# A Stochastic Seasonal Model for Commodity Option Pricing

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January 15, 2003

**ABSTRACT**

In this paper we develop a two-factor model to value options on commodities in the presence of a stochastic seasonal factor which affects the growth and production of the underlying commodity. Our analysis is based on a stochastic mean reversion model for the natural factor and an extension of a Geometric Brownian Motion (GBM) for the commodity spot price. Using this model we provide some numerical simulations illustrating the effect of the seasonal factor on the term structure of the futures and forwards as well as on their volatility.
**Key Words:** Option pricing, Stochastic ordinary differential equations

## 1 Introduction

Commodities are usually raw products such as precious metals, oil, forest and agricultural goods. Some of them are traded on specialized exchanges (e. g. The Chicago Board of Trade, or the Chicago and New York Mercantile Exchanges). Commodity deals are usually done on the futures market and they are closed out some time before the delivery is due. These contracts face uncertainty generated by market forces, growth and production. The production uncertainty is generated by several factors, such as weather, storage costs and capacities, production technology and capacities. Another important issue in futures contracts is the availability of data. Since these contracts imply future delivery, when pricing them one usually does not have access to spot prices. In these cases futures prices are used as proxies for spot prices, so in order to price these contracts it is crucial for us to find the relationship between spot and futures prices. The main products traded on commodity exchanges are

1. futures - deals to buy or sell commodities delivered at some time in the future;

2. call/put options - agreements giving the holder the right, but not the obligation, to buy/sell the commodity at or before some moment in the future.

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One can observe that growth of agricultural commodities is directly influenced by the environmental conditions. The environmental factors we consider in this paper have annual or multi-annual periodicity. Among these, temperature and precipitation are annual factors, while El Niño and La Niña are multi-annual factors. For an illustration of these variations, Figure 1 contains a plot of the futures prices of soybeans with expiration date January 2002 source: the Chicago Board of Trade.


Figure 1: Futures prices for soybeans (source: The Chicago Board of Trade - agricultural futures)

The commodity pricing models developed in the literature until now have Black's model [4] as a starting point, which is an application of the methodology introduced by Black and Scholes [5] extended to futures and forward contracts. In a partial equilibrium framework, Black [4] developed the cost-of-carry formula when the only source of uncertainty is the spot price process, which is exogeneously determined. Further research on commodity pricing led to the concept of convenience yield. As defined by Brennan and Schwartz [9], the convenience yield of a commodity is a stream of benefits which accrue (in the same way as dividends) to the owner of the physical commodity, and are not obtained by the the holder of the futures contract. Therefore commodity pricing models were extended to two state variables, the spot price and the convenience yield. As Gibson, Schwartz [13] and Brennan [9] prove, the empirical estimations of the benefit from holding the physical commodity displays a seasonal behaviour. At the same time they also find evidence that the convenience yield fluctuates around a long term value, which justifies the choice of modelling it as a mean reverting stochastic process. However, the mean reversion towards a constant long-term equilibrium value does not capture properly the seasonal movements in commodity prices.

Extensions of the two-factor models have been developed in [3], [10] and [2] by introducing a third stochastic factor for the instantaneous riskless interest rate, whose process is exogenously given. The generalisation to the case of non-constant interest rates is particularly important because it allows us to make the distinction between forward and futures prices (which are otherwise equal, as proved in [1]). Further generalisations of the commodity pricing models as obtained by Miltersen and Schwartz [10], and Hilliard and Reis [2] introduce a jump diffusion process in the spot price of the commodities. All the models reviewed above are based on a stochastic convenience yield, for which in empirical applications a proxy must be determined. Moreover, since the convenience yield is not a traded asset, it always has a market price of risk associated with it, which must also be approximated empirically.

In this paper we develop a more general two-factor commodity pricing model by introducing, instead of the convenience yield, a stochastic factor based on a weather indicator which influences the commodity supply and demand. This model is particularly suitable for agricultural commodities such as wheat, rice, corn, soybeans, beef and dairy products (whose supply fluctuates seasonally), as well as for energy (whose demand fluctuates seasonally). In order to capture the seasonality of the commodity price we choose a general linear stochastic process for the weather factor. In our setting there will be two sources of uncertainty, the spot price and the weather.

The paper is organized as follows: Part 2 contains the description of a stochastic seasonal model for the weather factor on which we construct our pricing model, the estimation of parameters of the equation and some simulations of the estimated model. Part 3 contains the two factor model, the derivation of the forward price of the commodity underlying the model, and the results for European option pricing based on this model. Part 4 contains conclusions and implications for future work.

## 2 A climate model

Practically it is very difficult to find a type of business completely unaffected by weather conditions. During 1997-98 El Niño generated floods in southern US and brought drought to Eastern Australia, while in 1998-99 La Niña brought very warm winters. These particular events influenced to a great extent the revenues in the agriculture and energy industries in a number of countries. For a long time, futures contracts on agricultural commodities were used to hedge weather related risks. More recently, specialized instruments, such as weather derivatives and insurance emerged in order to hedge against weather uncertainty.

Until now the commodity option pricing theory was based on single or multi-factor models, where the possible state variables were the spot price of commodity, the convenience yield (the benefit from holding the commodity) and interest rates. However none of these models take into account explicitly the weather factors influencing the commodity. We are going to approach commodity pricing from a climate perspective and use a stochastic differential equation to capture the seasonal weather variations.

Using one climate indicator, for example temperature, which is measured and recorded with high frequency, we can observe that its daily average over time (years) follows approximately the shape of a sine wave function. In Figure 2 we plotted $\frac{maximum+minimum}{2}$ temperature (an indicator often used in weather contracts) for the Brisbane (Queensland, Australia) area from January 1996 till June 2000 (source: the Australian Bureau of Meteorology).

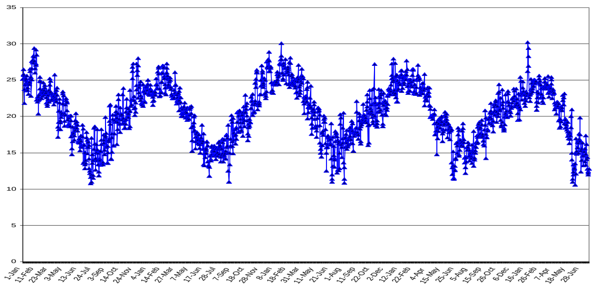

Figure 2: Observed temperatures for the Brisbane area, Jan 1996 - June 2000 (source: the Australian Bureau of Meteorology)

The plot appears to take the form of a sine function over which has been applied a stochastic noise. Therefore we choose to model the temperature as a SDE based on mean reversion with aditive noise. Denoting the temperature state variable by $X_t$, the form of the SDE is:

$$dX_t = \big(c_1 + c_2 \sin(2\pi t + c_3) - aX_t\big)dt + bdW_t \tag{2.1}$$

where $W_t$ is a Brownian motion. This means that $E[W_t] = 0$, $Var[W_t] = t$ and for any $j$

$$\frac{W_{t_{j+1}} - W_{t_j}}{\sqrt{t_{j+1} - t_j}}$$

are independent and identically distributed, following a normal distribution $N(0,1)$.

As Kloeden and Platen [11] show, the solution of a linear Itô stochastic differential equation

$$dX_t = \big(a(t)X_t + c(t)\big)dt + \big(b(t)X_t + d(t)\big)dW_t,$$

$$X(0) = X_0$$

is

$$X_t = \Phi_{t,0}\left(X_0 + \int_0^t \Phi_{s,0}^{-1}\big(c(s) - b(s)d(s)\big)ds + \int_0^t \Phi_{s,0}^{-1}d(s)dW_s\right), \tag{2.2}$$

where

$$\Phi_{t,0} = \exp\left(\int_0^t \left(a(s) - \frac{1}{2}b(s)^2\right)ds + \int_0^t b(s)dW_s\right).$$

Identifying and expanding the terms in (2.2), the solution of (2.1) becomes

$$X_t = e^{-at}\Bigg(X_0 + \frac{c_1}{4\pi^2 + a^2}\left(ae^{at}\sin(2\pi t + c_3) - 2\pi e^{at}\cos(2\pi t + c_3) + 2\pi\right) + \frac{c_2}{a}(e^{at} - 1) + b\int_0^t e^{as}dW_s\Bigg). \tag{2.3}$$

Since the equation (2.3) is difficult to simulate in this form, we used a discrete version of (2.1) based on the Euler-Maruyama method [11] in order to estimate its parameters. Thus equation (2.1) becomes

$$X_{t+\Delta t} = (1 - a\Delta t)X_t + \big(c_2\cos(c_3)\sin(2\pi t) + c_2\sin(c_3)\cos(2\pi t) + c_1\big)\Delta t + b\epsilon_t \tag{2.4}$$

where $\epsilon_t$ is a random error term distributed $N(0, \Delta t)$.

We used the least squared estimation for regression with lagged variables [8] and obtained the parameters of the equation above. The equation to which we applied the least squared estimator is

$$X_{t+\Delta t} = \alpha_1 X_t + \alpha_2\sin(2\pi t) + \alpha_3\cos(2\pi t) + \alpha_4 + b\epsilon_t.$$

After the estimation, the values of the annualised parameters are (the number of observation used for regression is N=7504 and the timestep used for annualisation is $\Delta t$=1/365 because the data are recorded daily and we consider the unit is one year):

| | | | | |
|---|---|---|---|---|
| $\alpha_1$ | 0.6506 (0.1179) | a | $\frac{1-\alpha_1}{\Delta t}$ | 127.50 |
| $\alpha_2$ | 0.6521 (0.0288) | $c_2$ | $\sqrt{\frac{\alpha_2^2+\alpha_3^2}{\Delta t^2}}$ | 640.78 |
| $\alpha_3$ | 1.6300 (0.0453) | $c_3$ | $tan^{-1}\left(\frac{\alpha_3}{\alpha_2}\right)$ | 1.19 |
| $\alpha_4$ | 7.1516 (0.0087) | $c_1$ | $\frac{\alpha_4}{\Delta t}$ | 2610 |
| $\frac{\sum_{i=1,N} {e_i}^2}{N-4}$ | 1.3670 | b | $\sqrt{\frac{\sum_{i=1}^N {e_i}^2}{N-4}\frac{1}{\Delta t}}$ | 26.12 |

The second column of the table contains, in parenthesis, the estimated variances of the coefficients. Since the estimated coefficient of lagged variable of the regression, $X_{t-1}$ is 0.6506, then the root of the polynomial $(1 - \alpha_1 L)X_t$ is $\frac{1}{0.6506} = 1.53$, which is outside the unit circle, so the ordinary least square estimation we applied is consistent for our regression. Therefore we will maintain these results throughout the paper.

The errors of the estimated model are plotted in Figure 3. After the estimation, the mean of errors is zero and the estimated variance (4 is the number of parameters estimated)

$$\hat{s}^2 = \frac{1}{N-4} \sum_{i=1,N} {e_i}^2 = 1.8693$$

and $R^2 = 0.8823$. Applying the Box-Pierce and Ljung-Box tests [8] to the estimation errors up to the 4th lag, we obtain that the correlation is insignificant.

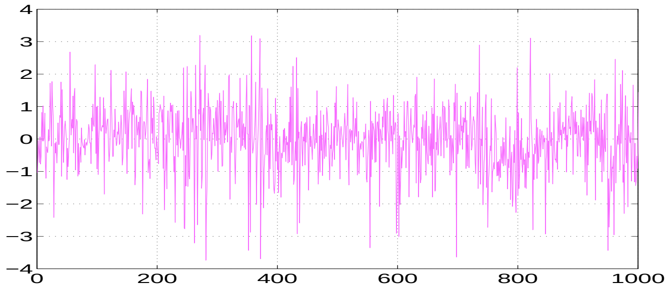


Figure 3: Errors of the estimated model

After annualizing (the last column of the table contains the annualized parameters) and substituting them back in (2.3) we performed some simulations and plotted the results in Figure 4.

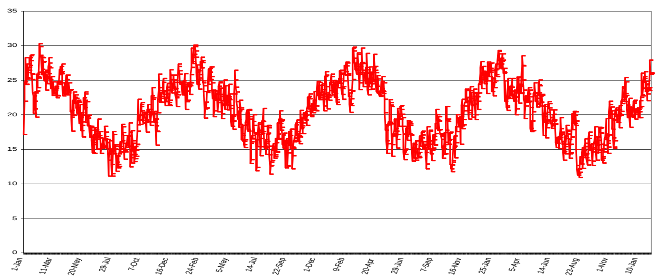


Figure 4: Simulation of the weather process described by equation 1.1

Comparing the plots in Figures 2 and 4 we can observe that our approximation fits reasonably well with the behaviour of temperature over time. Therefore we use this equation to price options on commodities in the next section. One important point of our model is that we want to model a stochastic process which has not only a trend given by the drift, but also a cyclical behaviour given by the weather factor factor. If the stock price equation is

$$dS_t = (\alpha + \beta X_t) S_t dt + \sigma S_t dW_{1t}$$

then if $X_t$ has only positive or negative values, the whole drift term will have a sign given by $\frac{\alpha}{\beta} + X_t$, or by the difference between $X_t$ and $-\frac{\alpha}{\beta}$. If we choose the parameters

$\alpha$ and $\beta$ in such way to obtain the desired fluctuations then if we drop $\beta$, the values for $\alpha$ will not be appropriate for comparison with the standard GBM

$$dS_t = \alpha S_t dt + \sigma S_t dW_t.$$

Therefore it is justified that the other factor of the model must display both negative and positive values. In order to translate the temperature model we obtained so far, which has an average of $+20°C$ in a stochastic seasonal equation with an average of zero we decided to drop the term $c_1$ from equation (2.1) and let the temperature model be a mean-reversion to a sine wave function with the overall average zero. Therefore the new equation which we are going to use throught the next section for pricing a two-factor model is:

$$dX_t = \big(c_1 \sin(2\pi t + c_2) - aX_t\big)dt + bdW_t. \tag{2.5}$$

# 3 The commodity option pricing

The stochastic behaviour of commodity prices has a great influence on the value of contingent claims on the commodity and on the investment to grow or produce the commodity. As Schwartz [3] points out, in an equilibrium framework, if the prices are high, higher cost producers will enter the market, increase the supply and move the prices down, while if the prices are low, the same producers will exit the market and reduce the supply, increasing the prices. In this setting, the relative price fluctuations induce supply changes. However, in the case of agricultural commodities, when the natural conditions dramatically affect the growth (and therefore supply), the seasonal discrepancy between supply and demand generates movements in commodity prices. For example Fama and French [6] found evidence that spot prices for agricultural product usually increase between harvests and fall across harvests. Therefore the futures prices also vary across seasons according to the natural growth cycles of the underlying commodities. The optimal investment decision in commodity based contracts, is determined by the stochastic process assumed for the underlying.

## 3.1 The two-factor model

Throughout this paper we will make several assumptions which will help us analyze the effect of seasonality on the futures and spot contracts on agricultural commodities. The assumptions underlying our model are the classical ones:

- trading takes place in continuous time and assets are infinitely divisible;
- there are no dividends, transaction costs, taxes and short-sale restrictions;
- there are no arbitrage opportunities (no possibility to make a positive profit starting with zero net investment);

• markets are efficient, so the asset prices reflect all available information on the market;

• the agents operate in a risk neutral world and the risk-free rate of interest is constant $r$.

Assuming that the riskless interest rates are constant simplifies our analysis, because it makes the futures and forward prices equal, as Cox, Ingersoll and Ross [1] prove. Based on these, we consider the spot option price $S_t$ described by the following equations:

$$dS_t = (\alpha + \beta X_t)S_t dt + \sigma S_t dW_{1t}, \tag{3.1}$$

where $X_t$ is a stochastic process

$$dX_t = (c_1 \sin(2\pi t + c_2) - aX_t)dt + bdW_{2t}, \tag{3.2}$$

and in the most general case we allow for correlation between the two Brownian increments $dW_{1t}$ and $dW_{2t}$, with the coefficient

$$corr(dW_{1t}, dW_{2t}) = \rho dt.$$

In the above equations, the variable $X_t$ denoting the weather factor follows a mean reverting process. The long term value around which it fluctuates is the sine function. We chose the sine function as a model for temperature as a climate indicator, with periodicity one year. The model can be easily modified for factors with other periodicity by replacing $2\pi$ with a different value determined from observations. In order to see the difference between the two-factor model and the standard GBM, we performed some simulations of both stochastic process and plotted the results in Figures 5 and 6. The simulation was based on the values of the parameters $\alpha = 0.05$, $\beta = 0.15$, $\sigma = 0.20$, $\rho = 0.20$, $S_0 = 0$, $c_1$,$c_2$, $a$ and $b$ as estimated in the previous section, time to maturity is T=2 and the time step $\Delta t = \frac{1}{5000}$. The simulation uses an exact path of the weather process (derived in Section 2) and a discretisation of the spot price process by Euler-Maruyama method [11]:

$$\Delta S_t = (\alpha + \beta X_t)S_t \Delta t + \sigma S_t \Delta W_{1t}$$

which becomes

$$S_{t+1} = (1 + \alpha + \beta X_t)S_t \Delta t + \sigma S_t \epsilon_{2t} \sqrt{\Delta t},$$

where $\epsilon_{2t}$ is a standard normal variable generated in correlation with $\epsilon_{1t}$ the normal variable used for the simulation of the Brownian increment of the weather process, with the correlation coefficient $\rho = 0.1$.

In Figure 6 we plotted the simulation of an exact GBM with $\alpha = 0.05$, $\beta = 0.15$, $\sigma = 0.20$, $T = 2$ and $X_0 = 0$.

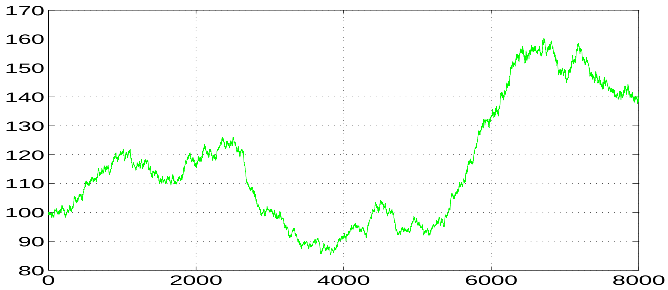


Figure 5: Simulation of the joint stochastic seasonal model

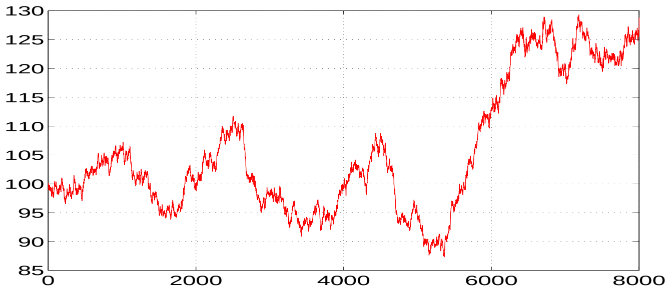


Figure 6: Simulation of a Geometric Brownian Motion

## 3.2 The general framework of pricing European type of contracts

In this sub-section we are going to focus on European put and call options. A European call (put) option is a contract giving the owner the right, but not the obligation to buy (sell) a certain good for a specified price. There are two moments in time involved in this contract, the present time, denoted by t, when the option is written (the contract is entered), and the expiration date, denoted by T, which is the time in the future when option can be exercised. The price for which the parties agree to buy (sell) the option is called the strike (or exercise) price and is denoted by K. A condition for buying the goods underlying the option is that the option strike or exercise price is lower than its market price of the goods at time T. So the payoff of a European call option is given by the following equation:

$$C(S,T) = \begin{cases} S_T - K, & \text{if } S_T > K \\ 0, & \text{otherwise.} \end{cases}$$

In a similar manner, the payoff of a European put option is

$$P(S,T) = \begin{cases} K - S_T, & \text{if } K > S_T \\ 0, & \text{otherwise.} \end{cases}$$

Both call and put options are called derivatives because their prices are derived from the spot price of the underlying commodity.

In commodity pricing models [3], in order to obtain the price of a derivative we must first find the price of the forward contract on the underlying with the same maturity as the call/put option. We are going to use the following notation:

| | |
|---|---|
| $t$ | time when the contract is written |
| $T_1$ | time when the futures contract expires |
| $T$ | time when the call option expires ($T \leq T_1$) |
| $P(t,T)$ | price at $t$ of \$1 to be delivered at $T$, here $e^{-r(T-t)}$ |
| $S_t$ | spot price at time t |
| $K$ | strike price |
| $F(t,T)$ | forward price for the contract with maturity at T. |

The relationship between the spot and the forward price is $S_t = F(t,t)$. In order to see the results of our stochastic seasonal model, in the following we will price two European types of contracts (on futures and on spot) based on this model and compare them with the similar results for a one-factor GBM commodity model.Thus sections 3.4 and 3.5 will contain a parallel of these results. The theoretical price of a European call option with the strike price K on the spot $S_t$ is

$$C(S,K,t,T) = E_t\big[P(t,T)\max(S_T - K, 0)\big],$$

where the expectation is taken under the risk-neutral distribution of $S_t$. For the same underlying, the price of a European call option with strike K on the futures contract is

$$C(S,K,t,T,T_1) = E_t\big[P(t,T)\max(F(T,T_1) - K, 0)\big].$$

## 3.3 Derivation of the analytical expression of the futures prices

In this section we present an introduction to the framework for pricing commodity options. According to Clewlow and Strickland [15], there are two streams to the pricing literature. The first one starts with the stochastic model for the stock prices (one or multi-factor) and derives the prices of the contingent claims consistent with the spot prices (such as in [13], [3], [2]). The second one models directly the evolution of the forward or futures curve (as in [14], [7]). The two approaches are ultimately related. Our approach is a mixture of both. Although it is closer to the second one, it also contains a few features of the first stream. We begin by deriving the SDE for the forward or futures curve and then solve it in two ways. The first one will give us the analytical solution of the forward price as function of the spot price, while the second one will bring information about the distribution of the forward prices and consequently the spot prices. This information is crucial for finding the formula for pricing European contracts on the spot and forward or futures prices.

One of the most important assumptions introduced by Black and Scholes and used in the derivation of option and futures prices is the no arbitrage condition. This means it is impossible for one agent to make a riskless positive profit with zero net investment. As we already know, the futures and forward contracts involve no payment at time $t$. Since these contracts do not require any initial investment, in a risk neutral world the expected change in the forward price must be zero. In terms of SDE this is equivalent to the drift of the forward or futures price equation being equal to zero.

In order to obtain the analytical expression for $F(t,T)$ we apply first Itô's lemma to $F(t,T,S,X)$, and obtain under an equivalent martingale measure (here we write $F$ as a function of $t$ and $T$ only for the simplification of notations):

$$dF(t,T) = \big[F_t + \frac{1}{2}F_{XX}b^2 + \frac{1}{2}F_{SS}\sigma^2S^2 + F_{SX}\sigma b\rho S + F_S(\alpha+\beta X_t)S_t + F_X\big(c_1\sin(2\pi(T-t)+c_2) - aX - \lambda\big)\big]dt + F_S\sigma S dW_{1t} + F_X b dW_{2t}, \tag{3.3}$$

where $\lambda$ is the price of risk associated with the weather (as the weather cannot be hedged, it has a price of risk associated with it). According to Clewlow and Strickland [15], in a risk neutral world the expected return on a futures or forward contract must be zero, otherwise there would be arbitrage oportunities. Therefore in the equation (3.3) the drift of $dF(t,T)$ must be zero. In this case we obtain a parabolic partial differential equation for F(t,T):

$$F_t + \frac{1}{2}F_{XX}b^2 + \frac{1}{2}F_{SS}\sigma^2S^2 + F_{SX}\sigma b\rho S + F_S(\alpha+\beta X_t)S_t + F_X\big(c_1\sin(2\pi(T-t)+c_2) - aX - \lambda\big) = 0 \tag{3.4}$$

with the initial condition $F(t,t) = S_t$.

Substituting and verifying if this solves the above equation, we obtain the following expression for $F(t,T)$:

$$F(t,T) = S_t\exp\left(\beta X_t\frac{1-e^{-a(T-t)}}{a} + A(t,T)\right), \tag{3.5}$$

where $A(\cdot,\cdot)$ is a function of $t$ and $T$ only. Calculating the partial derivatives of F(t,T) with respect to S, t and X, reducing the terms in S and X, and dividing all the equation by F(t,T), we obtain:

$$\frac{\partial A(t,T)}{\partial t} = \frac{1}{2}\left(\beta\frac{1-e^{-a(T-t)}}{a}\right)^2 b^2 + \left(\beta\frac{1-e^{-a(T-t)}}{a}\right)\sigma b\rho + \alpha +$$

$$\left(\beta\frac{1-e^{-a(T-t)}}{a}\right)(c_1\sin(2\pi(T-t)+c_2)-\lambda).$$

After integrating the previous equation with respect to t and applying the initial condition $F(t,t)=S_t$, equivalent to $A(t,t)=0$, gives

$$A(t,T)=\frac{1}{2}\frac{\beta^2b^2}{a^2}\left(T-t+\frac{2}{a}e^{-a(T-t)}-\frac{1}{2a}e^{-2a(T-t)}\right)+\alpha(T-t)+$$

$$\left(\frac{\sigma b\rho\beta}{a}+\frac{\beta\lambda}{a}\right)\left(T-t+\frac{1}{a}e^{-a(T-t)}\right)+\frac{\beta c_1}{a}\left(-\frac{1}{2\pi}\cos(2\pi(T-t)+c_2)+\right.$$

$$\left.\frac{1}{a^2+4\pi^2}\big(a\sin(2\pi(T-t)+c_2)+2\pi\cos(2\pi(T-t)+c_2)\big)e^{-a(T-t)}\right)+K,$$

where:

$$K=-\frac{1}{2}\frac{\beta^2b^2}{a^2}\left(\frac{2}{a}-\frac{1}{2a}\right)-\frac{1}{a}\left(\frac{\sigma b\rho\beta}{a}-\frac{\beta\lambda}{a}\right)$$

$$+\frac{\beta c_1}{a}\left(\frac{1}{2\pi}\cos(c_2)-\frac{1}{4\pi^2+a^2}(a\sin(c_2)+2\pi\cos(c_2)\right).$$

After combining the two previous expression, we obtain:

$$A(t,T)=\left(\frac{1}{2}\frac{b^2\beta^2}{a^2}+\frac{\sigma b\rho\beta}{a}+\frac{\beta\lambda}{a}+\alpha\right)(T-t)-\left(\frac{b^2\beta^2}{a^3}+\frac{\sigma b\rho\beta}{a^2}-\frac{\beta\lambda}{a^2}\right)(1-e^{-a(T-t)})$$

$$+\frac{b^2\beta^2}{4a^3}(1-e^{-2a(T-t)})+\frac{\beta c_1}{a}\left(\frac{1}{2\pi}\big(\cos(c_2)-\cos(2\pi(T-t)+c_2)\big)\right.$$

$$\left.+\frac{e^{-a(T-t)}}{a^2+4\pi^2}\Big(a\big(\sin(2\pi(T-t)+c_2)-\sin(c_2)\big)+2\pi\big(\cos(2\pi(T-t)+c_2)-\cos(c_2)\big)\Big)\right).$$

Using the formula we derived for the term structure of the forward and futures prices, we plot the result in Figure 7 using the same parameters as in the simulations of the GBM and the joint seasonal stochastic process in section 3.1.

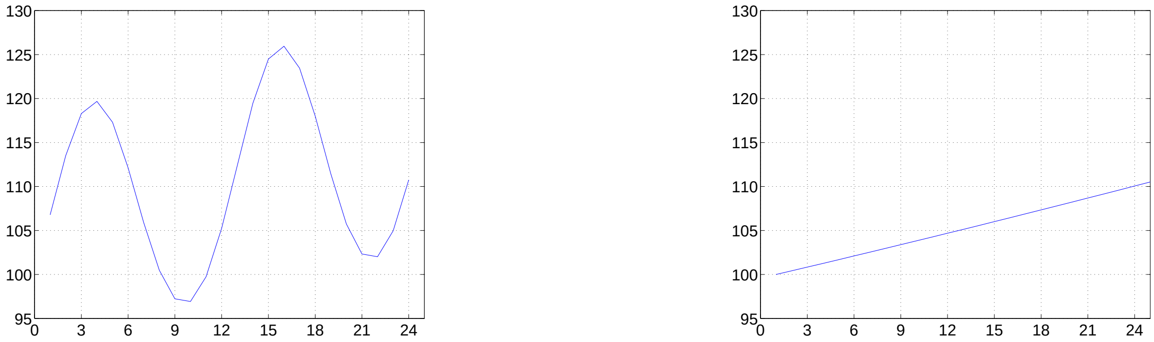


Figure 7: The term structure of forward price for maturities up to 24 months left hand figure the two-factor model, right hand figure GBM

The plot is for $F(0,T)$, for each month up to two years of maturity. In the formula for $F(t,T)$ we have a term depending on $\lambda$, which must be estimated from the real data. However, for simplicity here we can ignore it because for our parameters the coeficient of $\lambda$ is $\frac{\beta}{a}$ which is relatively small, approximately 0.001. This expression for the forward price is useful for price evaluation. However, it is not sufficient to allow us to write a Black-Scholes type of analytical formula for pricing European call and put options. For this purpose we must examine the distribution of the forward and spot prices and afterwards calculate the expectation of the price over the whole life of the option. We can do this by going back to equation (3.3), and setting the drift equal to zero:

$$\frac{dF(t,T)}{F(t,T)} = \sigma dW_{1t} + \left( b\beta \frac{1-e^{-a(T-t)}}{a} \right) dW_{2t}. \tag{3.6}$$

Integrating the preivious equation [14] we obtain that $F(t,T)$ is log-normally distributed, and

$$F(t,T) = F(0,T) \exp\left( \sum_{i=1,2} \left[ -\frac{1}{2} \int_0^t \sigma_i(u,T)^2 du + \int_0^t \sigma_i(u,T) dW_i^*(u) \right] \right), \tag{3.7}$$

where

$$\sigma_1(t,T) = \sigma + \beta b \rho \frac{1-e^{-a(T-t)}}{a},$$

$$\sigma_2(t,T) = \sqrt{1-\rho^2}\beta b \rho \frac{1-e^{-a(T-t)}}{a}$$

and where $dW_1^*$ and $dW_2^*$ are two independent Wiener increments.

Setting $t = T$ we obtain $S(t) = F(t,t)$,

(3.8)

$$S(t) = F(t,t) = F(0,t)\exp\left(\sum_{i=1,2}\left[-\frac{1}{2}\int_0^t \sigma_i(u,t)^2 du + \int_0^t \sigma_i(u,t)dW_i^*(u)\right]\right).$$

so

$$\ln S_t = \ln F(0,t) + \sum_{i=1,2}\left[-\frac{1}{2}\int_0^t \sigma_i(u,t)^2 du + \int_0^t \sigma_i(u,t)dW_i^*(u)\right]$$

which means that $\ln S_t$ follows a normal distribution with mean

$$\ln F(0,t) - \frac{1}{2}\sum_{i=1,2}\int_0^t \sigma_i(u,t)^2 du$$

and variance

$$\sum_{i=1,2}\int_0^t \sigma_i(u,t)^2 du.$$

The last equation is equivalent to $S_t$ being a log-normal variable.

## 3.4 Pricing standard European Options on the Spot

The last result of the above section gives us the distribution of the spot price $S_t$. We can now use it in order to find the price of a standard European call option, which becomes [15]

(3.9)
$$C(S_t, K, t, T) = P(t,T)\big[F(t,T)N(h) - KN(h-w)\big],$$

where $P(t,T)$ is the time discount factor, in our case $e^{-r(T-t)}$, and $w$ and $h$ are:

$$w^2 = \sum_{i=1,2}\int_t^T \sigma_i(u,T)^2 du =$$

$$= \left(\sigma^2 + \frac{2\sigma\beta\rho b}{a} + \frac{b^2\beta^2}{a^2}\right)(T-t) - \frac{2b\beta}{a^2}\left(\sigma\rho - \frac{b\beta}{a}\right)\left(1 - e^{-a(T-t)}\right) + \frac{b^2\beta^2}{2a^3}\left(1 - e^{-2a(T-t)}\right)$$

and

$$h = \frac{\ln\frac{F(t,T)}{K} + \frac{w^2}{2}}{w},$$

respectively, and $N(\cdot)$ is the cumulative normal distribution function defined as

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{1}{2}s^2} ds.$$

Now we want to compare the above two-factor model with the standard Geometric Brownian Motion (GBM)

$$dS_t = \alpha S_t dt + \sigma S_t dW_t$$

which has the solution

$$S_t = \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma \int_0^t dW_u\right).$$

The price of a standard European call option with the same maturity and strike price, on a commodity following GBM is given by

$$C(S_t, K, t, T) = S_t N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{3.10}$$

where

$$d_1 = \frac{\ln \frac{S}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

For the same underlying commodities, the put option can be calculated from the put-call parity [12]

$$P(S_t, K, t, T) = C(S_t, K, t, T) - S_t + Ke^{-r(T-t)}.$$

In the following table we list some results for the price of a European call option calculated using equations (3.9) and (3.10).

| $S_0$ | T=maturity (years) | two-factor model | GBM | $S_0$ | T=maturity (years) | two-factor model | GBM |
|---|---|---|---|---|---|---|---|
| 80 | T=0.25 | 0.454 | 0.004 | 80 | T=0.75 | 0.000 | 0.391 |
| 100 | T=0.25 | 13.544 | 3.635 | 100 | T=0.75 | 3.810 | 7.133 |
| 120 | T=0.25 | 29.881 | 21.254 | 120 | T=0.75 | 14.567 | 23.940 |
| 80 | T=0.5 | 0.000 | 0.112 | 80 | T=1 | 0.000 | 0.804 |
| 100 | T=0.5 | 10.970 | 5.527 | 100 | T=1 | 8.647 | 8.591 |
| 120 | T=0.5 | 25.587 | 22.578 | 120 | T=1 | 21.304 | 25.296 |

As an illustration of pricing European options on the spot, in the following figures we plotted some simulations of the prices of the call options on the current spot price.

The figures show the dependence of the call price on the spot price for different maturities of the contracts: 3, 6, 9 and 12 months. The continuous line corresponds to the call option price of GBM, while the dotted one corresponds to the two-factor model. Both models have the same parameters as in the above simulations. In each figure on the X-axis we have the spot price $S_t \in [60, 150]$, and on the Y-axis we have the the call option price. In all cases, the strike price used was $K = 100$. The results in Figure 8 show that the weather component generates oscillations around the Black-Scholes (BS) price of the European call options on the spot, depending on the maturity of the contracts. For certain maturities (i. e. 3, 6, 15 and 18 months) our model gives higher values for the European call option than the BS formula, while for others (i. e. 9 or 12 months) it gives lower values than the BS formula.

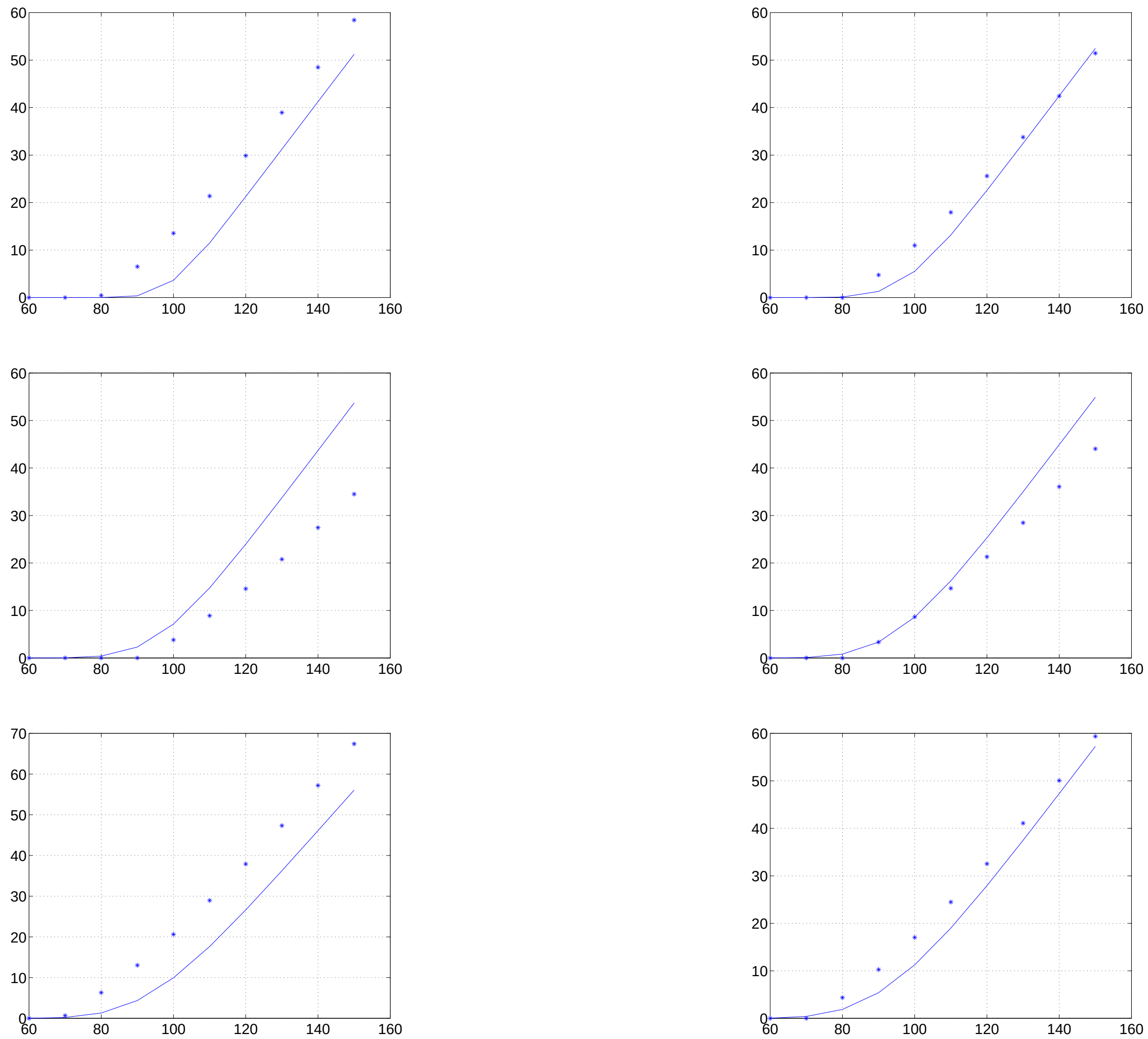


Figure 8: European call option prices on the spot as function of $S_t$, $K = 100$ at 3 monthly maturity periods, from 3 to 18 months

## 3.5 Pricing European Options on the Forward and Futures

In this section we will extend the formula for pricing European call and put options on the spot to futures and forwards with the same underlying commodities such as in the previous section. According to Hilliard and Reis [2], the call option price of a two-factor model can be written as ($T_1 \geq T$):

$$C(S_t, K, t, T_1, T) = P(t,T)\big[F(t,T_1)N(h) - KN(h-w)\big], \tag{3.11}$$

where $P(t,T)$ and $N(\cdot)$ are defined as in the previous section, and $w$ and $h$ are given by:

$$w^2 = \sum_{i=1,2}\int_t^T \sigma_i(u,T_1)^2 du =$$

$$= \left(\sigma^2 + \frac{2\sigma\beta\rho b}{a} + \frac{b^2\beta^2}{a^2}\right)(T-t) - \frac{2b\beta}{a^2}\left(\sigma\rho - \frac{b\beta}{a}\right)\left(e^{-a(T_1-t)} - e^{-a(T_1-T)}\right) +$$

$$\frac{b^2\beta^2}{2a^3}\left(e^{-2a(T_1-t)} - e^{-2a(T_1-T)}\right)$$

and

$$h = \frac{\ln\frac{F(t,T_1)}{K} + \frac{w^2}{2}}{w},$$

respectively. From the put-call parity the price of a European put option on the same underlying is:

$$P(S_t, K, t, T_1, T) = P(t,T)\big[KN(-h+w) - F(t,T_1)N(-h)\big] \tag{3.12}$$

with the same parameters as above.

In this setting, the volatility of the forward price returns is not constant and equal to the volatility of the spot price returns as in [4], but is $\sigma_F$, which is a function of the time to maturity. We can see the volatility of forward returns is a decreasing function of time, as plotted on a monthly basis in Figure 8 for the same parameters $\alpha$, $\beta$, $a$ and $b$ as before, until maturity (24 months). For an asset following a GBM, the price of a European call option on a futures contract on this asset is [4] is given by the equation

$$C(S_t, K, t, T_1, T) = e^{-r(T_1-t)}\big(S_tN(d_1) - KN(d_2)\big),$$

where

$$d_1 = \frac{\ln\frac{S}{K} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

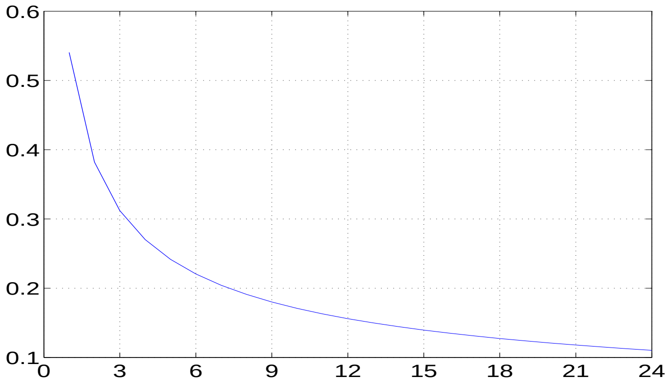

Figure 9: Futures return volatility up to 24 months maturity

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Options on forwards are influenced by two different maturities, the maturity of the call/put option and the maturity of the forward contract. Therefore it is interesting to see their effects on the option price. In Figure 10 we plotted the European call option prices with maturities $T = 3$ and 6 months, on forwards with maturity $T_1 =$ 9, 12, 18 and 24 months. We chose these particular maturities because the maturity of the forward contact must be longer than that of the option, and most of the commodities traded on exchanges have maturities which are multiples of 3 months. The instantaneous riskless interest rate is $r = 0.05$, the strike price $K = 100$ and all the parameters of the model are the same as before.

As we can observe, for maturities up to one year the BS price is higher than the two-factor model one, and for maturities longer than one year the BS price is lower. For the one year maturity the effect is ambiguous, for at-the-money options the BS price is slightly smaller than the two-factor model, while for in-the-money options it is higher. From these results we can conclude that our two-factor seasonal model price is different from the BS option pricing. However, the direction and magnitude of the difference depend on the specific parameters of the option, such as the maturity and the strike price. As in the previous subsection, in all graphs in Figure 10 the BS call option price is plotted with a continuous line, and the two-factor model price with a dotted line.

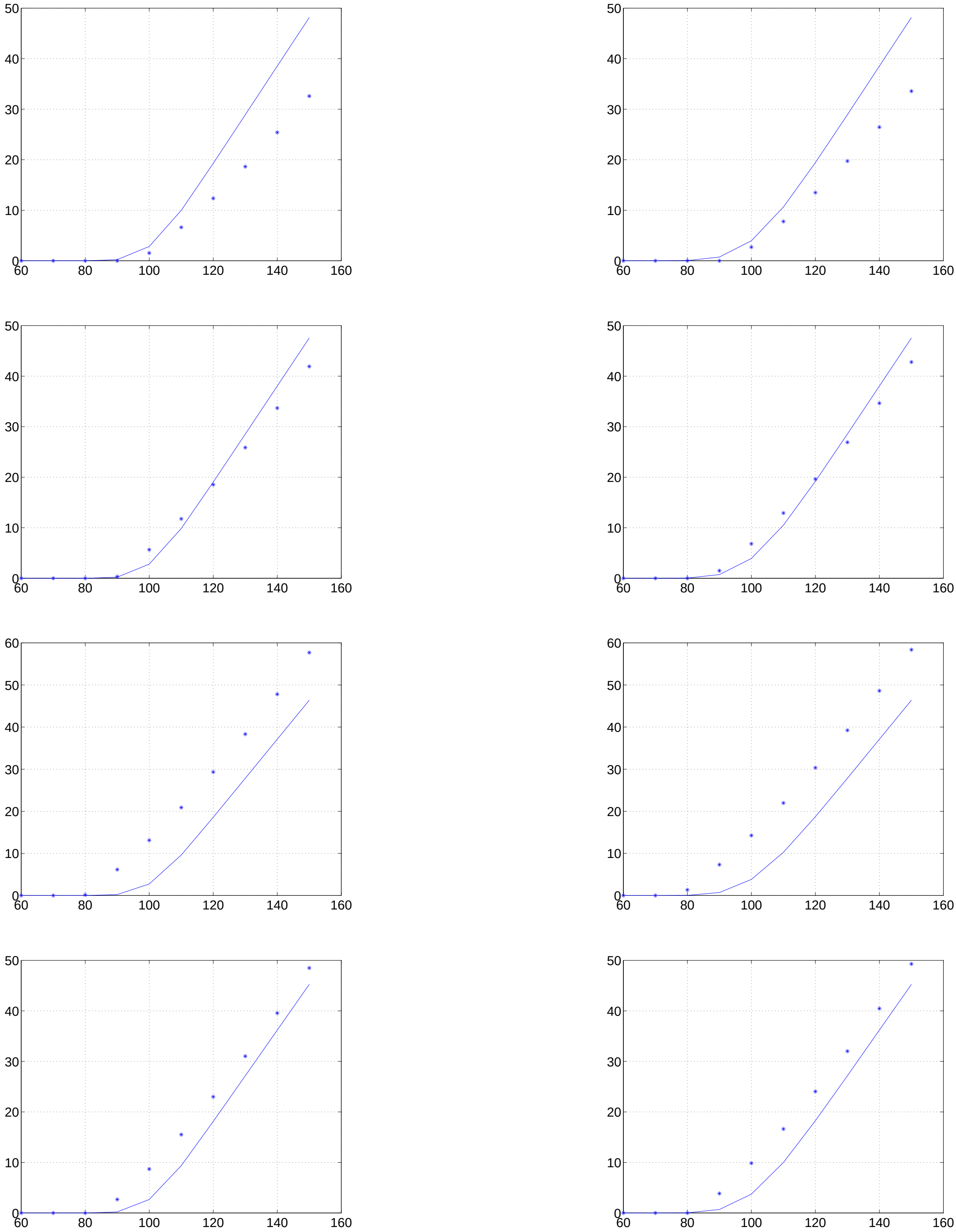


Figure 10: European call option prices on futures as function of $S_t$, $K = 100$
left hand figures $T$=3 months, $T_1$=9,12,18,24 months
right hand figures $T$=6 months, $T_1$=9,12,18,24 months

## 4 Conclusion

In this paper we present a two-factor theoretical model to value options on commodity spot and futures based on the assumption that the spot commodity price and the weather component which determines the growth and production follow a joint general linear stochastic process. This allows us to find the analytical solution for European type of options on commodities, both on the spot and on futures and forwards. Using some numerical examples we show that the introduction of a stochastic seasonal factor in the model can have a significant effect on forward and futures prices, as well as on European option prices.

This model can be further developed to allow for stochastic interest rates and for the possibility of jumps in commodity prices, due to unexpected market shocks. Moreover, we can extend the theory developed in this paper to more complicated derivatives, such as American options and exotic options, and investigate numerical methods for evaluating the cases where analytical solutions are not available.

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