned}\hat{\xi}_1 &= (1+\phi)^{-1}(\gamma_1 a_{11} + \gamma_2 a_{21}) \\ \hat{\xi}_2 &= (1+\phi)^{-1}(\gamma_1 a_{12} + \gamma_2 a_{22})\end{aligned}$$

The parameters in a Misspecification Equilibrium are given by

$$\begin{bmatrix} 1+n_1^*\phi & \phi n_1^*\rho \\ \phi(1-n_1^*)\tilde{\rho} & 1+(1-n_1^*)\phi \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = A'\gamma, \tag{16}$$

where $n_1^* \in N_\alpha$. We saw that a non-trivial solution to (16) exists for all $\phi \geq 0$ and is given by

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \frac{1}{\triangle} \begin{bmatrix} (1+(1-n_1^*)\phi)(\gamma_1 a_{11} + \gamma_2 a_{21}) - \phi n_1^*\rho(\gamma_1 a_{12} + \gamma_2 a_{22}) \\ (1+n_1^*\phi)(\gamma_1 a_{12} + \gamma_2 a_{22}) - \phi(1-n_1^*)\tilde{\rho}(\gamma_1 a_{11} + \gamma_2 a_{21}) \end{bmatrix}$$

where $\triangle = (1+n_1^*\phi)(1+(1-n_1^*)\phi) - \phi^2 n_1^*\rho\tilde{\rho}$.

Clearly the REE parameters $(\hat{\xi}_1, \hat{\xi}_2)'$ differ from the ME parameters $(\xi_1, \xi_2)'$. For example, consider the case when the random variables $z_{1,t}, z_{2,t}$ are uncorrelated. Then

$$\begin{aligned}\xi_1 &= (1+n_1^*\phi)^{-1}(\gamma_1 a_{11} + \gamma_2 a_{21}) \\ \xi_2 &= (1+(1-n_1^*)\phi)^{-1}(\gamma_1 a_{12} + \gamma_2 a_{22}).\end{aligned}$$

3.4 Discussion of Intuition

The intuition behind Condition P and the existence of Intrinsic Heterogeneity is subtle. In a cobweb model the exogeneous shocks z have both a direct and an indirect effect on price. The direct effect is simply the $\gamma' z_t$ term. The indirect effect depends on the way in which agents incorporate z into their expectations. It is the interplay between the direct and indirect effects that makes intrinsic heterogeneity possible. In this subsection we illustrate the intuition through a simple example.

Suppose that the components $z_{1,t}, z_{2,t}$ are uncorrelated. Then the RPE is given by

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} (1+n_1\phi)^{-1} & 0 \\ 0 & (1+(1-n_1)\phi)^{-1} \end{bmatrix} \begin{bmatrix} \gamma_1 a_{11} + \gamma_2 a_{21} \\ \gamma_1 a_{12} + \gamma_2 a_{22} \end{bmatrix}$$

Recall that

$$p_t = \xi_1 z_{1,t-1} + \xi_2 z_{2,t-1} + \eta_t$$

Now set $\phi = 0$. This is the case where there is no feedback from expectations to price. In this special case

$$\begin{aligned}\xi_1 &= (\gamma_1 a_{11} + \gamma_2 a_{21}) \\ \xi_2 &= (\gamma_1 a_{12} + \gamma_2 a_{22})\end{aligned}$$

The parameters ξ_1, ξ_2 are completely determined by the direct effect $\gamma' A$. Suppose that $\phi > 0$. The RPE parameters are now

$$\begin{aligned} \xi_1 &= (1 + n_1\phi)^{-1} (\gamma_1 a_{11} + \gamma_2 a_{21}) \\ \xi_2 &= (1 + (1 - n_1)\phi)^{-1} (\gamma_1 a_{12} + \gamma_2 a_{22}) \end{aligned}$$

The parameters depend on the direct effect $\gamma' A$ and the indirect effect of expectations parameterized by n_1, ϕ. Note in particular that as $n_1 \to 1$ we have $|\xi_1(n_1)| \downarrow$ and $|\xi_2(n_1)| \uparrow$. For a fixed ϕ the indirect effect depends on n_1. As agents mass onto a particular predictor it diminishes the effect of that variable. This is because of the self-referential feature of the cobweb model that leads to an indirect effect on prices opposite to the direct effect of that variable. This makes $z_{1,t}$ a less useful predictor than before and the $z_{2,t}$ component becomes more profitable. The opposite happens as $n_1 \to 0$ and there is a unique n_1 where both predictors fare equally well in terms of mean profits. This proportion is the limit point of Intrinsic Heterogeneity.

Condition P places conditions on the indirect and direct effects and on the relative importance of the two exogeneous variables. In our simple example of uncorrelated shocks Condition P is equivalent to

$$\frac{(\gamma_1 a_{11} + \gamma_2 a_{21})^2}{(1+\phi)^2 (\gamma_1 a_{12} + \gamma_2 a_{22})^2} < Q < \frac{(1+\phi)^2 (\gamma_1 a_{11} + \gamma_2 a_{21})^2}{(\gamma_1 a_{12} + \gamma_2 a_{22})^2}$$

where $Q = \frac{Ez_2^2}{Ez_1^2}$. When there is no feedback ($\phi = 0$) there does not exist a matrix A and Σ_ε which satisfies Condition P. Intrinsic Heterogeneity does not exist in this instance. Because there is no indirect effect from expectations, and expectations has no bearing on price, agents will choose that model which forecasts price best. As ϕ increases the range of admissible Q increases.

3.5 Numerical Examples

We now turn to some specific numerical examples. Figure 1 illustrates the T-map for various values of α. The upper most part of the figure are T-maps corresponding to (starting from $n_1 = 0$ and moving clockwise) $\alpha = 2, \alpha = 20, \alpha = 50, \alpha = 100, \alpha = 200, \alpha = 2000$. We set

$$A = \begin{bmatrix} .3 & .10 \\ .10 & .7 \end{bmatrix}$$

$\gamma' = [.7, .5]$, the covariance matrix for ε_t, the white noise component of z_t is

$$\Sigma_\varepsilon = \begin{bmatrix} .7 & .2 \\ .2 & .6 \end{bmatrix}$$

and $\phi = 2$. The bottom portion of the figure is the profit difference function $F(n_1)$.

INSERT FIGURE 1 HERE

The matrix A and parameter ϕ have been chosen so that Condition P holds. For $n_1 = 0$ we have $F(0) > 0$ and for $n_1 = 1$ $F(1) < 0$. The proof of Proposition 8 shows that as $\alpha \to \infty$

$$H_\alpha(x) \to \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x < 0 \\ 1/2 \text{ if } x = 0 \end{cases}$$

and clearly this will govern the behavior of $T_a = H_\alpha \circ F$. Figure 1 illustrates how as α increases the inverse S-shape becomes more pronounced. The dashed line is the 45-degree line and all fixed points of the T-map will interesect this line. As α increases the fixed point declines from above .5 to about .22 which is the point at which $F(\hat{n}_1) = 0$. The Misspecification Equilibrium continues to exhibit heterogeneity even as $\alpha \to \infty$.

Figures 2 illustrates how the heterogeneity may disappear as $\alpha \to \infty$. Now we set

$$A = \begin{bmatrix} .93 & .10 \\ .10 & .2 \end{bmatrix}$$

so that Condition P does not hold but condition P1 does. For low values of α agents are not fully optimizing since some agents will continue to use z_2 even though it returns a lower expected payoff. But, as $\alpha \to \infty$ all agents behave optimally and the proportion using z_2 goes to zero.

INSERT FIGURE 2 HERE

Figure 4 shows the role ϕ plays in the degree of Intrinsic Heterogeneity. This graph depicts the T-map for various increasing values of ϕ. Notice that as ϕ increases the fixed point of the T-map moves further to the left. In this example, z_1 has a stronger influence on the price than z_2. When z_2 has a stronger effect, the fixed point will move to the right.

INSERT FIGURE 3 HERE

It is important to note that in a Misspecification Equilibrium in a model with Intrinsic Heterogeneity all predictors have the same average return as α becomes large. When α, the 'intensity of choice' is finite then there can be dispersions in the relative performance of predictors. For the case $\alpha = +\infty$ agents forecast optimally given their misspecification. If agents fully optimize and predictors are costless, the only way there will be heterogeneity is if all predictors have the same expected return. Heterogeneity arises in the costless case of (Brock and Hommes 1997) only in the steady-state in which different predictors make identical forecasts. Our results arise in a stochastic equilibrium in which different predictors produce different forecasts. Because of the importance of both exogeneous variables and expectational feedback both predictors will be in use. Although the forecasts are different, each predictor has identical mean forecast performance.

4 Stability under Real-Time Learning

In this section we address whether the equilibrium is stable under real-time learning. In a Misspecification Equilibrium agents misspecify, however, their beliefs are optimal linear projections given their underparameterization. They choose which component of the exogeneous process to underparameterize based on the unconditional mean. We now substitute optimal linear projections with real-time estimates formed via recursive least squares (RLS).[11] We also assume that agents choose their model each period based on a real-time estimate of mean profits.

Prices now depend on time-varying parameters

$$p_t = \xi_1(b^1_{t-1}, n_{1,t-1})z_{1,t-1} + \xi_2(b^2_{t-1}, n_{1,t-1})z_{2,t-1} + \eta_t$$

where b^1_{t-1}, b^2_{t-1} are updated by RLS

$$\begin{aligned} b^1_t &= b^1_{t-1} + t^{-1}R^{-1}_{1,t}z_{1,t-1}\left(p_t - b^1_{t-1}z_{1,t-1}\right) \\ b^2_t &= b^2_{t-1} + t^{-1}R^{-1}_{2,t}z_{2,t-1}\left(p_t - b^2_{t-1}z_{2,t-1}\right) \end{aligned}$$

where

$$\begin{aligned} R_{1,t} &= R_{1,t-1} + t^{-1}(z^2_{1,t-1} - R_{1,t-1}) \\ R_{2,t} &= R_{2,t-1} + t^{-1}(z^2_{2,t-1} - R_{2,t-1}) \end{aligned}$$

The $R_{j,t}, j = 1, 2$ are recursive estimates of the covariance matrix of the explanatory variables z_j.

Given these beliefs agents estimate the mean profits associated with each model

$$\begin{aligned} \hat{E}\pi_{1,t} &= \hat{E}\pi_{1,t-1} + t^{-1}\left(\pi_{1,t} - \hat{E}\pi_{1,t-1}\right) \\ \hat{E}\pi_{2,t} &= \hat{E}\pi_{2,t-1} + t^{-1}\left(\pi_{2,t} - \hat{E}\pi_{2,t-1}\right) \end{aligned}$$

The mean profits map into predictor proportions according to the law of motion

$$n_{j,t} = \frac{\exp\left[\alpha\hat{E}\pi_{j,t}\right]}{\sum^2_{j=1}\exp\left[\alpha\hat{E}\pi_{k,t}\right]}$$

[11] For an overview of stability under RLS in dynamic macroeconomics see (Evans and Honkapohja 2001).

The dynamic version of the model exhibits real-time learning in the sense that agents adaptively update previous estimates of their belief parameters and the mean profits from those beliefs. Agents now choose their model in each time period based on these recursive estimates. We are interested in whether the sequence of estimates b_t^1, b_t^2 and predictor proportions $n_{1,t}$ converge to the Misspecification Equilibrium.[12] Our aim is to use numerical illustrations to show that the equilibrium can be stable under real-time learning. It is beyond the scope of this paper to establish analytical convergence results for this learning rule.

We continue with a particular parameterization that generated Intrinsic Heterogeneity in the previous section. We set

$$A = \begin{bmatrix} .3 & .1 \\ .1 & .7 \end{bmatrix}$$

$\gamma' = [.7, .5]$, $\phi = 2$, and

$$\Sigma_\varepsilon = \begin{bmatrix} .7 & .2 \\ .2 & .6 \end{bmatrix}$$

We simulate the model for 100,000 time periods. We set the initial value of the VAR equal to a realization of its white noise shock, i.e., $z_0 = \varepsilon_0$. The initial value for $n_{1,0} = .82$ a value that was chosen to lie away from the end points and the ME. Initial estimated mean profits are equal to the realized profits under the initial conditions. The initial belief parameters were set to $b_0^1 = 1, b_0^2 = 2$. The initial estimated covariance matrices $R_{1,0}, R_{2,0}$ are the identity matrices. We chose $\alpha = 100$.

Figure 4 illustrates the results of the simulation. The top panel plots the simulated proportion $n_{1,t}$ against time. The middle and bottom panels plot the simulated belief parameters $b^{1,t}, b^{2,t}$, respectively. In each plot the solid horizontal line represents the respective variables' values in the Misspecification Equilibrium with Intrinsic Heterogeneity. As can be seen in each plot, there appears to be convergence to the ME. Initially there is considerable volatility in the proportion of agents who choose predictor 1. This volatility gradually dampens until the proportion approaches its equilibrium value. The dampening is much quicker in belief parameters as they approach their equilibrium values in a short period of time. Similar stability results were obtained for other parameter settings but the qualitative results were affected by α. For larger values of α it takes longer for the predictor proportions to settle down near the equilibrium values.

INSERT FIGURE 4 HERE

The intuition behind the stability is as follows. In our parameterization there is a unique ME with Intrinsic Heterogeneity. The uniqueness and heterogeneity arises

[12] Since we conduct the analysis numerically, we are being deliberately vague in what sense these sequences converge.

because Condition P guarantees that under, say, z_1 homogeneity agents will have an incentive to mass on z_2, and vice-versa. For large α agents mass on the predictor that returns the highest mean profit. In our simulations the proportions of agents are initially away from the ME. This implies that one predictor has a higher profit than the other. In the next period agents mass onto that predictor. Because of Condition P in the next period agents will mass onto the other predictor. As the rapid switching occurs agents update parameter estimates, which converge quickly, and accumulate data on relative forecast performance. As they learn about mean relative forecast performance, the volatility in predictor selection dampens and there is convergence towards the Misspecification Equilibrium.

In the light of (Brock and Hommes 1997) our results may seem surprising. However, in (Brock and Hommes 1997) the model is determinstic, the predictor choice is between a costly stabilizing predictor and a costless destabilizing predictor, and predictor fitness is the most recent period's realized profits. The stability results in our model are the result of agents looking at the mean relative performance of the predictors using the whole history of profits. This seems most appropriate within the stochastic model we examine.

5 Conclusion

This paper demonstrates how to obtain heterogeneous expectations as an equilibrium outcome in a model with optimizing agents. Our set-up is the standard cobweb model in which rational expectations was originally developed. We obtain our results with a discrete choice model for predictors, when agents are constrained to choose from a set of misspecified models. As in (Brock and Hommes 1997) the proportion of agents using the different predictors depends on their relative performance according to an 'intensity of choice' parameter. As the 'intensity of choice' increases agents will select only the most successful predictors. In (Brock and Hommes 1997) heterogeneity of expectations is a reflection of finite intensities of choice and disappears in the neoclassical limit. One contribution of this paper is to show that heterogeneity may remain for high intensities of choice as a result of the use of misspecified models.

Because of limits in cognition, knowledge of the economy, degrees of freedom, etc., we assume that agents must underparameterize by neglecting a variable or lag from their forecasting model. The importance of misspecification is widely recognized in applied econometrics and one that we believe should be reflected in realistic models of bounded rationality. Although we constrain agents to choose from a list of misspecified models, at the same time we require that the parameters of each model chosen are formed optimally in the sense that forecast errors are orthogonal to the explanatory variables of that model.

Our major theoretical contribution is to obtain existence results for a Misspecification Equilibrium within this framework and to obtain a suitable condition under which heterogeneous expectations persists for high intensities of choice. When this condition is satisfied we say the model exhibits Intrinsic Heterogeneity.

Our central finding that misspecification can lead to heterogeneous expectations is not at all obvious. If the intensity of choice is large, a key requirement for this possibility is that the model be self-referential, i.e., that there be feedback from expectations to actual outcomes. Heterogeneous expectations are not a necessary outcome when the intensity of choice is large, but do arise under a suitable joint condition on the model and the exogeneous driving processes. We illustrate the results in a simple bivariate model. In particular, we show that, *ceteris paribus*, Intrinsic Heterogeneity arises when the parameter governing the self-referential extent of the model is sufficiently large. This surprising feature of self-referential models has not been noted in previous work.

In this paper we have focused on the cobweb model. In future work, we will examine the framework in a Lucas-type monetary model. The Lucas-type model shares a similar reduced-form as the cobweb model. However, expectations have a positive feedback on price. The self-referential feature of these models are essential and thus a model with positive feedback may yield distinct results from those in this paper.

A Appendix

Proof of Proposition 1. Consider the matrix

$$[\triangle] = \left(I + \phi \sum_{j=1}^{K} n^j \Omega' u^{j'} \left(u^j \Omega u^{j'} \right)^{-1} u^j \right).$$

The absolute value of the indicated sum has a maximum value when considered as a function of $n \in S$. Hence for $|\phi|$ sufficiently small $[\triangle]$ is strictly diagonally dominant (see Horn and Johnson (1985), pg. 302) for all $n \in S$. Strictly diagonally dominant matrices have non-zero determinants and hence are invertible. ■

Proof of Theorem 3. Suppose to the contrary that the model does not exhibit intrinsic heterogeneity. From Theorem 2 we know that a ME exists for every α. Since the model does not have intrinsic heterogeneity, then for all $\bar{n} < 1$ there are infinitely many α such that $n_k^* > \bar{n}$ for some component $k = 1, \ldots, K$ where $n^* \in N_\alpha$. Hence there exists a sequence indexed by $\hat{s}$ such that $\alpha(\hat{s}) \to \infty$ with fixed points $n^*(\hat{s})$ satisfying $n^*_{k(\hat{s})}(\hat{s}) \to 1$. It follows that for some $i \in \{1, \ldots, K\}$ there exists a subsequence indexed by s such that $\alpha(s) \to \infty$ and $n_i^*(s) \to 1$. The expected profit functions $\tilde{F}_j(n)$ are continuous and hence for this sequence

$$E\pi^k(s) - E\pi^i(s) \to \tilde{F}_k(e_i) - \tilde{F}_i(e_i),$$

for all $k = 1, \dots, K$, where e_i is the unit coordinate vector with component i equal to one. However, condition P implies that there exists $j \neq i$ such that $\tilde{F}_j(e_i) - \tilde{F}_i(e_i) > 0$. It follows from (11) that

$$n_i^*(s) = \frac{1}{1 + \sum_{k \neq i} exp\{\alpha(s)(E\pi^k(s) - E\pi^i(s)\}}.$$

Thus $n_i^*(s) \to 0$ as $s \to \infty$. This contradicts $n_i^*(s) \to 1$ and hence the model must exhibit intrinsic heterogeneity. ■

Proof of Proposition 4. We want to show

$$\triangle = (1 + n_1\phi)\left((1+\phi) - \phi n_1\right) - \phi^2 \rho\tilde{\rho}\left(n_1 - n_1^2\right) > 0$$

or equivalently

$$\triangle = \phi^2 \left(\rho\tilde{\rho} - 1\right) n_1^2 + \phi^2 \left(1 - \rho\tilde{\rho}\right) n_1 + (1 + \phi)$$

The equation $\triangle$ is a quadratic concave in ϕ. Evaluated at the end points ($n_1 = 0$ and $n_1 = 1$) $\triangle > 0$. The quadratic is maximized at $n_1 = 1/2$ and returns a value of $\triangle(1/2) = (1/2)\phi^2 + (1 + \phi) > 0$. Since $\triangle$ is concave and is positive at both its extrema, we conclude that Condition $\triangle$ is satisfied. ■

Proof to Lemma 5. Define $S(n_1) = [\triangle]$. We can rewrite (13) as

$$S(n_1)\xi = A'\gamma \tag{17}$$

Somewhat abusing notation, it is now convenient to rewrite $F(n_1)$ as $F(\xi(n_1))$ thus

$$\frac{dF}{dn_1} = F'(\xi(n_1)\frac{d\xi}{dn_1}(n_1)$$

To establish the result examine $DF'\frac{d\xi}{dn_1}$

Differentiating (17) leads to

$$Sd\xi + DS\xi dn_1 = 0$$

or

$$\frac{d\xi}{dn_1} = -S^{-1}DS\xi \tag{18}$$

One can verify that

$$DF(\xi) = \phi D \begin{bmatrix} & \xi_1 E z_1^2(1 - r^2) \\ - & \xi_2 E z_2^2(1 - r^2) \end{bmatrix} \tag{19}$$

where $r^2 = \rho\tilde{\rho}$. (19 can be expressed as

$$DF' = \xi'\phi D(1 - r^2)\begin{pmatrix} Ez_1^2 & 0 \\ 0 & -Ez_2^2 \end{pmatrix} \tag{20}$$

Combining (18) and (20) we have

$$\begin{aligned} DF' \frac{d\xi}{dn_1} &= -\phi D(1-r^2)Ez_1^2\xi' \begin{pmatrix} 1 & 0 \\ 0 & -Q \end{pmatrix} S^{-1}DS\xi \\ &= -Ez_1^2\phi^2 D(1-r^2)\gamma' A \left(S^{-1}\right)' \begin{pmatrix} 1 & 0 \\ 0 & -Q \end{pmatrix} A'\gamma \\ &= -Ez_1^2\phi^2 D(1-r^2)\gamma' AKA'\gamma \end{aligned}$$

The remainder of the proof will establish K is positive definite. Observe that K is (2×2) and positive definiteness is equivalent to $K(1,1) > 0$ and $det(K) > 0$.

Consider the sign of $K(1,1)$. The numerator of $K(1,1)$ (its denominator is always positive) can be written[13]

$$num(K(1,1)) = 1 + \left(2 - r^2 + 2n_1(r^2-1)\right)\phi - (n_1-1)^2(r^2-1)\phi^2$$

When $n_1 = 1$

$$num(K(1,1)) = 1 + r^2\phi$$

which is positive for all $\phi > -1$. Furthermore, it can be verified the derivative of $num(K(1,1))$ is

$$\frac{d(num(K(1,1)))}{dn_1} = 2\phi\left(r^2-1\right)(1+\phi(1-n_1))$$

This quantity is negative for all n_1 provided $\phi > 0$. negative for all n_1 provided $\phi > 0$. For the case $\phi > -1$ $K(1,1)$ evaluated when $n_1 = 0$ is positive and its derivative is

$$\frac{d(num(K(1,1)))}{dn_1} = (1+\phi)(1+(1-r^2)\phi) > 0$$

These imply that the quadratic in $num(K(1,1))$ is concave and always positive for $n_1 \in [0,1]$.

The determinant is given by

$$det(K) = \frac{Q^2(1-r^2)}{\left(1+\phi+(n_1-1)n_1(r^2-1)\phi^2\right)^2}$$

which is always positive. This establishes the lemma. ■

[13] All simplifications were done in Mathematica. The programs are available from the authors upon request.

Further Details For Section 3.2. Using the profit functions derived above we can find

$$\begin{aligned}\frac{F(1)}{Ez_1^2} &= -\phi D\{(\xi_1^2(1)\tilde{\rho} - \xi_2^2(1)\rho)\rho + (1/2)(\xi_2^2(1) - \\ &\quad \tilde{\rho}^2\xi_1^2(1))Q - (1/2)(\xi_1^2(1) - \rho^2\xi_2^2(1))\} \\ \frac{F(0)}{Ez_2^2} &= \phi D\{\tilde{\rho}[\xi_2^2(0)\rho - \xi_1^2(0)\tilde{\rho}] + (1/2)[(\xi_1^2(0) - \\ &\quad \xi_2^2(0)\rho^2)Q^{-1} - (\xi_2^2(0) - \xi_1^2(0)\tilde{\rho}^2)]\}\end{aligned}$$

Thus, for example,

$$F(1) < 0 \text{ if } \left[\xi_1^2(1) - \xi_2^2(1)\right]\left(Q\tilde{\rho}^2 - 1\right) > 0.$$

Using $Q\tilde{\rho}^2 = r^2 < 1$ it follows that

$$F(1) < 0 \text{ if } \left[\xi_1^2(1) - \xi_2^2(1)\right] < 0.$$

■

Proof of Lemma 8. Take part (1), which states that Condition P implies Intrinsic Heterogeneity. We will establish that (i) for each α, $\exists n_1^*(\alpha) \in N_\alpha$ uniquely, (ii) $\exists \{\alpha(s)\}_s$ *s.t.* $\alpha(s) \to \infty \Rightarrow n_1^*(\alpha(s)) \to \hat{n}_1$ where $\hat{n}_1 \in N_\infty \equiv \{n_1 \in [0,1] : \text{ for } \alpha \to \infty\ n_1 = T_\alpha(n_1)\}$ and (iii) $F(\hat{n}_1) = 0$.

Claim (i) that there exists a unique fixed point $n_1^*(\alpha)$ for each α comes directly from Theorem 6.

Claim (ii) is that there is a sequence of α's indexed by s defined so that as $\alpha(s) \to \infty$ the corresponding sequence of fixed points from claim (i) $n_1^*(\alpha(s)) \to \hat{n}_1$. That there exists a sequence $\alpha(s) \to \infty$ and a similarly corresponding sequence $n_1^*(\alpha)$ follows from claim (i) and since $\alpha \in \mathbb{R}_+$ there are infinitely many such sequences. Theorem 6 used Brouwer's theorem and Lemma 5 to establish that there exists a unique fixed point for each α. Hence there exists a limit to the sequence of fixed points indexed by s and define it to be $n_1^*(\alpha(s)) \to \hat{n}_1$. By construction, $\hat{n}_1 \in N_\infty$.

Claim (iii) is that $F(\hat{n}_1) = 0$. Assume $\hat{n}_1 \in N_\infty$, Condition P, and $F(\hat{n}_1) \neq 0$. It follows that $F(\hat{n}_1) > 0$ or $F(\hat{n}_1) < 0$. Recall, $n_1(\alpha) = H_\alpha(F(n_1))$. By definition, as $\alpha \to \infty$

$$H_\alpha(x) \to \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x < 0 \\ 1/2 \text{ if } x = 0 \end{cases}$$

So we have $n_1^*(\alpha) \to \hat{n}_1 \in \{0,1\}$. But, assuming Condition P implies $F(1) < 0$ and $F(0) > 0$. Hence, $\hat{n}_1$ is not an ME which contradicts our initial assumption. It must be the case that, with Condition P, $F(\hat{n}_1) = 0$.

Note now that Lemma 5 establishes $\hat{n}_1$ is the unique point where $F(\hat{n}_1) = 0$. Thus, we conclude that Condition P implies $n_1^*(\alpha) \to \hat{n}_1$ where $F(\hat{n}_1) = 0$.

A similar argument establishes parts (2) and (3) of the proposition. Note that Condition P1 implies $F(1) > 0$ and $F(0) > 0$ and Condition P0 has $F(1) < 0$ and $F(0) < 0$. The monotonicity of F means that $\forall n_1, \alpha \; F(n_1(\alpha)) \neq 0$ and the result follows immediately from above. ■

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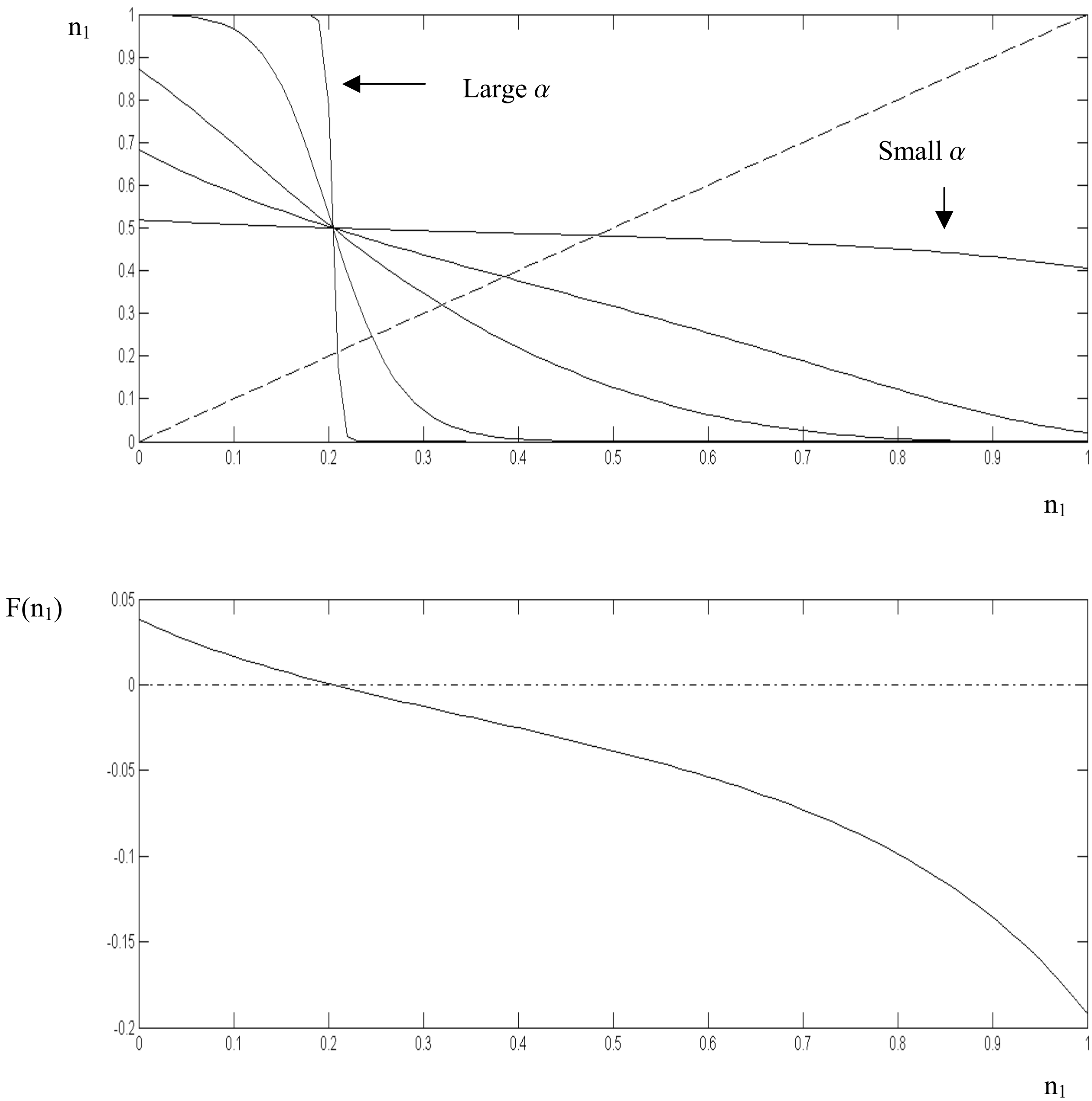

Figure 1: T-map for various values of α and ϕ =2.

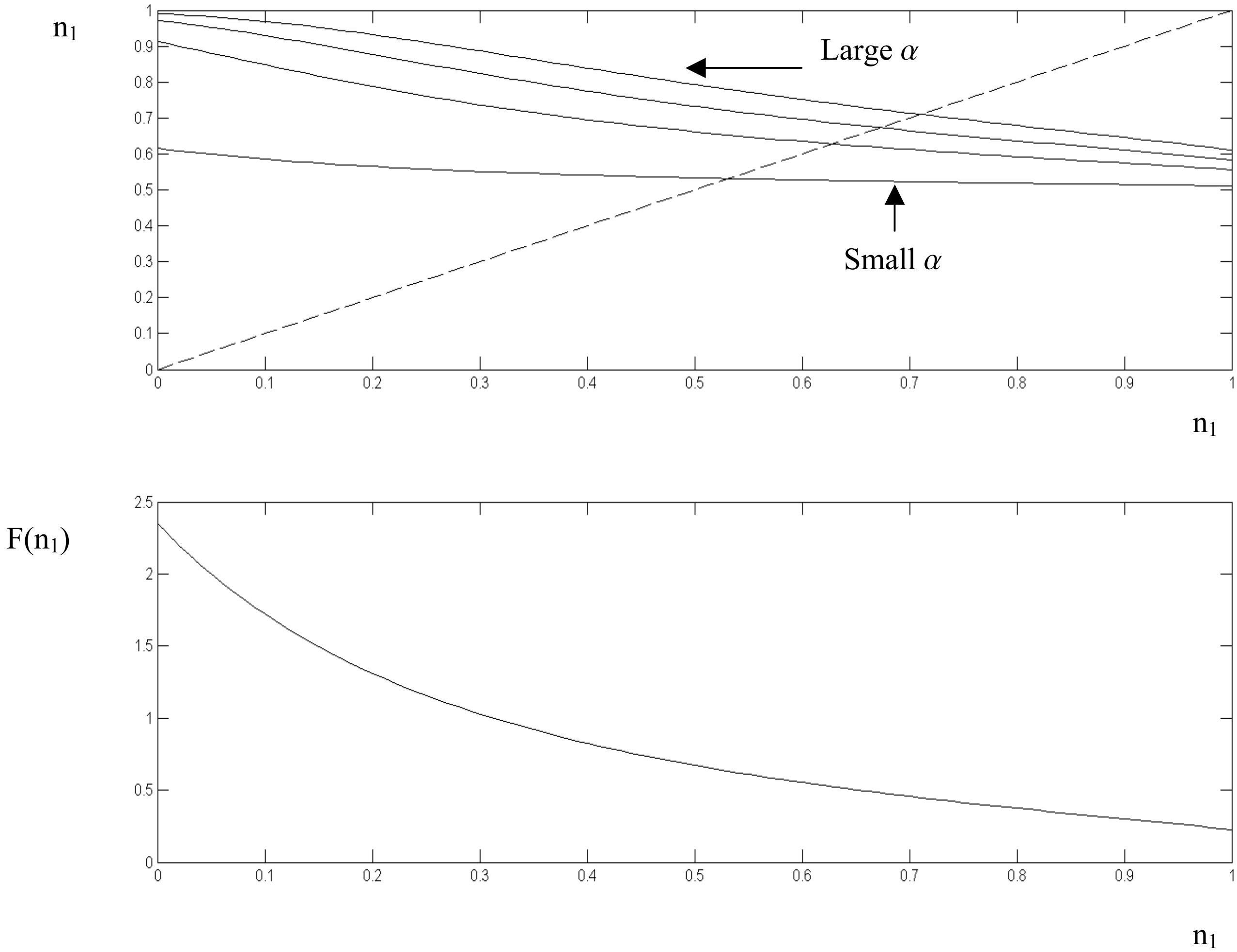

Figure 2. T-map for various values of α and ϕ=2 for the case of no Intrinsic Heterogeneity and predictor 1 homogeneity.

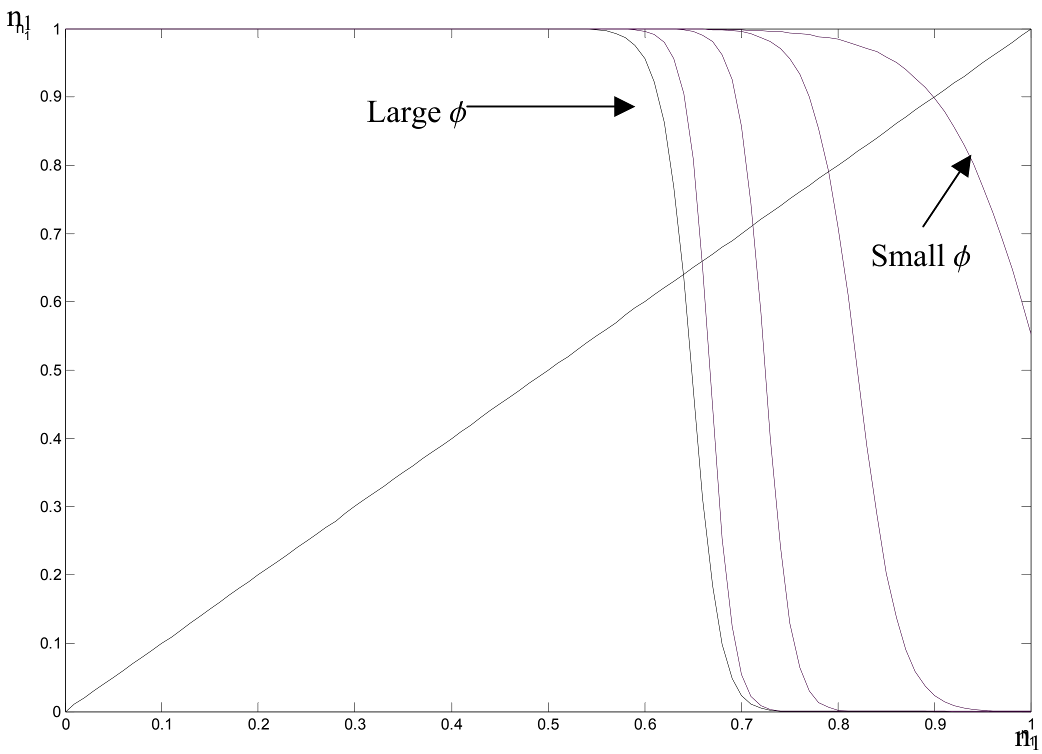

Figure 3. T-map for α=2000 and ϕ=.5,1,2,5,10,20 for the case of Intrinsic Heterogeneity. Note that as ϕ increases the fixed point of the T-map.

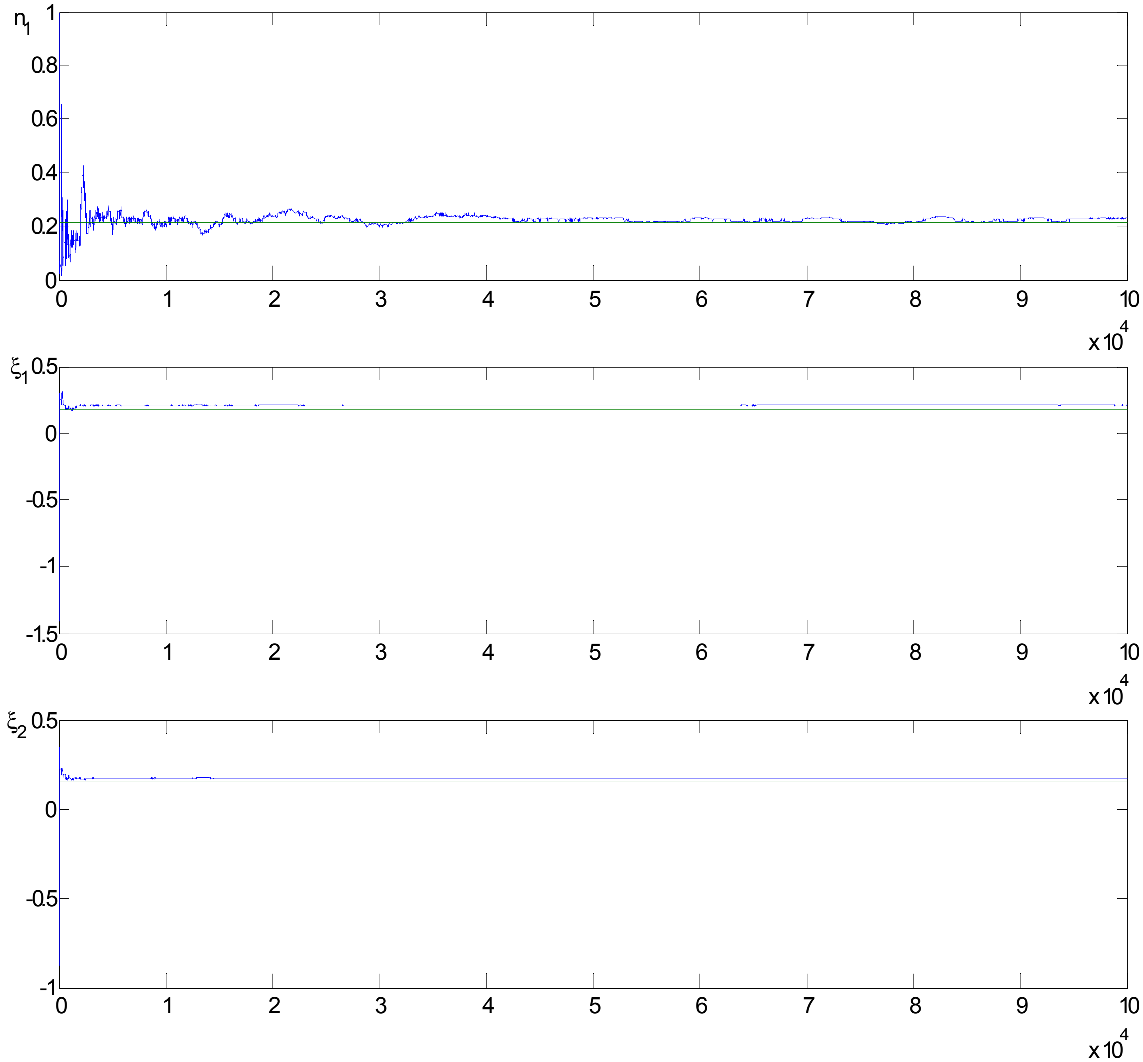

Figure 4. Real-time learning simulations.