Long Memory Models and Tests for Cointegration: A Synthesizing Study

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Abstract

A pervasive literature documents the potential downfall for cointegration methodology when the underlying relationships are long memory variables. In particular, cointegration procedures such as those proposed by Johansen (1988) and Engle and Granger (1987) can fail in at least three ways. First, one may fail to find cointegration when it in fact exists. Second, one may fail to find the correct cointegrating relationship. Third, cointegration tests may find an equilibrium relationship when none exists. To our knowledge, no existing study considers each of these potential failures using a common methodology. In this paper, we use Monte Carlo methodology to tie together the existing literature by considering all three cases in one study. We are able to show that the problems are severely exacerbated should the individual variables or equilibrium relationships be distributed as long memory processes. We further extend the literature by considering a new class of long memory models that allows for periodic movement in the autocorrelation function of data, namely the GARMA model. We are able to show that cointegration methodology suffers from many of the same concerns under this framework, albeit less problems exist when estimating cointegrating vectors. The GARMA model allows cyclical long memory, and it shown that the introduction of this type of model has important implications for long memory models unavailable with existing models. In particular, we extend the analysis of Baillie and Bollerslev (1994) to show that a collection of nominal exchange rates are individually I(1) but share a long memory component that dissipates at a cyclical rate. Johansen cointegration fails to find a connection and fractional cointegration produces connection that is not non-stationary.

Keywords: Long memory; GARMA; Gegenbauer process; ARFIMA; Johansen cointegration tests; Engle-Granger cointegration tests.
JEL classifications: C15, C22, C32

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1 Introduction

An important strain of modern empirical macroeconomics is the question of whether or not important economic variables are bound together by an equilibrium relationship. The methodology of detecting equilibrium relationships in macroeconomics was fundamentally changed with the seminal work of Engle and Granger (1987). The detection of unit roots in time series data has led to an abundance of studies using the techniques of cointegration. According to the definition of Engle and Granger, two variables that are integrated of order d are cointegrated if there exists a strictly positive value b such that a linear combination of these variables are integrated of order d-b. Generally speaking, cointegration seeks to find linear combinations of variables that reduce the variance of the individual series. In the vast majority of cases, it is assumed that d=b=1, and the researcher attempts to determine if linear combinations of infinite variance unit root processes create a finite variance series. The two most commonly employed procedures for detecting cointegration are the univariate regression techniques of Engle and Granger (1987) and the full system's approach of Johansen (1988) and Johansen and Juselius (1990). These procedures are generally employed under the assumption that d=b=1.

Recent research has questioned the assumption that economic data satisfy the property that d=b=1. A proliferation of research has suggested that many economic variables and their equilibrium relationships exhibit characteristics that are more consistent with long memory. A long memory process has the characteristic that the correlation of a variable with itself over time decays very slowly; in other words, the autocorrelation function of a long memory process typically decays at a hyperbolic rate. In particular, a long memory process is one in which the autocovariances of the process are not absolutely summable. Alternatively, the notion of long memory can be expressed in the frequency domain by noting that the spectrum of a long memory process is unbounded for some frequency between 0 and π. Granger and Joyeux (1980) and Hosking (1981) popularized the notion of long memory with their paper on fractional processes (ARFIMA processes). A long memory ARFIMA process must be differenced d times, where d can take on non-integer values, to achieve a stationary ARMA process. The long memory ARFIMA process also has the property that its spectrum in unbounded at the origin. The notion of fractional integration has proven to be quite important in modelling macroeconomic data and their relationships. In particular researchers have used fractional models to study interest rates (Barkoulas, Baum, and Oguz (2001), Baum and Barkoulas (2002), and Dueker and Startz(1998)), aggregate output (Diebold and Rudebusch (1989) and Sowell (1992)), inflation (Baillie, Chung, and Tieslau (1996)), unemployment (Diebold and Rudebusch (1989)), monetary aggregates (Barkoulas, Baum, and Caglayan (1999) and Bae and Jensen (1999)), and exchange rate dynamics (Cheung and Lai (1993), Cheung (1993), Baillie and Bollerslev (1994), Diebold, Husted, and Rush (1991)). An alternative long memory model is the k-factor GARMA model, which was first studied by Woodward et. al. (1998) and more recently by Smallwood and Beaumont (2002). The k-frequency GARMA model generalizes the fractional model by allowing asymmetric periodic decay in the autocorrelation function of data. Arteche and Robinson (2000) suggest that the GARMA model may be quite important for studying variables with highly dependent seasonal characteristics, and it is likely a better alternative than seasonal differencing. Baum, Barkoulas, and Oguz (2001) suggest that the equilibrium relationship between international interest rates has properties that are more consistent with a GARMA model as compared to a fractional one. Baillie and Bollerslev reach the same conclusion when analyzing equilibrium relationships between nominal exchange rates. The GARMA model has been used to successfully model inflation rates (Chung (1996 b) and Arteche and Robinson (2000)), real interest rates (Smallwood and Norrbin (2002)), the error correction mechanism for interest rates (Ramachandran and Beaumont(2001)), and financial aggregates (Smallwood and Beaumont (2002)).

Given the abundance of research in the area of long memory, one may naturally challenge the usefulness of existing cointegration tests if the underlying variables and/or the equilibrium relationships among variables, are long memory. Cheung and Lai (1993) have demonstrated through limited Monte Carlo analysis that the Engle-Granger procedure has low power if the underlying equilibrium relationship among I(1) variables is distributed as a fractional process. Andersson and Gredenhoff (1999) show the Johansen procedure yields an extremely biased estimate of the components of a fractional error correction mechanism when the equilibrium relationship is distributed as a fractional process. Elliott (1998) demonstrates analytically that the Wald test associated with the null hypothesis that the cointegrating vector takes on a certain form does not have the typical chi square distribution if at least one variable in the system has a first order autoregressive coefficient that is marginally less than one. On the other hand, Gonzalo and Lee (1998; 2000) demonstrate analytically that the Johansen procedure is flawed for detecting cointegration among certain unrelated fractional processes. In particular, they show that under certain models, the Johansen test statistic for the null of no cointegration diverges as the sample size grows. In this vein, it is clear that cointegration tests can fail in three basic ways. First, cointegration tests may fail to find cointegration when the variables and particularly equilibrium relationships are long memory. Second, even when cointegration is correctly found, the research of Andersson and Gredenhoff suggests that inference regarding a particular cointegrating vector can be influenced by the presence of long memory components. Finally, the work of Gonzalo and Lee suggests that it is possible to find equilibrium relationships when in fact none exist.

In this paper we use Monte Carlo analysis to demonstrate the potential failure of existing cointegration tests. In particular, and unlike previous research, we consider each of the ways in which cointegration tests can fail using a common methodology. We also consider the use of Johansen's likelihood ratio test in regards to inference about a particular cointegrating vector. Finally, we consider the properties of inference for cointegration tests under a more general notion of long memory; in particular, we introduce the GARMA model and present the small sample properties of existing cointegration tests. Finally, to demonstrate that the GARMA model is more than a theoretical curiosity, we present several examples of existing economic data that likely have properties that are quite well described by the GARMA model. We find that cointegration tests can fail when the underlying process and/or equilibrium relationships are long memory. This failure can result in any of the three scenarios described above. When interest lies in a particular cointegrating vector, we show that cointegration tests very rarely find cointegration with the correct cointegrating vector. Further, we demonstrate that Gonzalo and Lee's notion of spurious cointegration applies to GARMA processes as well. In addition, we show that the GARMA model fits several variables quite well including the nominal euro-dollar exchange rate, and the equilibrium relationship among a group of nominal exchange rates.

The rest of the paper is organized as follows. In section 2, we present the k-frequency GARMA model and its special cases, the GARMA model and the ARFIMA model. We demonstrate that existing economic data is quite well described by the GARMA model. In section 3, we present the Monte Carlo results for all three cases described above. We concentrate on those processes that most likely result in failure of cointegration tests and unit root tests. Section 4 presents concluding remarks and ideas for future research.

2 Long Memory Processes

There are several classes of long memory processes considered here. We are particularly interested in analyzing fractional and k-frequency GARMA models. The best way to introduce the properties of k-frequency GARMA models is to show how they generalize ARIMA, ARFIMA and ARMA models. Consider first the process

$$\phi(L)(1-L)^d(x_t-\mu)=\theta(L)\,\varepsilon_t. \tag{1}$$

where $\phi(L)$ and $\theta(L)$ are polynomials in the lag operator L such that $\phi(z)=0$ and $\theta(z)=0$ have roots outside the unit circle and $\{\varepsilon_t\}$ is a white noise disturbance sequence. The model reduces to an ARMA model when the differencing parameter $d=0$ and to an ARIMA process when $d=1$. ARFIMA models are stationary for $d<\frac{1}{2}$ and mean-reverting for $d<1$ so when $\frac{1}{2}<d<1$ we get the interesting result that the process is nonstationary yet mean-reverting. Granger and Joyeux (1980) show that the autocorrelations, ρ_k, for an ARFIMA process for large k and $d<\frac{1}{2}$ are given by the following approximation

$$\rho_k \simeq \frac{\Gamma(1-d)}{\Gamma(d)}k^{2d-1}, \tag{2}$$

from which the monotonic, hyperbolic decay of the autocorrelation function can be seen. Further, it is well known that the spectrum behaves like ω^{-2d} as $\omega\to 0$. In particular, the spectrum is unbounded at the origin. In contrast, for ARIMA processes, the spectrum behaves like ω^{-2}, while the theoretical autocorrelations are 1 at all time horizons (Granger and Joyeux (1980)).

Long memory models differ from ARMA models in several important respects. As seen above, for the classes of ARFIMA and ARIMA models, the spectrum is unbounded at the origin. The spectral density function for stationary ARMA models is bounded at all frequencies. Further long memory processes, as described above, have the characteristic that their autocovariances are not absolutely summable (c.f. McLeod and Hippel (1978)). On the other hand, stationary ARMA processes have autocorrelations that decay at a geometric rate to zero. Because of these properties, and given the fact that ARFIMA models can often capture the statistical properties of a variable in a more parsimonious manner than ARMA models, ARFIMA models have been successfully used in many disciplines as referred to above. The success of ARFIMA processes in modelling economic and financial time series has been very well documented (c.f. Baillie (1996)), and therefore, we will not consider estimation of these models in this section.

The k-frequency GARMA model generalizes the ARFIMA model, by allowing for periodic or quasi-periodic movement in the data. The multiple frequency GARMA model is defined as follows.

$$\phi(L)\prod_{i=1}^{k}\left(1-2\eta_i L+L^2\right)^{d_i}(X_t-\mu)=\theta(L)\,\varepsilon_t \tag{3}$$

where the parameters η_i provide information concerning the periodic movement in the data, and again all roots to $\phi(z)=0$ and $\theta(z)=0$ lie outside the unit circle. If there exists a single value $\eta_j=1$, then the model has a fractional component as described above. This model was initially proposed by Gray et. al. (1989). Recently, Artche and Robinson (2000) consider a semiparametric approach to estimating the above model, while Smallwood and Beaumont (2002) calculate the statistical properties of the time domain quasi maximum likelihood estimator. In this paper, we place particular interest on the case where k=1. The single frequency GARMA model, or more simply the GARMA model, has been studied extensively by Chung (1996 a and b) and Ramachandran and Beaumont (2001).

For a single frequency GARMA model, when $\eta=1$, the model reduces to an ARFIMA$(p,2d,q)$ model, and when $\eta=1$ and $d=\frac{1}{2}$, the process is an ARIMA model. Finally, when $d=0$ we get a stationary ARMA model. The GARMA model is stationary when $|\eta|<1$ and $d<\frac{1}{2}$ or when $|\eta|=1$ and $d<\frac{1}{4}$ (see Gray et. al., 1989). The model exhibits long memory when $d>0$ and is anti-persistent when $d<0$. Note that the polynomial $(1-2\eta z+z^2)=0$ has a pair of complex conjugate roots with length one. Chung (1996 b) calculates the spectral density function and shows that for d>0, the spectral density function has a pole at $\upsilon=\cos^{-1}(\eta)$, which ranges from 0 to π. Further, Chung shows that for large k, the autocorrelation function ρ_k for a GARMA(0,0) model with $|\eta|<1$ and $0<d<\frac{1}{2}$ can be approximated as

$$\rho_k \approx K^*\cos(k\,\upsilon)\,k^{2d-1} \tag{4}$$

where K^* does not depend upon k. This expression makes clear the hyperbolically damped sinusoidal pattern of the autocorrelation function of a stationary GARMA model with $|\eta| < 1$. Note that the autocorrelation function will decay symmetrically about zero in this case.

The GARMA model may prove to be an important long memory model in that it relaxes several aspects of the ARFIMA model. First, it is clear that the GARMA model allows for more diversity in the covariance structure of a variable witnessed both through the autocorrelation function and the spectral density function. Further, the GARMA model is stationary for a large class of processes. For example, let x_t and y_t be stochastic sequences defined as (1-L)$^{2*.4}x_t = \varepsilon_{1t}$ and (1-2*.9995L+L^2)$^{.4}x_t = \varepsilon_{2t}$, where ε_{1t} and ε_{2t} are unrelated martingale difference sequences.[1] These models are nearly identical from a parametric standpoint. In fact, the second model has a cycle of over 196 periods. However, by relaxing the assumption that $\eta = 1$, the second model is in fact covariance stationary, while the first is non-stationary. One of the primary objectives of this paper is to analyze how the relaxation of this assumption affects inference as related to the standard cointegration tests the modern econometrician has at their disposal.

As already alluded to, the GARMA model and its generalization, the k-frequency GARMA model, has already been used to successfully model inflation, the real interest rate, financial aggregates, and equilibrium among international interest rates. However, the GARMA model is a relatively new tool, and as such, has not been as widely applied as the ARFIMA model. In the following subsection, we demonstrate that the GARMA model can be used to successfully model the nominal euro dollar exchange rate and the equilibrium relationship among several nominal interest rates as studied in Baillie and Bollerslev (1994).

2.1 Empirical Applications of the GARMA Model

In this subsection, we analyze several important macroeconomic series and show that the GARMA model captures the statistical properties of these variables. In particular, we are able to show that the GARMA model captures important cyclical components of the daily euro-dollar exchange rate, and the model also captures important periodic movement in the equilibrium relationship between seven nominal exchange rates. In addition, for each of these scenarios, we show that the ARFIMA model is inappropriate and can result in particularly misleading conclusions. For example, and as alluded to below, we demonstrate that when an ARFIMA model is fit to the equilibrium relationship between the nominal exchange rates described below, the result is a non-stationary process with an estimated value of d equal to approximately .92. In contrast, the estimated GARMA model is significantly stationary, and further, we are able to reject the null hypothesis that the underlying process is distributed as an ARFIMA process when the GARMA alternative is considered.

[1]The first model has been written as such to highlight the fact that when $\eta = 1$ for a GARMA model, the result is an ARFIMA model where the differencing parameter, d, is multiplied by 2.

The data for the nominal euro exchange rate are taken from the Federal Reserve Board. The data are daily noon buying rates in New York, and the sample runs from December 1, 2000 to August 2, 2002. We therefore have a moderately large sample consisting of 419 observations. To estimate the model we use a constrained sum of squares that exploits the varying convergence rates of the parameters of the GARMA model. Technical details can be found in Smallwood and Norrbin (2002), and we thus omit most of the details. The standard errors are calculated using the distributions found in Chung (1996 b). It should also be noted that the asymptotic distribution of η is non-standard, and the convergence rate is T or T^2, depending on whether or not η is less than one. We further note that under the assumption of normality in the residuals, there are at least two ways to determine if the underlying process is "significantly" GARMA. Recall that if $\eta = 1$, the model reduces to an ARFIMA model, and for better results, such a model should be estimated. First, one can test this restriction by looking at the asymptotic confidence intervals associated with η.[2] If the value of unity lies outside the 95% confidence interval for the estimated value of η, then one can reject the null hypothesis that $\eta = 1$. Conversely, one can obtain the sum of squared errors from a restricted model (the ARFIMA model) and compare them to the sum of squared errors from an unrestricted (the GARMA model). Under normality of the residuals, the standard Wald test can be constructed.

Before we consider estimation of a GARMA model, first consider the first 300 autocorrelations of the log of the nominal euro-dollar exchange rate, which are depicted in figure 1. The figure also depicts the 95% confidence intervals associated with the autocorrelations, which appear above and below the associated autocorrelations. The most glaring aspect of the figure is the clear cyclical correlation structure. Casual visual inspection of the figure indicates the existence of a cycle lasting between 170 to 200 days. The results of GARMA estimation, which are reported in table 1, confirm these results.[3] The estimated value of η is equal to .99947. This corresponds to a cycle of about 193.40 periods. As seen in table 1, the results indicate that the 95% confidence interval of η does not contain the value 1.[4] Further, the p-value associated with the null hypothesis that the underlying process was generated from an ARFIMA model versus the alternative of a GARMA model is .0197. We should further note that the estimated value of d (.4759) is within two standard deviations of .5 (the boundary for non-stationary processes). Together these results allow us to conclude that the GARMA model provides a superior in-sample fit relative to

[2]Chung (1996 a) shows that the asymptotic distribution of η converges at either rate T or rate T^2 to various quotients of integrals of Brownian motion processes. Chung reports simulated asymptotic confidence intervals both for the case $|\eta| < 1$, and $|\eta| = 1$.

[3]We considered various criteria for selection of p and q (the number of autoregressive and moving average components). In all cases, the various information criteria, including the Schwarz Bayesian Information Criterian and the Akaike Information Criterian, selected p=q=0.

[4]We have reported the 95% confidence interval associated with the hypothesis that $|\eta| = 1$. The 95% confidence interval associated with the hypothesis that $|\eta| < 1$ is [.9990,.9999], and hence does not contain the value of unity either.

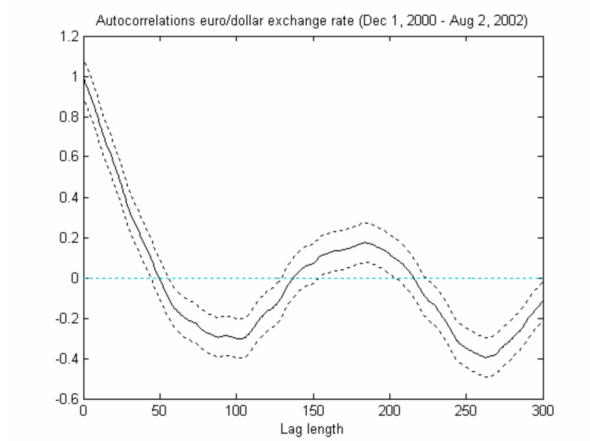

Figure 1:

the ARFIMA model. The estimated value of d is within stationary ranges, but application of the appropriate standard errors indicates that a non-stationary GARMA model may also be appropriate.[5]

As a second application, we consider the equilibrium relationship among the same nominal exchange rates studied by Baillie and Bollerslev (1994). In their paper, they point out that the nominal exchange rates of Canada, Germany, France, Italy, Switzerland, Japan, and the United Kingdom vis-a-vis the U.S. dollar appear to have unit roots, although a linear combination found through OLS appears to be distributed as a long memory variable with integration order significantly less than 1. Although compelling, Baillie and Bollerslev are forced to admit that their findings come with some caution, since the equilibrium relationship, although mean-reverting, is itself non-stationary. Further, the authors note that there appears to be a strong cyclical relationship among the exchange rates that may well be described by a GARMA model. Given the dearth of the literature at the time, the authors do not consider estimation of this model.

We use the same exchange rates considered by Baillie and Bollerslev, although we update their sample. The data come from the St Louis Federal Reserve Board (FRED) and are daily buying rates at noon in New York. The sample extends from January 2, 1990 to December 31, 1998. This gives us a total of 2,264 observations. Figure 2 depicts the first 500 autocorrelations of the German DM. The figure displays the linear decay typical of a unit root

[5]To further validate these results, we applied the following linear filter to the log of the nominal eurodollar exchange rate:

$(1\text{-}2{*}.99947L + L^2)^{.2}$.

Note, in this case, differencing the data is inappropriate, since the value of $\eta \neq 1$. The results are nearly identical to the ones reported in table 1.

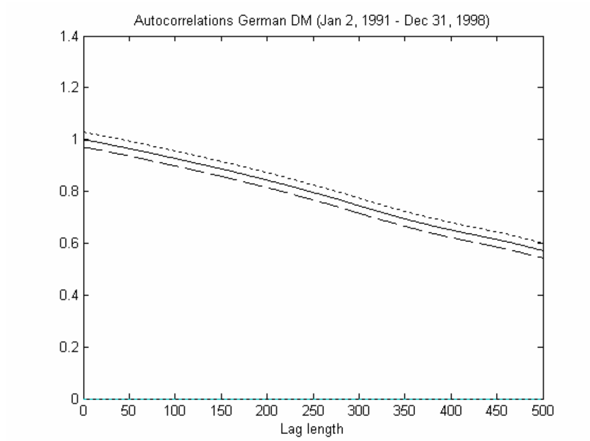

Figure 2:

process. The remaining exchange rates produce similar pictures; these figures are omitted for reasons of space.

Unit root tests were run on each of the exchange rates described above. In particular, we consider both an Augmented Dickey Fuller (ADF) test (1979) and a Kwiatkowski-Phillips-Schmidt-Shin (KPSS, 1992) test. The null hypothesis of the ADF test is a unit root, while the KPSS has a stationary null. In every case, we fail to reject the null hypothesis for the ADF test, while we reject the null for the KPSS test. As further evidence, we fit a GARMA (0,0) model to each of the exchange rates, and in no case could we reject the null hypothesis that $\eta = 1$ and $d = .5$. Thus, the individual exchange rates may well be described as unit root processes. As a benchmark for cointegration, we also ran the Johansen cointegration test. The results were quite robust, and we were never able to reject the null hypothesis of 0 cointegrating relationships. For example, when an intercept was included in the cointegrating relationship, the trace statistic associated with the null of 0 cointegrating relationships was 92.33, while the maximum eigenvalue statistic achieved a value of 43.33. The 5% critical values are 124.24 and 45.28 respectively, and hence our failure to reject.

Baillie and Bollerslev conjecture that Johansen's procedure is unable to capture equilibrium relationships should the cointegrating relationship be distributed as a long memory process. They run OLS, using the German DM as a dependent variable.[6] We consider the same methodology, and regress the log

[6]Cheung and Lai (1993) are among the first to show that if the errors from such a cointegrating regression are distributed as long memory fractional processes, the regression results in parameters that are indeed consistent. However, the limiting distribution is not standard, and the convergence rate depends on the unknown differencing parameter. Marinucci and Robinson (2001) show that if the errors in such a cointegrating regression are distributed as fractional processes, then frequency based regression analysis also results in consistent

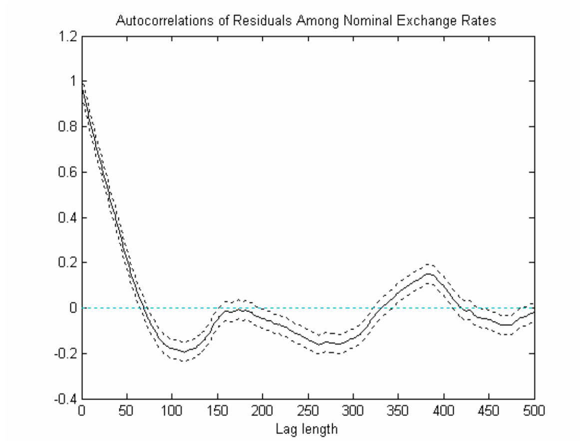

Figure 3:

of the DM on a constant and the log of the remaining six exchange rates. The first 500 autocorrelations of the residuals from such a regression are depicted in figure 3. These autocorrelations stand in stark contrast to the autocorrelations depicted in figure 2 and suggest that the equilibrium relationship may best be defined by a GARMA model. The estimation results for the residual series are depicted in table 2. The results are quite strong and clearly suggest that the GARMA model fits the data well. In particular, the estimated value of η (.99957), while close to 1, is significantly less than 1. Furthermore, the p-value associated with the null of an ARFIMA process is .000011, clearly indicating that the null of an ARFIMA process can be rejected in favor a GARMA model. In addition, the estimated value of η corresponds to a cycle of roughly 214.13 days. This is consistent with the autocorrelations of figure 3. Finally, we point out that the estimated value of d (.4554) is significantly less than $\frac{1}{2}$. The evidence here clearly suggests that the 7 nominal exchange rates are individually non-stationary, but that at least one linear relationship exists such that the cointegrating error is stationary. This stands in contrast to both the Johansen cointegration test as described above, where no cointegration could be found, and the results of ARFIMA estimation. An ARFIMA model applied to the residuals resulted in an estimated value of d that is equal to .92. Thus, the evidence from fractional cointegration techniques would yield results suggesting that the equilibrium relationship is non-stationary. The implications of this result are quite disparate from the results found from GARMA estimation.

estimates albeit to a non-standard distribution. To our knowledge, no research has been conducted in the case where the residuals are distributed as GARMA processes.

3 Monte Carlo Results

A pervasive literature dealing with the potential for long memory among variables with equilibrium relationships has emerged in the past decade. The vast majority of this literature has dealt with ARFIMA processes. As alluded to in section 2, there are interesting economic processes that appear to be better defined by models that accommodate periodic or quasi-periodic movement. The results of section 2 and the existing literature also suggest that many variables under question are distributed as near I(1) processes. However, anecdotal evidence, such as the example related to nominal exchange rates above, suggests that treating these variables and or their equilibrium relationships as strictly I(1) or I(0) variables can lead to the incorrect conclusion regarding equilibrium relationships. Furthermore, as suggested by Anderrson and Gredenhoff (1999), misspecification of the underlying equilibrium relationship can affect inference regarding the parameters of an error correction model.

In this section, we more fully consider the implications of misspecification on cointegration tests using extensive Monte Carlo analysis. We are primarily concerned with three cases. First, as evidenced above, it is plausible that a vector of I(1) variables has an equilibrium relationship that is defined as either and ARFIMA or GARMA process. Second, the strong GARMA affects and the potential for non-stationarity in the euro-dollar nominal exchange rate bring into question the use of cointegration techniques. It should be noted that for small samples, Ramachandran and Beaumont (2001) document the low power of the Dickey-Fuller procedure for near unit root GARMA processes. Thus the second case involves detecting equilibrium relationships among various long memory processes, which can be described as near unit root processes. The final case involves the use of cointegration techniques among long memory variables when no equilibrium relationship exists. Our findings suggest that cointegration techniques can often lead to misleading results supporting the anecdotal evidence often cited in the long memory literature regarding cointegration. These results also apply to many GARMA processes, where the relaxation of the assumption $\eta = 1$, can have important consequences as documented above.

3.1 I(1) Variables with Long Memory Equilibrium Relationships

In the section, we review the problems that occur with cointegration testing when two I(1) variables share an equilibrium relationship that is itself a long memory process. We will place particular emphasis on the region in which cointegration tests fail. We are also interested in extending the analysis of Andersson and Gredenhoff (1999) to evaluate how inference regarding a particular cointegrating vector is affected when the equilibrium relationship is distributed as a long memory variable. Theoretically, the Johansen cointegration test attempts to find maximum canonical correlations between the residuals of certain VAR equations. Let $\mathbf{y_t}$ denote an n×1 vector of strictly I(1) variables, $\mathbf{u}_t$ the residuals from $\mathbf{\Delta y}_t$ regressed on $(p-1)$ lags of itself and potentially a constant/trend, and let $\mathbf{v}_t$ denote the residuals from $\mathbf{y}_{t-1}$ regressed on $(p-1)$ lags of $\mathbf{\Delta y}_t$ and perhaps a constant/trend. The canonical correlations are formed from the eigenvalues of the following matrix:

$$\hat{\mathbf{\Sigma}}_{vv}^{-1}\hat{\mathbf{\Sigma}}_{vu}\hat{\mathbf{\Sigma}}_{uu}^{-1}\hat{\mathbf{\Sigma}}_{uv}, \tag{5}$$

where $\hat{\mathbf{\Sigma}}_{vv}$ denotes the variance covariance matrix of $\mathbf{v}_t$, with similar notation for the remaining variance covariance matrices. If there exists r<n eigenvalues from the above matrix that are "large" in magnitude, then this signifies the existence of a reduced rank regression, which can represented in vector form as follows:

$$\mathbf{\Delta y}_t = \boldsymbol{\delta} + \sum_{j=1}^{p-1} \mathbf{\Phi}_j \mathbf{\Delta y}_{t-j} + \boldsymbol{\beta}\boldsymbol{\alpha}' \mathbf{y}_{t-1} + \mathbf{e}_t, \tag{6}$$

where $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ are $n \times r$ matrices with rank strictly equal to r. In this case, $\boldsymbol{\alpha}$ is said to the matrix of cointegrating vectors, and $\boldsymbol{\beta}$ denotes a matrix of coefficients describing the speed of reversion to equilibrium. The above equation is known as the error correction representation of a cointegrated system.

Often, researchers are interested in a particular form for $\boldsymbol{\alpha}$. For example, in the purchasing power parity literature, researchers have sought a cointegrating vector of $[1\ \text{-}1\ 1]'$ between the nominal domestic price of a foreign currency, the domestic price level, and the foreign price level. The form of the cointegrating vector has also been important in studying foreign market efficiency (c.f. Norrbin and Reffett (1996) and Zivot (2001)), the Fisher hypothesis (c.f. Lewis and Evans (1995)), and real interest rate parity (c.f. Goodwin and Grennes (1994)) along with numerous other applications. One perceived advantage of the Johansen procedure is the ability to implement tests regarding the cointegrating vector. In particular, for the case n=2 and r=1, suppose that one wishes to test the hypothesis that the cointegrating vector $\boldsymbol{\alpha} = \mathbf{D}$. Then from above, the matrices $\hat{\mathbf{\Sigma}}_{vv}$, $\hat{\mathbf{\Sigma}}_{uv}$, and $\hat{\mathbf{\Sigma}}_{vu}$ are replaced with the following elements:

$$\begin{aligned} \tilde{\mathbf{\Sigma}}_{vv} &= \mathbf{D}'\hat{\mathbf{\Sigma}}_{vv}\mathbf{D} \\ \tilde{\mathbf{\Sigma}}_{uv} &= \hat{\mathbf{\Sigma}}_{uv}\mathbf{D},\ \tilde{\mathbf{\Sigma}}_{vu} = \tilde{\mathbf{\Sigma}}_{uv}'. \end{aligned} \tag{7}$$

Canonical correlations are constructed with the new elements in precisely the same way described above. From these values, one can construct a likelihood ratio test that the true cointegrating vector is given by $\mathbf{D}$. Given that the test involves only stationary values, the distribution of the test statistic is $\chi^2(1)$ (see Hamilton, 1994 for details).

The Engle-Granger (1987) methodology, on the other hand, is based on the fact that linear combinations of n $I(1)$ variables are $I(0)$ if and only if cointegration exists. Engle and Granger show that if indeed such a linear combination exists, the cointegrating relationship can be estimated consistently using ordinary least squares (OLS). A unit root test can then be applied to the residuals to determine if cointegration indeed exists. Further, Hamilton (p 603,

1994) suggests using either a standard t or F-statistic to test hypothesis about the true value of a cointegrating vector if the residuals from the unit process in a triangular representation are unrelated to the residuals from the cointegrating relationship.

For this experiment, we generated sample sizes of 100, 200, and 500 observations. In each case two unit root processes, x_t and y_t were generated such that x_t-y_t produced a series that was distributed as a long memory process.[7] For each simulated series, we perform 5000 replications. Our goal is to demonstrate that even when an equilibrium relationship is correctly found among related variables, both the Engle-Granger and Johansen procedure fail to isolate the correct equilibrium vector. As documented above, the vector of interest is $[1 - 1]'$. For the Johansen procedure,[8] we consider up to 4 lags in the VAR equation in levels, while for the Engle-Granger procedure, we use the augmented Dickey Fuller statistic and consider up to 4 lags for the ADF equation in difference form.[9] To be conservative, we report the findings associated with the number of lags that yield the most favorable results for the individual tests. In this section, for example, we select the number of lags that minimizes a general failure. A general failure occurs, in this section, when we either fail to reject the null of no cointegration or we reject the null that the cointegrating vector is $[1 - 1]'$.

Table 3 presents the results for an equilibrium process defined as an ARFIMA process, while table 4 presents the results associated with GARMA residuals. For the sake of brevity, we report only the results for 200 observations. In general, the rejection rates in table 3 for the Johansen procedure are quite comparable to those reported in Andersson and Gredenhoff. In particular, when

[7]Following Cheung and Lai (1993), Andersson and Gredenhoff (1999), and Marinucci and Robinson(2001), we generated an equilibrium relationship from the following bivariate system:

$x_t + 2y_t = e_{1t}$

$x_t - y_t = e_{2t}$,

where e_{1t} is defined as a unit root process, and e_{2t} is distributed as the long memory process in question. To generate e_{2t} with mean zero, we generate the autocovariances for either an ARFIMA process (see Sowell (1992)) or the autocovariances for a GARMA process (see Chung (1996 a). We then generate a $T \times 1$ (T=100, 200, or 500) vector, ε_t of normal variates. Let C denote the Cholesky factorization of the autocoviances. Then,

$e_{2t} = C\ \varepsilon_t$.

Previous researchers have generated long memory data by expanding the infinite order polynomials in L described in equations 1 and 3, and then truncating the polynomial for the desired sample size. We considered both approaches, and determined that our approach is superior in that it produces long memory data that more closely reflects the statistical properties of the desired series. In addition, Woodward et. al. (1998) criticize the use of truncation methods for GARMA models, suggesting that the Gegenbauer polynomials are poorly approximated in finite samples. To generate non-stationary data, we generate a series that is borderline non-stationary using the procedure described above, and then apply a linear filter based on the appropriate Maclaurin series. We find this procedure provides better results than the simple truncation method.

[8]There was very little disparity between the results using the trace statistic and the maximum eigenvalue statistic. In this section, we report the results for the maximum eigenvalue statistic. Results for the trace statistic are available upon request.

[9]The critical values for the Engle Granger test were calculated using simulations of length 50,000.

$d < .6$, the rejection rates are manageable. The Engle-Granger test results in slightly higher rejection rates throughout. Further, it should be pointed out that the rejection rates are small only for equilibrium relationships that are themselves non-stationary. However, when one considers the use of a log likelihood ratio statistic or t-statistic for particular cointegrating relationships, the results are no longer acceptable. In particular, for $d = .4$, one is able to reject the correct cointegrating relationship 53.68% and 78.76% of the time using the LR test statistic and t-statistic respectively. This results in a general failure of 55.06% and 78.76% respectively. As expected, the results are worse as $d \to 1$. In addition the biases associated with the cointegrating vector worsen as $d \to 1$, for both the Engle-Granger procedure and the Johansen procedure. In particular, when estimating the cointegrating vector, $\boldsymbol{\alpha} = [1, \alpha_2]\prime$, the mean bias from the Johansen procedure for α_2 is 2.8008 for d=.9, while the Engle-Granger procedure results in a mean bias of -0.7428.

In table 4, the results are slightly different than those of table 3. In particular, when the equilibrium relationship among two I(1) variables is distributed as a GARMA(0,0) process, cointegration is rejected too often even for stationary processes. In addition, there is no discernible pattern about the nature of the bias in the cointegrating vector using the Johansen procedure, although the median bias tends to positive. In addition, the mean and median bias are smaller than their counterparts in table 4. It is rather surprising to note that for $\eta < .996$, the power of the Johansen procedure generally increases as d approaches .5. For $\eta > .996$, the Johansen procedure performs poorly as d approaches .5. For all of the results, the Johansen procedure and Engle-Granger procedure perform poorly as η approaches 1. For example, if the equilibrium relationship is defined as a GARMA(0,0) model with $\eta = .9995$ and d=.45 (a stationary process), the Johansen procedure fails to find cointegration with the correct cointegrating vector with probability 0.9948, while the Engle-Granger procedure fails with probability 0.9952. Unlike the system's approach to cointegration, the Engle-Granger procedure results in a discernible mean bias which is negative and grows with both η and d. Finally, we should point out that the results do improve for a sample size of 500, although the rejection rates are considerably large when $\eta > .996$. In addition, inference regarding the cointegrating vector can actually worsen for the Engle-Granger procedure.

3.2 Long Memory Variables with Long Memory Relationships

In this section, we consider equilibrium relationships among non-stationary long memory variables. The equilibrium relationships are themselves long memory, and hence these results encompass the widely studied concept of fractional cointegration. We are mainly interested in studying cointegration tests for variables that are best defined as near unit root processes, processes that would typically escape the scrutiny of a unit root test. We again employ a cointegrating vector of $[1, -1]\prime$. We consider a system based on 100, 200, and 500 observations and again base our results on 5,000 simulations.[10] The results for ARFIMA models are reported in table 5. In the top row of each section of table 5, we report the value of d for the original series in levels. The second column records the value of d for the equilibrium residuals. We also consider equilibrium residuals that are distributed as an AR(1) processes. For this example, we choose a value of $\phi = 0.70$. The results suggest that both the Johansen trace statistic and the Engle-Granger ADF statistic have remarkable power in detecting an equilibrium relationship among these series. For many simulations, the test statistics reject the false null with probability 1. However, if interest lies in a particular cointegrating relationship, the results of table 5 indicate that cointegration tests can lead to quite dubious conclusions. For example, when $d = .5$ and the equilibrium relationship is defined as a stationary AR(1) process, the Engle-Granger procedure results in a rejection of the proper cointegrating vector 43.96% of the time. In fact, the use of either method results in a general failure that typically exceeds 50%. The exception to the rule occurs when the Johansen trace statistic and accompanying likelihood ratio statistic are applied to processes having an equilibrium relationship defined as an AR(1) process.

The associated results for the GARMA model are presented in tables 6 and 7. Table 6 reports the cointegration results for the Johansen trace statistic, while table 7 yields results for the Engle-Granger procedure. In both tables, the value of η in the parent series is allowed to range from .995 to .999, while d is equal to either .50 or .55. The equilibrium residual has the same value of η as the parent series, while d is allowed to range from 0 to 0.40. When $d = 0$, the result is again an AR(1) model with an autoregressive coefficient equal to 0.70. The results indicate that the power to detect an equilibrium relationship decreases as d and η increase. When the equilibrium relationship is indeed defined as an AR(1) process, the results are quite favorable and suggests that the Johansen trace statistic results in little size distortion. The results are poorest when the equilibrium residuals are GARMA variables with a value of d=0.40. Here the absolute value of the mean bias of the cointegrating relationship are generally highest, and both high rejection rates and a large general failure result. For example, when the parent series and residual series are distributed as GARMA processes with η/d pairs given by (0.999,0.50) and (0.999,0.40) respectively, the rejection rate of the null of no equilibrium relationship is 0.6370, while we reject the null that the cointegrating relationship is equal to $[1, -1]'$ with probability 0.2940. This results in a general failure of 0.6072. The results associated with the Engle-Granger test, which are reported in table 7, are significantly worse. Here, we generally observe a negative bias associated with the second element of the cointegrating vector, and we frequently reject the null of no cointegration. Again, the results are worse as d and η increase. For example, for the same case described above, the Engle-Granger procedure results in a general failure 95.30% of the time.

[10] We follow Marinucci and Robinson(2001) and first generate a non-stationary series x_t. We then generate a stationary series ε_t. The series y_t is then generated as, $y_t = x_t - \varepsilon_t$.

3.3 Spurious Cointegration

We have already documented the poor performance of cointegration tests when variables share an equilibrium relationship that is defined as a long memory variable. Gonzalo and Lee (1998, 2000) consider cointegration tests among certain unrelated fractional processes and show that the likelihood ratio test statistic associated with the Johansen trace and eigenvalue statistic diverges to $+\infty$ as the sample size grows. In particular, when two unrelated fractional processes are distributed as $I(d)$ process with $d > 1.5$, the authors demonstrate that the largest eigenvalue does not go to zero as the sample size grows. In addition, they show that for $1 < d < 1.5$, the true convergence rate associated with the largest eigenvalue is less than the sample size (T) resulting in a test statistic that is too large when multiplied by T. Further, the authors show that the nominal size of the Johansen tests is large for unrelated non-stationary, mean reverting ARFIMA processes. They demonstrate these results in small samples and show that the frequency of rejecting the true null of no cointegration among unrelated ARFIMA processes in too large. In this section, we extend the results of Gonzalo and Lee to include GARMA processes. In addition, we consider several sample sizes, although here, we report only the findings for $T = 200$.

We use the same method described above to generate unrelated long memory processes. Obviously, no equilibrium relationship exists, and hence we do not report statistics associated with inference about the true cointegrating vector. The results for unrelated ARFIMA processes are reported in table 8; the results for the GARMA processes are reported in table 9.[11] Our findings in table 8 are quite comparable to those reported in Gonzalo and Lee, although we document a slightly lower rejection rate for the Johansen test. In particular, we find that the rejection rates are consistently higher than the theoretical size of 0.05. For example, when two variables are distributed as unrelated fractional processes with $d = .5$, the rejection rates of the Johansen test and Engle Granger test are 0.5584 and 0.6702 respectively. It is not unsuspected that the rejection rates are higher for small d. In every case, as documented by Gonzalo and Lee, the rejection rates increase as the sample size increases. The common sense explanation that is demonstrated theoretically by Gonzalo and Lee is that the cointegration tests described here are based on the assumption that the underlying variables are distributed as I(1) variables. If the underlying variables are near unit root processes, the results indicate that a larger sample size allows these tests to distinguish cointegrating residuals from I(1) processes. It is also interesting to note that the best results occurred for a large number of lags. This is again consistent with the findings of Gonzalo and Lee (1998).

The results of table 9 are again considerably worse than the results of table 8. This is not surprising, since the processes represented in table 9 are generally

[11] The results using the Johansen procedure for table 8 are associated with the Johansen eigenvalue statistic, while the results of table 9 are based on the trace statistic. These selections are made to coincide with the most favorable results for the cointegration tests (in this case the test that yields the lowest rejection rates).

stationary. Similar to table 8, the results generally improve as d grows, although when $\eta > .996$, the Johansen procedure worsens as d grows, while the rejection rates fall as η increases for a fixed value of d. We document a similar finding for η. It is important to note, however, that the rejection rates are typically unacceptable. This is especially true for the Johansen procedure, where even when $\eta = .9995$ and $d = .50$, the resulting rejection rate is .4374. These two unrelated GARMA processes are arbitrarily close to unit root processes. We should point out that the Engle-Granger procedure outperforms the Johansen procedure in every case documented in table 9. These results are similar to those found in Gonzalo and Lee for unrelated fractional processes. Again, these results do not improve asymptotically, as the rejection rates increase throughout as we increase the sample size from 200 to 500.

4 Conclusion

In this paper, we aim to document the poor performance of existing cointegration tests when the variables and equilibrium errors are distributed as long memory processes. Although existing studies have considered the performance of cointegration tests in this environment, we are unaware of any study that documents more than one problem with these tests in a single study. In addition, we are unaware of any study that documents the performance of cointegration tests among a more general class of long memory models than the ARFIMA model. To this end, we use Monte Carlo analysis to document the performance of cointegration tests when the underlying variables and or equilibrium relationships are distributed as GARMA processes. The GARMA process may be an important extension to the ARFIMA model and seems to capture the statistical properties of several important macroeconomic variables better than the ARFIMA model. In particular, we are able to show that the GARMA model captures the properties of the euro-dollar nominal exchange rate quite well. In addition, the equilibrium relationship among several important nominal exchange rates appears to be well defined by a GARMA model.

The main conclusion from this paper, simply put, is that if the underlying variables or equilibrium errors are distributed as long memory processes, then existing cointegration tests fail far too often. This failure results in one of three ways as documented in section 3. First, the rejection rates of the null of no cointegration are too low when an equilibrium relationship exists, but the equilibrium errors are distributed as long memory processes. This is true whether the original variables are I(1) processes or are themselves long memory processes. Secondly, one too frequently rejects the correct cointegrating vector under the same scenarios using conventional techniques associated with the Johansen and Engle-Granger tests. If interest lies in detecting a particular cointegrating relationship, these two cases suggest that existing cointegration techniques may not be very useful. Finally, cointegration tests often detect cointegration among unrelated long memory variables. This is especially true for stationary GARMA processes that are near unit roots.

Given the poor performance of the widely applied Johansen and Engle-Granger procedures, research needs to be applied to new techniques. A promising research area has emerged in the area of fractional cointegration (c.f. Dueker and Startz (1998), Marinucci and Robinson(2001), and Robinson and Yajima (2002)). These techniques could likely be extended to GARMA processes. If interest lies in detecting a particular cointegrating vector, then a natural method is too individually estimate the parameters of the original processes. These estimates can then be compared to the estimates from the relationship one has in mind. This suggests at least two avenues for future research. A semi-parametric frequency based cointegration test could be developed for GARMA processes along the lines of the estimator proposed for fractional processes by Marinucci and Robinson. Finally, one could compare the results of direct estimation of long memory processes and proposed equilibrium relationships with the techniques described in this paper.

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Table 1: Estimation of Nominal Euro Rate

	η	d	$\hat{\eta}$ Lower CI	$\hat{\eta}$ Upper CI	μ
Estimates	0.99947	0.4759	0.99918	0.99952	-.1040
Std Errors		[0.0194]			[0.0084]
OBS	419				
Max CSS Value	1651.36				
SSE	0.01851				

Table 2: Estimation of Equilibrium Relationship

	η	d	$\hat{\eta}$ Lower CI	$\hat{\eta}$ Upper CI	μ
Estimates	0.99957	0.4554	0.999560	0.999571	1.02e^{-12}
Std. Errors		[0.0084]			[0.0008]
OBS	2264				
Max. CSS Value	12010.383				
SSE	0.0065				

Table 3: Power of Johansen Cointegration Tests. I(1) Processes with Long Memory Fractional Equilibrium Errors. Sample Size Equal to 200

d	**.40**	**.50**	**.60**	**.70**	**.80**	**.90**	**1.00**
Johansen Test							
Rej. Rate of Null	0.9850	0.8766	0.6000	0.2872	0.1128	0.0572	0.0536
Rej. Rate $\alpha' = [1, -1]$	0.5368	0.5738	0.5596	0.5012	0.4254	0.3782	0.3560
General Failure	0.5506	0.6620	0.8136	0.9262	0.9792	0.9876	0.0536
Mean Bias α'	-0.0052	-0.0173	-0.1391	0.4216	0.8746	2.8008	N/A
Median Bias α'	-0.0019	-0.0019	-0.0007	0.0261	0.1499	0.6422	N/A
# of Lags in VAR	1	1	1	1	1	1	1
Engle-Granger							
	.40	**.50**	**.60**	**.70**	**.80**	**.90**	**1.00**
Rej. Rate of Null	0.9996	0.9790	0.8628	0.6052	0.2794	0.1146	0.0494
Rej. Rate $\alpha' = [1, -1]$	0.7876	0.8632	0.9094	0.9322	0.9406	0.9588	0.9988
General Failure	0.7876	0.8674	0.9264	0.9642	0.9854	0.9974	0.0494
Mean Bias α'	-0.0113	-0.0162	-0.0318	-0.0846	-0.2375	-0.7428	N/A
Median Bias α'	-0.0065	-0.0107	-0.0249	-0.0688	-0.2388	-0.7108	N/A
# of Lags	0	0	0	0	0	0	0

Table 4: Power of Cointegration Tests. I(1) Processes with Long Memory GARMA Equilibrium Errors. Sample Size Equal to 200

Johansen Test	d/η	**.990**	**.992**	**.994**	**.996**	**.998**	**.9995**
Rej. Rate of Null	**.40**	0.5854	0.3548	0.2254	0.1680	0.1172	0.0904
Rej. Rate $\alpha' = [1, -1]$	**.40**	0.0086	0.0120	0.0142	0.0242	0.0482	0.1198
General Failure	**.40**	0.4228	0.6562	0.7852	0.8448	0.9066	0.9506
Mean Bias α'	**.40**	-0.0095	0.0144	0.9412	-0.0180	0.0237	0.0808
Median Bias α'	**.40**	-0.0012	0.0003	0.0020	0.0010	0.0001	0.0112
# of Lags in VAR	**.40**	4	4	1	1	1	1
		.990	**.992**	**.994**	**.996**	**.998**	**.9995**
Rej. Rate of Null	**.45**	0.7798	0.5074	0.2336	0.0740	0.0400	0.0514
Rej. Rate $\alpha' = [1, -1]$	**.45**	0.0056	0.0072	0.0110	0.0208	0.0648	0.1608
General Failure	**.45**	0.2254	0.4984	0.7746	0.9372	0.9780	0.9896
Mean Bias α'	**.45**	-0.0028	-0.0115	-0.0251	0.4960	-0.2025	-1.6575
Median Bias α'	**.45**	0.0007	0.0000	0.0004	0.0033	0.0193	0.0874
# of Lags	**.45**	4	4	4	4	1	1
		.990	**.992**	**.994**	**.996**	**.998**	**.9995**
Rej. Rate of Null	**.50**	0.8962	0.7426	0.4606	0.1618	0.0364	0.0684
Rej. Rate $\alpha' = [1, -1]$	**.50**	0.0040	0.0050	0.0070	0.0160	0.0440	0.2076
General Failure	**.50**	0.1076	0.2614	0.5450	0.8458	0.9802	0.9948
Mean Bias α'	**.50**	-0.0088	-0.0157	-0.0221	0.0287	0.1814	0.9793
Median Bias α'	**.50**	0.0002	0.0000	0.0005	0.0038	0.0171	0.3358
# of Lags		4	4	4	4	4	1
Engle-Granger							
	d/η	**.990**	**.992**	**.994**	**.996**	**.998**	**.9995**
Rej. Rate of Null	**.40**	0.9286	0.7682	0.5502	0.4624	0.3444	0.2460
Rej. Rate $\alpha' = [1, -1]$	**.40**	0.5028	0.5126	0.5430	0.5738	0.6326	0.7400
General Failure	**.40**	0.5376	0.6178	0.7498	0.8080	0.8842	0.9528
Mean Bias α'	**.40**	-0.0563	-0.0581	-0.0637	-0.0707	-0.0816	-0.1078
Median Bias α'	**.40**	-0.0243	-0.0241	-0.0270	-0.0304	-0.0368	-0.0528
# of Lags in ADF eq.	**.40**	4	4	0	0	0	0
		.990	**.992**	**.994**	**.996**	**.998**	**.9995**
Rej. Rate of Null	**.45**	0.9580	0.8352	0.4902	0.1284	0.0586	0.0408
Rej. Rate $\alpha' = [1, -1]$	**.45**	0.5918	0.5994	0.6296	0.6490	0.6972	0.7852
General Failure	**.45**	0.6056	0.6536	0.8062	0.9538	0.9856	0.9952
Mean Bias α'	**.45**	-0.1299	-0.1364	-0.1423	-0.1678	-0.2067	-0.2955
Median Bias α'	**.45**	-0.0530	-0.0572	-0.0646	-0.0723	-0.0926	-0.1495
# of Lags	**.45**	4	4	4	4	0	0
		.990	**.992**	**.994**	**.996**	**.998**	**.9995**
Rej. Rate of Null	**.50**	0.9622	0.8830	0.6466	0.2136	0.0204	0.0138
Rej. Rate $\alpha' = [1, -1]$	**.50**	0.6870	0.6966	0.7098	0.7442	0.7780	0.8394
General Failure	**.50**	0.6946	0.7220	0.7930	0.9440	0.9954	0.9988
Mean Bias α'	**.50**	-0.2329	-0.2461	-0.2605	-0.3075	-0.3826	-0.5687
Median Bias α'	**.50**	0.1025	-0.1099	-0.1229	-0.1478	-0.1976	-0.3434
# of Lags		4	4	4	4	4	0

Table 5: Rejection Rates of False Null of No Equilibrium Relationship among ARFIMA processes. 200 OBS

Johansen Test		d orig	**.90**	**.80**	**.70**	**.60**	**.50**
	d resid						
Rej. Rate of Null	**.40**		0.9986	0.9986	0.9972	0.9952	0.9988
Rej. Rate $\alpha' = [1, -1]$	**.40**		0.8402	0.8162	0.8136	0.8144	0.8318
General Failure	**.40**		0.8410	0.8172	0.8148	0.8174	0.8330
Mean Bias α'	**.40**		0.0032	0.0170	-0.0513	-0.0296	-0.1286
Median Bias α'	**.40**		0.0013	-0.0017	-0.0099	-0.0284	-0.0734
# of Lags in VAR	**.40**		4	4	4	4	4
			.90	**.80**	**.70**	**.60**	**.50**
Rej. Rate of Null	**.20**		0.9992	0.9998	0.9994	0.9996	0.9996
Rej. Rate $\alpha' = [1, -1]$	**.20**		0.8748	0.7910	0.7324	0.6936	0.6818
General Failure	**.20**		0.8754	0.7912	0.7330	0.6940	0.6822
Mean Bias α'	**.20**		0.0017	0.0010	-0.0008	-0.0062	-0.0048
Median Bias α'	**.20**		0.0015	0.0015	0.0008	-0.0017	-0.0084
# of Lags	**.20**		4	4	4	4	4
			.90	**.80**	**.70**	**.60**	**.50**
Rej. Rate of Null	**AR 1**		0.9998	0.9996	0.9996	1.0000	1.0000
Rej. Rate $\alpha' = [1, -1]$	**AR 1**		0.0590	0.0566	0.0590	0.0674	0.0996
General Failure	**AR 1**		0.0592	0.0570	0.0594	0.0674	0.0996
Mean Bias α'	**AR 1**		0.0002	-0.0002	-0.0034	-0.0401	0.0090
Median Bias α'	**AR 1**		0.0000	0.0000	-0.0001	-0.0002	0.0015
# of Lags	**AR 1**		1	1	1	1	1
Engle Granger		d orig	**.90**	**.80**	**.70**	**.60**	**.50**
	d resid						
Rej. Rate of Null	**.40**		1.0000	1.0000	1.0000	1.0000	1.0000
Rej. Rate $\alpha' = [1, -1]$	**.40**		0.8416	0.8446	0.8488	0.8536	0.5838
General Failure	**.40**		0.8416	0.8466	0.8488	0.8536	0.8538
Mean Bias α'	**.40**		0.0400	0.0176	0.0383	0.0852	0.1970
Median Bias α'	**.40**		0.0045	0.0103	0.0222	0.0482	0.1007
# of Lags in ADF Eq	**.40**		0	0	0	0	0
			.90	**.80**	**.70**	**.60**	**.50**
Rej. Rate of Null	**.20**		1.0000	1.0000	1.0000	1.0000	1.0000
Rej. Rate $\alpha' = [1, -1]$	**.20**		0.4566	0.4606	0.4664	0.4770	0.4896
General Failure	**.20**		0.4566	0.4606	0.4664	0.4770	0.0390
Mean Bias α'	**.20**		0.0008	0.0023	0.0063	0.0162	0.1011
Median Bias α'	**.20**		0.0004	0.0009	0.0026	0.0070	0.0184
# of Lags	**.20**		0	0	0	0	0
			.90	**.80**	**.70**	**.60**	**.50**
Rej. Rate of Null	**AR 1**		1.0000	1.0000	1.0000	1.0000	1.0000
Rej. Rate $\alpha' = [1, -1]$	**AR 1**		0.4396	0.4804	0.5560	0.6800	0.8278
General Failure	**AR 1**		0.4396	0.4804	0.5560	0.6800	0.8278
Mean Bias α'	**AR 1**		-0.0063	-0.0159	–0.0380	-0.0846	-0.1717
Median Bias α'	**AR 1**		-0.0008	-0.0025	0.0089	-0.0311	-0.1085
# of Lags	**AR 1**		0	0	0	0	0

Table 6: Power of Johansen Cointegration Tests. GARMA Processes with Long Memory GARMA Equilibrium Errors. 200 OBS

Johansen Test		d resid	**.40**	**.30**	**.20**	**.10**	**AR** 1
	η/d orig.						
Rej. Rate of Null	**.995/.55**		0.9756	0.9996	1.0000	1.0000	1.0000
Rej. Rate $\alpha' = [1, -1]$	**.995/.55**		0.3800	0.2804	0.2088	0.1346	0.0692
General Failure	**.995/.55**		0.4038	0.2808	0.2088	0.1346	0.0692
Mean Bias α'	**.995/.55**		-0.0181	0.0123	-0.0028	-0.0005	-0.0020
Median Bias α'	**.995/.55**		-0.0258	-0.0059	-0.0013	-0.0003	-0.0003
# of Lags in VAR	**.995/.55**		1	1	1	1	1
			.40	**.30**	**.20**	**.10**	**AR** 1
Rej. Rate of Null	**.997/.55**		0.9070	0.9956	0.9998	1.0000	1.0000
Rej. Rate $\alpha' = [1, -1]$	**.997/.55**		0.3418	0.2988	0.2334	0.1302	0.0492
General Failure	**.997/.55**		0.4282	0.3032	0.2336	0.1302	0.0492
Mean Bias α'	**.997/.55**		0.0548	-0.0100	0.0021	-0.0008	0.0004
Median Bias α'	**.997/.55**		-0.0260	-0.0072	-0.0021	-0.0007	-0.0001
# of Lags	**.997/.55**		1	1	2	2	1
			.40	**.30**	**.20**	**.10**	**AR** 1
Rej. Rate of Null	**.999/.55**		0.5358	0.8914	0.9916	1.0000	0.9998
Rej. Rate $\alpha' = [1, -1]$	**.999/.55**		0.1358	0.0460	0.0484	0.0548	0.0590
General Failure	**.999/.55**		0.5952	0.1546	0.0568	0.0548	0.0592
Mean Bias α'	**.999/.55**		0.0311	0.0014	0.0008	0.0005	-0.0002
Median Bias α'	**.999/.55**		0.0042	0.0015	0.0008	0.0004	0.0000
# of Lags			4	1	3	4	2
Johansen Test		d resid	**.40**	**.30**	**.20**	**.10**	**AR** 1
	η/d orig.						
Rej. Rate of Null	**.995/.50**		0.9512	0.9988	1.0000	1.0000	1.0000
Rej. Rate $\alpha' = [1, -1]$	**.995/.50**		0.3570	0.2828	0.1902	0.1200	0.0456
General Failure	**.995/.50**		0.4036	0.2840	0.1902	0.1200	0.0456
Mean Bias α'	**.995/.50**		-0.4339	-0.0247	-0.0046	-0.0003	-0.0012
Median Bias α'	**.995/.50**		-0.0596	-0.0113	-0.0022	-0.0005	-0.0006
# of Lags in VAR	**.995/.50**		1	1	2	2	1
			.40	**.30**	**.20**	**.10**	**AR** 1
Rej. Rate of Null	**.997/.50**		0.8982	0.9840	0.9998	1.0000	1.0000
Rej. Rate $\alpha' = [1, -1]$	**.997/.50**		0.3134	0.2394	0.1994	0.1194	0.0486
General Failure	**.997/.50**		0.4084	0.2550	0.1996	0.1194	0.0486
Mean Bias α'	**.997/.50**		0.1444	-0.0088	-0.0032	0.0001	-0.0005
Median Bias α'	**.997/.50**		-0.0304	-0.0106	-0.0033	-0.0011	-0.0002
# of Lags	**.997/.50**		2	2	2	3	2
			.40	**.30**	**.20**	**.10**	**AR** 1
Rej. Rate of Null	**.999/.50**		0.6370	0.7922	0.9688	1.0000	1.0000
Rej. Rate $\alpha' = [1, -1]$	**.999/.50**		0.2940	0.2394	0.1924	0.1316	0.0622
General Failure	**.999/.50**		0.6072	0.4356	0.2252	0.1316	0.0622
Mean Bias α'	**.999/.50**		-0.1753	0.0185	0.0029	0.0011	-0.0004
Median Bias α'	**.999/.50**		0.0621	0.0127	0.0036	0.0011	0.0000
# of Lags			4	4	4	4	1

Table 7: Power of Engle Granger Cointegration Tests. GARMA Processes with Long Memory GARMA Equilibrium Errors. 200 OBS

Engle-Granger		d resid	**.40**	**.30**	**.20**	**.10**	**AR** 1
	η/d orig.						
Rej. Rate of Null	**.995/.55**		0.7394	0.9996	1.0000	1.0000	1.0000
Rej. Rate $\alpha' = [1, -1]$	**.995/.55**		0.8358	0.7218	0.5722	0.3410	0.4462
General Failure	**.995/.55**		0.8934	0.7220	0.5722	0.3410	0.4462
Mean Bias α'	**.995/.55**		0.0123	-0.0034	-0.0046	-0.0044	-0.0081
Median Bias α'	**.995/.55**		0.0227	0.0017	-0.0014	-0.0018	-0.0036
# of Lags in VAR	**.995/.55**		0	0	0	0	0
			.40	**.30**	**.20**	**.10**	**AR** 1
Rej. Rate of Null	**.997/.55**		0.5986	0.9992	1.0000	1.0000	1.0000
Rej. Rate $\alpha' = [1, -1]$	**.997/.55**		0.8288	0.7370	0.5846	0.3256	0.4466
General Failure	**.997/.55**		0.9118	0.7372	0.5846	0.3256	0.4466
Mean Bias α'	**.997/.55**		0.0056	-0.0017	-0.0029	-0.0031	-0.0066
Median Bias α'	**.997/.55**		0.0150	0.0030	-0.0001	-0.0010	-0.0031
# of Lags	**.997/.55**		0	0	0	0	0
			.40	**.30**	**.20**	**.10**	**AR** 1
Rej. Rate of Null	**.999/.55**		0.5168	0.9988	1.0000	1.0000	1.0000
Rej. Rate $\alpha' = [1, -1]$	**.999/.55**		0.6350	0.5314	0.4344	0.2834	0.4488
General Failure	**.999/.55**		0.8064	0.5322	0.4344	0.2834	0.4488
Mean Bias α'	**.999/.55**		-0.0256	-0.0093	-0.0050	-0.0037	-0.0057
Median Bias α'	**.999/.55**		-0.0143	-0.0050	-0.0026	-0.0018	-0.0027
# of Lags			0	0	0	0	0
Engle Granger		d resid	**.40**	**.30**	**.20**	**.10**	**AR** 1
	η/d orig.						
Rej. Rate of Null	**.995/.50**		0.7234	0.9992	1.0000	1.0000	1.0000
Rej. Rate $\alpha' = [1, -1]$	**.995/.50**		0.8298	0.7224	0.5884	0.3856	0.4976
General Failure	**.995/.50**		0.8908	0.7230	0.5884	0.3856	0.4976
Mean Bias α'	**.995/.50**		0.0068	-0.0104	-0.0104	-0.0095	-0.0171
Median Bias α'	**.995/.50**		0.0251	-0.0009	-0.0041	-0.0043	-0.0086
# of Lags in VAR	**.995/.50**		0	0	0	0	0
			.40	**.30**	**.20**	**.10**	**AR** 1
Rej. Rate of Null	**.997/.50**		0.5748	0.9988	1.0000	1.0000	1.0000
Rej. Rate $\alpha' = [1, -1]$	**.997/.50**		0.8074	0.7368	0.5898	0.3658	0.4888
General Failure	**.997/.50**		0.9092	0.7368	0.5898	0.3658	0.4888
Mean Bias α'	**.997/.50**		-0.0077	-0.0085	-0.0076	-0.0072	-0.0142
Median Bias α'	**.997/.50**		0.0078	0.0000	-0.0023	-0.0030	-0.0071
# of Lags	**.997/.50**		0	0	0	0	0
			.40	**.30**	**.20**	**.10**	**AR** 1
Rej. Rate of Null	**.999/.50**		0.4930	0.9950	1.0000	1.0000	1.0000
Rej. Rate $\alpha' = [1, -1]$	**.999/.50**		0.8976	0.7962	0.6526	0.4164	0.4734
General Failure	**.999/.50**		0.9530	0.7972	0.6526	0.4164	0.4734
Mean Bias α'	**.999/.50**		-0.0804	-0.0244	-0.0104	-0.0065	-0.0096
Median Bias α'	**.999/.50**		-0.0721	-0.0190	-0.0066	-0.0035	-0.0045
# of Lags			0	0	0	0	0

Table 8: Rejection Rates of True Null Hypothesis of No Cointegration among unrelated Fractional Processes

d	**.40**	**.50**	**.60**	**.70**	**.80**	**.90**	**1.00**
Johansen Test							
Rej. Rate of Null	0.9780	0.5584	0.2806	0.1392	0.0742	0.0746	0.0516
# of Lags in VAR	4	4	4	4	4	2	1

Engle-Granger							
	.40	**.50**	**.60**	**.70**	**.80**	**.90**	**1.00**
Rej. Rate of Null	0.8058	0.6702	0.5208	0.3334	0.1892	0.1050	0.0494
# of Lags in ADF	4	4	4	4	4	4	4

Table 9: Rejection Rate of True Null of No Cointegration Among Unrelated GARMA Processes. Sample Size Equal to 200

Johansen Test	d/η	**.990**	**.992**	**.994**	**.996**	**.998**	**.9995**
Rej. Rate of True Null	**.40**	0.8816	0.8134	0.7216	0.5404	0.3726	0.2046
# of Lags in VAR	**.40**	4	3	2	4	4	4
		.990	**.992**	**.994**	**.996**	**.998**	**.9995**
Rej. Rate of True Null	**.45**	0.8362	0.7908	0.7126	0.6204	0.4820	0.3660
# of Lags	**.45**	1	2	2	3	4	4
		.990	**.992**	**.994**	**.996**	**.998**	**.9995**
Rej. Rate of True Null	**.50**	.7994	0.7668	0.7178	0.6292	.5082	0.4374
# of Lags		1	1	1	2	4	4
Engle-Granger							
	d/η	**.990**	**.992**	**.994**	**.996**	**.998**	**.9995**
Rej. Rate of False Null	**.40**	0.7492	0.6318	0.5056	0.3338	0.1942	0.1184
# of Lags in ADF eq.	**.40**	1	2	2	4	4	4
		.990	**.992**	**.994**	**.996**	**.998**	**.9995**
Rej. Rate of False Null	**.45**	0.4106	0.3304	0.2646	0.1714	0.0880	0.0462
# of Lags	**.45**	1	1	1	2	4	4
		.990	**.992**	**.994**	**.996**	**.998**	**.9995**
Rej. Rate of False Null	**.50**	0.1482	0.1068	0.0916	0.0528	0.0372	0.0262
# of Lags	**.50**	1	0	0	0	0	2