Optimal Experimentation and the Perturbation Method.[*]

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Abstract: The perturbation method is used to approximate optimal experimentation problems. The approximation is in the neighborhood of the linear regulator problem which has a well-defined solution procedure. The first order perturbation of the optimal decision under experimentation is a combination of the linear regulator solution and a term that captures the impact of the uncertainty on the agent's value function. An algorithm is developed to quickly implement this procedure on the computer. As a result, the impact of optimal experimentation on an agent's decisions can be quantified and estimated for a large class of problems encountered in economics.

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1 Introduction

Economic agents generally find themselves in circumstances in which they have to maximize their welfare while at the same time they must learn about fundamental relations that influence their payoff. Optimal experimentation arises when the actions of the agents impact their information set.[1] In these circumstances the agents trade off optimal control with optimal experimentation. In general these problems are difficult to solve since the actions of the agents affect the distribution of payoffs. This paper demonstrates how to use the perturbation method to approximate the solution to these problems.

Wieland (2000a) shows how to use value and policy function iteration to examine the behavior of optimal experimentation problems.[2] He introduced the optimal experimentation problem into otherwise static optimization problem.[3] Within the context of these problems he was able to identify as much as a 52% increase in the agent's value from experimentation.

Extending this procedure to more general problems or using it for the estimation of experimentation effects is problematic. In the case of a continuous distribution in which the agent is learning about four state variables Wieland's computer program takes as much as a week to find a solution. Wieland (2002) evaluates a more complex macroeconomic problem in which agents learn about nine state variables. However, computational time, about 60 hours, limits the discussion to pairwise evaluation of the complete optimal experimentation problem. This limits the application of this procedure to calibration exercises and precludes the estimation of models of optimal experimentation.[4] In addition optimal experimenting about multiple parameters or more complex dynamic settings is prohibitive. This paper provides an alternative procedure for approximating optimal experimentation problems which can handle more complex problems, as well as the estimation of experimentation effects.

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[1] Balvers and Cosimano (1990,1994), Keller and Rady (1999), and Wieland (2000a) provide discussions of the literature on optimal experimentation. Kendrick (2002) in his 2002 keynote address for the Society of Computational Economics discusses the role of optimal experimentation in stochastic control problems.

[2] Beck and Wieland (2002) use the same technique when the state variable is dependent on one lag of the dependent variable.

[3] Balvers and Cosimano (1990, 1994) and Wieland (2000b) provide examples in which this approach may be applied.

[4] Wieland (2000b) provides a good example of a calibration exercise.

This alternative procedure is based on the perturbation method of Judd and Gaspar (1997), Judd (1998), and Jin and Judd (2002). The perturbation method is useful when a more general problem reduces to a simpler problem under some well-defined circumstances. In addition the simpler problem has a well-developed solution method. The perturbation method proceeds to introduce parameters such that the general problem reduces to the simpler problem when these parameters are zero. The more general problem is then approximated by taking a Taylor series approximation around zero values for the perturbation parameters.

In most applied problems on optimal experimentation the objective is quadratic and the equations of motions are linear. In addition it is usually possible to introduce parameters which remove the optimal experimentation problem. For example in Weiland's (2000b) optimal monetary policy problem the optimal experimentation issue would not be present when the central bank knows the impact of interest rates on inflation. Consequently, the optimal experimentation may be removed by attaching a parameter to the error term for this slope coefficient and setting the parameter to zero.[5]

Without the optimal experimentation these problems fall into the general rubric of the discounted stochastic regulator problem. The procedures for solving these problems have been developed by Hansen and Sargent (1998), and Anderson, Hansen, McGrattan and Sargent (1996). As a result we can use the perturbation method to approximate the optimal decision of an agent in the presence of experimentation. In this paper the second order perturbation to the experimentation problem is found. The optimal decision starts with the optimal decision found in Anderson *et. al.*. Appended to this decision is a term which captures the effect of the conditional variance-covariance matrix on the agents' optimal decisions.

This additional impact on the optimal decisions of the agent is akin to Ito's lemma in continuous time stochastic control problems.[6] The effect of optimal experimentation works through the second order effect of the variance-covariance matrix on the agent's marginal valuation of each state. This effect may be decomposed into two parts based on the chain rule.

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[5] A similar set up could be used to apply the perturbation method to Balvers and Cosimano (1990, 1994).

[6] Keller and Rady (1999) provide an example of an optimal experimentation problem in continuous time.

The variance-covariance matrix first changes how each state variable impacts the equation of motion for the state variables. This part occurs because this change influences the Kalman gain. The second part consist of the marginal impact of the state variables on the agent's valuation. The perturbation method develops a systematic way to measure these effect of optimal experimentation which is added to the optimal decision from the linear regulator problem.[7]

There are several benefits to this procedure. First, it builds on a well-developed procedure for handling a large class of economic problems. This class of problems includes those found in the literature on optimal experimentation as well as problems with more complex dynamics. Second, the procedure can be implemented on a computer in a timely manner. Finally, by suitably modifying the estimation procedure of Anderson *et. al.* it is now feasible to estimate optimal decision rules of agents in the presence of optimal experimentation. For example Sargent (1999) develops an optimal monetary control problem which fits into the Anderson *et. al.* framework.[8] This optimal monetary control problem is a more general version of Wieland (2000b). Thus, it would be feasible to estimate an optimal central bank reaction function in the presence of optimal experimentation.

The main drawback of this procedure is that it assumes that the value function and optimal decisions are differentiable. There are at least two reasons found in the literature when the value function is not differentiable. First, Balvers and Cosimano (1993) develop an example in which the objective of the agent is convex. Earlier Easley and Kiefer (1988) showed that the value function is convex in the conditional distribution of the shocks. As a result, the Bellman equation is convex. Thus, the optimal decision is a corner solution. Balvers and Cosimano show that the value function is not differentiable when the agents switches between corner solutions. This problem would not occur when the one period reward function is sufficiently concave relative to the convex effect of experimentation.[9] This condition may be checked in

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[7] Easley and Kiefler (1988) develop the qualitative characteristics.

[8] Wieland's (2002) model of monetary policy under uncertainty about the natural rate of unemployment and Woodford's (2002) model of imperfect common knowledge could also fit within this class of models.

[9] Keller and Rady (1999) analyze a similar problem when the agent's objective is concave. In this case corner solutions arise when the experimentation effect is dominate. They also find the value function is not differentiable

the perturbation method by examining whether a particular matrix is negative definite.[10]

Keller and Rady (1999) and Wieland (2000a) show that the value function is not differentiable when the agent has incorrect limit beliefs. Following Kiefer and Nyarko (1989) Wieland identifies three properties of limit beliefs and optimal decisions. First beliefs must be self-reinforcing. Second, given the limit beliefs the reward function should be optimized. Third, if the control variable is held constant, the agent would learn the mean value of the latent variables. The possible solution to these conditions are incorrect when the expected values of the latent parameters do not converge to their true value. In the simulations Keller and Rady (1999) and Wieland (2000a) find that the value function and optimal decisions are not differentiable when the latent parameter corresponds to an incorrect limit belief. Thus, the procedure needs to avoid these possibilities.

Using the perturbation method in the neighborhood of the linear regulator problem to approximate optimal experimentation problems helps to mitigate the convergence to incorrect limit beliefs. Starting with Marcet and Sargent (1989) the literature on optimal learning has developed conditions under which the learning procedure converges to the true parameter values.[11] Hansen and Sargent (1998) use the Kalman filtering procedure to represent optimal learning in stochastic linear regulator problems. They also show that the Kalman filtering procedure is the dual for the linear regulator problem. As a result, the conditions for the convergence of the Kalman filter are identical to those for a stable linear regulator problem. In this paper the optimal experimentation problem is approximated by developing a parameterization which collapses the optimal experimentation problem to the optimal learning problem in linear regulator problems. When beliefs converge the expectation of the latent parameter will satisfy the mean prediction property. In addition, the linear regulator problem will be optimized. The difference is that the Kalman gain will be smaller so that the conditional variance-covariance matrix will converge faster to a larger value. Thus, the perturbation method applied to optimal

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in these cases.

[10] This matrix is effectively an approximation of the Hessian matrix for the Bellman equation.

[11] In optimal learning problems the agent's actions do not affect the information used to update forecast, while it does under optimal experimentation. See Evans and Honkapohja (2001) for a detail survey and discussion of the work on optimal learning.

experimentation problems in the neighborhood of the optimal learning problem will converge to correct beliefs as long as the linear regulator problem is stable.

The next section summarizes the procedures to solve augmented linear regulator problems following Anderson *et. al.* (1996). Section 3 develops the parameterization of the conditional variance-covariance matrix so that the optimal experimentation problem reduces to the Kalman filtering problem when the parameters are set equal to zero. Section 4 derives the formula for optimal conditions in the presence of a second order perturbation of the optimal experimentation problem. Section 5 illustrates the procedure. The analysis of Balvers and Cosimano (1990) is applied to a bank with some monopoly power that does not know the demand for loans or the supply of deposits. This example is a generalization of Balvers and Cosimano in that optimal experimentation for two separate relations is undertaken. In addition the state vector includes eleven variables, including lagged dependent variables, which moves stochastically over time. The results of Balvers and Cosimano are quantified in that both the loan and deposits rates slowly adjust to changes in market conditions. The final section concludes the paper.

2 The Augmented Linear Regulator Problem

The perturbation method is used to solve a general experimentation problem by approximating the problem around a simpler problem with a known solution. In the optimal experimentation problem the augmented linear regulator problem is taken as the simpler problem. In this section we summarize the augmented linear regulator problem as well as its solution following Anderson, *et. al.* (1996). The agent is assumed to choose a sequence $\{u_t\}$ to maximize

$$-E\left(\sum_{t=0}^{\infty}\beta^t\left[u_t'Ru_t + y_t'Q_{yy}y_t + 2y_t'Q_{yz}z_t + z_t'Q_{zz}z_t + 2u_t'W_y'y_t + 2u_t'W_z'z_t\right]|\mathcal{F}_0\right)$$

subject to

$$x_{t+1} \equiv \begin{pmatrix} y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} A_{yy} & A_{yz} \\ 0 & A_{zz} \end{pmatrix}\begin{pmatrix} y_t \\ z_t \end{pmatrix} + \begin{pmatrix} B_y \\ 0 \end{pmatrix} u_t + \begin{pmatrix} G_{yy} & G_{yz} \\ 0 & G_{zz} \end{pmatrix}\begin{pmatrix} w_{yt+1} \\ w_{zt+1} \end{pmatrix} =$$

$$Ax_t + Bu_t + Gw_{1t+1},$$

Here $\{\mathcal{F}_t : t = 0, \cdots\}$ is an increasing sequence of information sets which is based on a martingale difference process $w'_{1t+1} \equiv (w_{yt+1}, w_{zt+1})'$ such that $E\left(w_{1t+1}|\mathcal{F}_t\right) = 0$ and $E\left(w_{1t+1}w'_{1t+1}|\mathcal{F}_t\right) = I$. u_t is the control vector, which may influence the endogenous state vector y_t but does not effect the exogenous state vector, z_t. Each of the matrices are conformable to these vectors.

To solve this problem the cross product terms and the discount factor are eliminated by defining the selection matrices $U_y \equiv [I, O]$ and $U_z \equiv [0, I]$ such that $U_z A U'_y = 0$, $U_z G U'_y = 0$, and $U_z B = 0$. Next let

$$y_t \equiv \beta^{t/2} U_y x_t, \quad z_t \equiv \beta^{t/2} U_z x_t, \quad v_t \equiv \beta^{t/2}(u_t + R^{-1} \begin{pmatrix} W'_y & W'_z \end{pmatrix} x_t), \tag{1}$$

$\begin{pmatrix} A^0_{yy} & A^0_{yz} \\ 0 & A^0_{zz} \end{pmatrix} \equiv \beta^{1/2}\left(A - BR^{-1}W'\right)$, $B^0_y \equiv \beta^{1/2} U_y B$, and $\begin{pmatrix} Q^0_{yy} & Q^0_{yz} \\ Q^{0\prime}_{yz} & Q^0_{zz} \end{pmatrix} \equiv Q - WR^{-1}W'$.[12] The solution to this augmented regulator problem is given by

$$v_t = -F_y y_t - F_z z_t,$$

where $F_y \equiv \left[R + B'_y P_y B_y\right]^{-1} B'_y P_y A_{yy}$ and $F_z \equiv \left[R + B'_y P_y B_y\right]^{-1} B'_y \left[P_y A_{yz} + P_z A_{zz}\right]$. The solution is found in two steps. First, P_y solves the Riccati equation

$$P_y = Q_{yy} + \left[A_{yy} - B_y F_y\right]' P_y \left[A_{yy} - B_y F_y\right] + F'_y R F_y,$$

and second, P_z satisfies the Sylvester equation

$$P_z = Q_{yz} + \left[A_{yy} - B_y F_y\right]' P_y A_{yz} + \left[A_{yy} - B_y F_y\right]' P_z A_{zz}.$$

Reversing the definitions in (1) the solution to the discounted regulator problem is

$$u_t = -\left[F_y + R^{-1}W'_y\right] y_t - \left[F_z + R^{-1}W'_z\right] z_t.$$

Under the augmented linear regulator problem the agent can learn about the economy independent of their optimal decisions. This result follows from the certainty equivalence property. Certainty equivalence is dependent on quadratic objectives, linear constraints and independence among the distribution of shocks and the agents choice.

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[12] To avoid unnecessary notation I delete the superscript on A and Q below.

In learning problems the agent observes signals which are a linear combination of the hidden state, control and random error vectors. For simplicity only the endogenous state vector is hidden from the agent. The endogenous state vector follows

$$y_{t+1} = A_{yy} y_t + A_{yz} z_t + B_y u_t + G^1 w_{1t+1},^{13}$$

while the agent observes each period t the signals

$$s_t = C_{sy} y_t + C_{sz} z_t + D u_t + H w_{2t}.$$

Assume

$$E \begin{pmatrix} w_{1t+1} \\ w_{2t} \end{pmatrix} \begin{pmatrix} w'_{1t+1} & w'_{2t} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

The agent is interested in forecasting the endogenous state vector $\hat{y}_t = E\left(y_t | u_t, z_t, s_t, s_{t-1}, \cdots, s_0, \hat{x}_0\right)$.[14] The Kalman Filter updates the agent's forecast according to

$$\hat{y}_{t+1} = A_{yy} \hat{y}_t + A_{yz} z_t + B_y u_t + K_t a_t,$$

where $a_t = s_t - \hat{s}_t = C_{sy}\left(y_t - \hat{y}_t\right) + H w_{2t}$. The Kalman Gain is defined by

$$K_t = A_{yy} \Sigma_t C'_{sy} \left(C_{sy} \Sigma_t C'_{sy} + H H'\right)^{-1},$$

and the conditional variance-covariance matrix of the state is updated according to

$$\Sigma_{t+1} = A_{yy} \Sigma_t A'_{yy} + G G' - A_{yy} \Sigma_t C'_{sy} \left(C_{sy} \Sigma_t C'_{sy} + H H'\right)^{-1} C_{sy} \Sigma_t A'_{yy}.$$

In the optimal experimentation literature the agent has some ability to manipulate the flow of information. This means that H is a function of the agent's decisions, so that the variance-covariance matrix for the signal is also a function of the control vector, u. As a result the agent's decision influences the distribution of the state vector. Thus, the certainty equivalence property no longer holds. This means that the agent's optimization problem cannot be separated from their forecasting problem. In the next section the optimal experimentation problem is formulated.

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[13] The superscript 1 refers to the column's of G associated with the endogenous state variables. Without loss of generality from now on I will delete the superscript on G.

[14] Ljungqvist and Sargent (2000, pp. 643-649) derives the Kalman Filter in a similar circumstances.

3 Optimal Experimentation

In the optimal experimentation problem the agent chooses a sequence $\{u_t\}$ to maximize

$$-E\left(\sum_{t=0}^{\infty}\beta^t\left[u_t'Ru_t+y_t'Q_{yy}y_t+2y_t'Q_{yz}z_t+z_t'Q_{zz}z_t+2u_t'W_y'y_t+2u_t'W_z'z_t\right]|\mathcal{F}_0\right)$$

subject to

$$x_{t+1}=Ax_t+Bu_t+Gw_{1t+1},$$

$$\hat{y}_{t+1}=A_{yy}\hat{y}_t+A_{yz}z_t+B_yu_t+K_ta_t=F\left[u_t,\hat{y}_t,z_t,\Sigma_t,\tau\right],$$

$$K_t=A_{yy}\Sigma_tC_{sy}'\left(C_{sy}\Sigma_tC_{sy}'+HH'\right)^{-1},$$

$$\Sigma_{t+1}=A_{yy}\Sigma_tA_{yy}'+GG'-A_{yy}\Sigma_tC_{sy}'\left(C_{sy}\Sigma_tC_{sy}'+HH'\right)^{-1}C_{sy}\Sigma_tA_{yy}'=G\left[u_t,z_t,\Sigma_t,\tau\right],$$

$$z_{t+1}=A_{zz}z_t+G_{zz}w_{zt+1}=Z(z_t)$$

and

$$E\left(\sum_{t=0}^{\infty}\left[|u_t|^2+|y_t|^2\right]|\mathcal{F}_0\right)<\infty.$$

In the optimal experimentation problem, the variance-covariance of the signal, HH', is a function of the current control u_t and state vector z_t. In particular, the uncertainty in the signals is a linear function of the control and state vectors. This effect may be represented by replacing Hw_{2t} with

$$Hw_{2t}+\tau_1u_t'\epsilon_{1t}+\tau_2z_t'\epsilon_{2t}$$

where w_{2t}, ϵ_{1t}, and ϵ_{2t} are not correlated. The variance-covariance matrix, HH', is now

$$V_t=HH'+\tau_1u_t'V_3\tau_1'u_t+\tau_2z_t'V_4\tau_2'z_t,$$

Where $V_3=E_t\left[\epsilon_{1t}\epsilon_{1t}'\right]$ and $V_4=E_t\left[\epsilon_{2t}\epsilon_{2t}'\right]$. The vectors τ_1' and τ_2' are perturbation vectors such that each element is equal to one under optimal experimentation.[15] In addition as both τ_1 and τ_2 approach zero the variance-covariance matrix approaches HH', so that the problem reduces to the linear regulator problem.

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[15] Ljungqvist and Sargent (2000, pp.643-649) allow for time varying variance-covariance matrix as long as the agent knows them.

In this case the Bellman equation becomes

$$V\left[\hat{y}_t, z_t, \Sigma_t, \tau\right] = E\left[\Pi\left[u_t, y_t, z_t, \tau\right] + \beta V\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] \left|\mathcal{F}_t\right.\right]. \tag{2}$$

Here

$$\Pi\left[u_t, y_t, z_t, \tau\right] = -\left(u_t' R u_t + y_t' Q_{yy} y_t + 2y_t' Q_{yz} z_t + z_t' Q_{zz} z_t + 2u_t' W_y' y_t + 2u_t' W_z' z_t\right).$$

This dynamic programming problem incorporates two new effects which are not present in the augmented linear regulator problem. The first effect measures the effect of the choice on the Kalman gain which in turn influences the conditional expectation of y_{t+1}. The second effect deals with the optimal choice on the conditional variance-covariance matrix for y_{t+1}. To analyze these effects the following results, proved in the appendix, are useful.

Lemma 1 *The partial derivates of the Kalman filter are* $\frac{\partial F}{\partial u_t} = vec\left(\left[B_y', 0'\right]\right)$, $\frac{\partial F}{\partial \hat{y}_t} = vec\left(\left[A_{yy}', 0'\right]\right) - \left(\begin{pmatrix} K_t \\ 0 \end{pmatrix} \otimes I_q\right)(C_{sy} \otimes I_q) vec\left(I_q\right)$, $\frac{\partial F}{\partial z_t} = vec\left(\left[A_{yz}', A_{zz}'\right]\right)$, $\frac{\partial F}{\partial \Sigma_t} \neq 0$, $\frac{\partial F}{\partial \tau_1} = 0$, $\frac{\partial F}{\partial \tau_2} = 0$, $\frac{\partial^2 F}{\partial u_t \partial \Sigma_t} = 0$, $\frac{\partial^2 F}{\partial \hat{y}_t \partial \Sigma_t} \neq 0$, $\frac{\partial^2 F}{\partial z_t \partial \Sigma_t} = 0$, $\frac{\partial^2 F}{\partial \Sigma_t^2} \neq 0$, $\frac{\partial G}{\partial u_t} = 0$, $\frac{\partial G}{\partial z_t} = 0$, $\frac{\partial G}{\partial \Sigma_t} \neq 0$, $\frac{\partial G}{\partial \tau_1} = 0$, $\frac{\partial G}{\partial \tau_2} = 0$, *and* $\frac{\partial^2 G}{\partial \Sigma_t^2} \neq 0$ *when the perturbation parameters are zero.*[16]

4 Perturbation Method

The optimal experimentation problem introduced in the previous section does not have an explicit solution. In this section the perturbation method is used to approximate this problem following the analysis of Judd and Gaspar (1997), Judd (1998), and Jin and Judd (2002).

The tensor notation is used extensively in the perturbation method. This notation may be illustrated by writing the quadratic form $x'Ax$ as $a_{ij} x^i y^j$ which means $\sum_i \sum_j a_{ij} x^i y^j$. As a result, a summation occurs whenever a superscript and a subscript match. The partial derivatives $\frac{\partial F[\hat{y}_t, z_t, u_t, \Sigma_t, \tau]}{\partial \hat{x}_t}$ are represented by F_j^i for each state vector. For example $F_j^i = a_{zz,j}^i$ for the exogenous state vectors i and j. In a similar way F_α^i would be the partial derivative of the i^{th} state variable with respect to the α^{th} control. $F_\alpha^i = 0$ for the exogenous state vectors.

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[16] To cut down on notation the functions F and Z have been stacked together and is called F. vec(A) stacks the columns of A in a vector starting with the first column.

F^i_I represents the partial derivative of the i^{th} state variable with respect to the I^{th} variance or covariance term. $F^i_I = 0$ for the exogenous state vectors. Finally, $F^i_{\mathcal{I}}$ represents the partial derivative of the i^{th} state variable with respect to the $\mathcal{I}^{th}$ perturbation parameter. $F^i_{\mathcal{I}} = 0$ for both state vectors.

Given this notation the Euler conditions may be written as

$$\begin{gathered} E\left[\Pi_\alpha\left[u\left[\hat{y}_t, z_t, \Sigma_t, \tau\right], y_t, z_t, \tau\right] + \right. \\ \beta V_i\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] F^i_\alpha\left[u\left[\hat{y}_t, z_t, \Sigma_t, \tau\right], \hat{y}_t, z_t, \Sigma_t, \tau\right] + \\ \left.\beta V_I\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] G^I_\alpha\left[u\left[\hat{y}_t, z_t, \Sigma_t, \tau\right], z_t, \Sigma_t, \tau\right] | \mathcal{F}_t\right] \leq 0 \end{gathered} \tag{3}$$

for each control α.

Solving the optimal learning problem (2) and (3) explicitly is problematic. The difficulty comes about because of the additional non-linearity introduced by the control variables influence on the Kalman Filter. However, the problem reduces to the augmented linear regulator problem when the perturbation vector, τ, is set equal to zero. As a result the perturbation method of Judd and Gaspar (1997), Judd (1998), and Jin and Judd (2002) may be applied to (2) and (3). Equation (3) implies an implicit function for the control vector, $u\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]$, so that equation (2) implies an implicit equation for $V\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]$. The perturbation method involves a taylor expansion of these functions around the known solution. In this case the expansion is around $\left[\hat{y}^{LR}_t, z_t, \Sigma^{LR}_t, 0\right]$, where the superscript LR refers to the linear regulator solution.[17] [18]

$$\begin{gathered} u^\alpha\left[\hat{y}_t, z_t, \Sigma_t, \tau\right] \approx u^\alpha\left[\hat{y}^{LR}_t, z_t, \Sigma^{LR}_t, 0\right] + u^\alpha_i\left[\hat{y}^{LR}_t, z_t, \Sigma^{LR}_t, 0\right]\left[\hat{x}_t - \hat{x}^{LR}_t\right]^i \\ + u^\alpha_I\left[\hat{y}^{LR}_t, z_t, \Sigma^{LR}_t, 0\right]\left[\Sigma_t - \Sigma^{LR}_t\right]^I. \end{gathered} \tag{4}$$

$$\begin{gathered} V\left[\hat{y}_t, z_t, \Sigma_t, \tau\right] \approx V\left[\hat{y}^{LR}_t, z_t, \Sigma^{LR}_t, 0\right] + V_i\left[\hat{y}^{LR}_t, z_t, \Sigma^{LR}_t, 0\right]\left[\hat{x}_t - \hat{x}^{LR}_t\right]^i \\ + V_I\left[\hat{y}^{LR}_t, z_t, \Sigma^{LR}_t, 0\right]\left[\Sigma_t - \Sigma^{LR}_t\right]^I + \frac{1}{2} V_{ij}\left[\hat{y}^{LR}_t, z_t, \Sigma^{LR}_t, 0\right]\left[\hat{x}_t - \hat{x}^{LR}_t\right]^i\left[\hat{x}_t - \hat{x}^{LR}_t\right]^j \\ + V_{iI}\left[\hat{y}^{LR}_t, z_t, \Sigma^{LR}_t, 0\right]\left[\hat{x}_t - \hat{x}^{LR}_t\right]^i\left[\Sigma_t - \Sigma^{LR}_t\right]^I + \frac{1}{2} V_{IJ}\left[\hat{y}^{LR}_t, z_t, \Sigma^{LR}_t, 0\right]\left[\Sigma_t - \Sigma^{LR}_t\right]^I\left[\Sigma_t - \Sigma^{LR}_t\right]^J. \end{gathered} \tag{5}$$

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[17] In the appendix it is shown that $\mathsf{V}_{\mathcal{I}} = 0$ and $\mathsf{u}^{\gamma}_{\mathcal{I}} = 0$.

[18] These approximation may be done to a higher order, however, only the first order terms in the optimal control solution should be important for empirical work.

As the perturbation vectors approach zero the variance-covariance term HH' approaches the linear regulator value so that the equation of motion for the variance-covariance matrix, $G\left[u_t, z_t\Sigma_t, \tau\right]$, approaches its linear regulator counterpart. Thus, Σ_t approaches Σ_t^{LR} as τ tends to zero. In addition the Kalman Gain, K_t, also approaches its value under the linear regulator problem so that $\hat{x}_t$ tends to $\hat{x}_t^{LR}$. This means that the first two terms in (4) are identical to the augmented linear regulator problem so that

$$u^{\alpha}\left[\hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0\right] + u_i^{\alpha}\left[\hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0\right]\left[\hat{x}_t - \hat{x}_t^{LR}\right]^i =$$

$$-\left[F_y + R^{-1}W_y\right]\hat{y}_t^{LR} - \left[F_y + R^{-1}W_y\right]\left[\hat{y}_t - \hat{y}_t^{LR}\right] - \left[F_z + R^{-1}W_z\right]z_t$$

A similar argument applied to (5) leads to

$$V\left[\hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0\right] + V_i\left[\hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0\right]\left[\hat{x}_t - \hat{x}_t^{LR}\right]^i + \frac{1}{2}V_{ij}\left[\hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0\right]\left[\hat{x}_t - \hat{x}_t^{LR}\right]^i\left[\hat{x}_t - \hat{x}_t^{LR}\right]^j$$

$$= \rho + \left[\hat{x}_t^{LR}\right]'\begin{pmatrix} P_y & P_z \\ P_z & P_{zz} \end{pmatrix}\left[\hat{x}_t^{LR}\right] + \left[\hat{x}_t - \hat{x}_t^{LR}\right]'\begin{pmatrix} P_y & P_z \\ P_z & P_{zz} \end{pmatrix}\left[\hat{x}_t - \hat{x}_t^{LR}\right].$$

Here P_{zz} was not needed for the solution to the linear regulator problem but can be found from the Riccati equation following Hansen and Singleton (1998, p. 162.)

$$P_{zz} = Q_{zz} + A_{yz}'P_yA_{yz} + A_{zz}'P_z'A_{yz} + A_{yz}'P_zA_{zz}$$

$$-\left[A_{yz}'P_yB_y + A_{zz}'P_z'B_y\right]\left[R + B_y'P_yB_y\right]^{-1}\left[B_y'P_yA_{yz} + B_y'P_zA_{zz}\right] + A_{zz}'P_{zz}A_{zz},$$

which is solved by iterating on P_{zz}. Finally, ρ is found by iterating on

$$\rho_{j+1} = \beta\rho_j + \beta\, trace\,(PGG').$$

The remainder of this section derives an expression for the last term in (4), $u_I^{\alpha}\left[\hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0\right]$. First the impact of the uncertainty on the value function is found by taking the total derivative of the value function (2) with respect to each of the $\frac{q(q+1)}{2}$ variance-covariance terms in Σ_{t+1}. Here q is the number of endogenous state variables which are hidden from the agent.

$$\begin{aligned} V_I\left[\hat{y}_t, z_t, \Sigma_t, \tau\right] = E_t\Big[&\beta V_j\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right]F_I^j\left[u\left[\hat{y}_t, z_t, \Sigma_t, \tau\right], \hat{y}_t, z_t, \Sigma_t, \tau\right] + \\ &\beta V_J\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right]G_I^J\left[u\left[\hat{y}_t, z_t, \Sigma_t, \tau\right], z_t, \Sigma_t, \tau\right]\Big]. \end{aligned} \tag{6}$$

In this equation all the terms are known for the linear regulator problem except V_I. As a result, these equations can be stacked into a vector of $\frac{q(q+1)}{2}$ first order linear difference equations which can be iterated on to yield $V_I\left[\hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0\right]$.

To find the second order effects in the appendix (6) is differentiated with respect to the $q+r$ state variables. When the perturbation vector is set to zero, these derivatives are reduced to

$$V_{Ik}\left[\hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0\right] = E_t\left[\beta V_j\left[\hat{y}_{t+1}^{LR}, z_{t+1}, \Sigma_{t+1}^{LR}, 0\right] F^j_{Ik} + \beta V_{Jl}\left[\hat{y}_{t+1}^{LR}, z_{t+1}, \Sigma_{t+1}^{LR}, 0\right]\left(F^l_k + F^l_\alpha u^\alpha_k\right) G^J_I\right]. \tag{7}$$

Consequently, if (7) are stacked together for each variance-covariance term, then (7) is a first order linear difference equation in the $\frac{q(q+1)(q+r)}{2}$ terms for $V_{Ik}\left[\hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0\right]$.

The results of a change in elements of the variance-covarince matrix on the optimal control can now be calculated. While the calculations are long, the results are simplified since the function G is not influenced by changes in the optimal controls when the perturbation vector is zero.[19] As a result,

$$\begin{aligned} u^\gamma_J\left[\hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0\right] = & -\left[E_t\left[\Pi_{\alpha,\gamma}\left[u_t^{LR}, y_t^{LR}, z_t, 0\right] + \beta V_{ik}\left[\hat{y}_{t+1}^{LR}, z_{t+1}, \Sigma_{t+1}^{LR}, 0\right] F^i_\alpha F^k_\gamma\right]\right]^{-1} \\ & \times\left[E_t\left[\beta V_{iK}\left[\hat{y}_{t+1}^{LR}, z_{t+1}, \Sigma_{t+1}^{LR}, 0\right] F^i_\alpha G^K_J\right]\right], \end{aligned} \tag{8}$$

where the inverse refers to the inverse tensor matrix.[20] When this matrix is negative definite, then the problem has an interior solution so that the bang-bang solution of Balvers and Cosimano (1993) is not present. By substituting (8) into (3) the linear approximation of the optimal controls is complete. Examination of (8) reveals that the variance-covariance matrix for the hidden state variables influences the decisions of the agent through its influence on the value function for the agent's problem. The chain rule implies that there are two parts to this effect. First, the change in uncertainty changes the variance-covariance matrix through the Kalman filter, G^K_J. Next the variance-covariance matrix impacts the evaluation of how the control vector influences the marginal future value of the state vector, $V_{iK}F^i_\alpha$. Both of these effects are manifested through the change in uncertainty on the marginal value of the state

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[19] Equation (8) is derived in the appendix.

[20] See Judd (1998, p. 500).

vector, V_{iK} based on (7). These effects work through the impact on the value function in this respect the results are similar to Ito's lemma in continuous time stochastic control.

5 An Example

Balvers and Cosimano (1990) use optimal experimentation to explain the slow adjustment of prices for a firm with some monopoly power. Subsequently, Cosimano, Emmons, Lee and Sheehan (2002) apply this argument to a bank to explain the slow adjustment of loan and deposit rates to changes in the treasury bill rate. The presence of monopoly power can be rationalized based on switching cost for the bank's customers along the lines of Klemperer (1995). The switching cost implies that the demand for loans is dependent on market share. The market share is represented by the presence of lagged loans and deposits in the demand for loans and supply of deposits, respectively. The demand for loans is also dependent on the bank's rate relative to the average loan rate in the market. The supply of deposits is also dependent on the relative deposit rate.[21]

These generalizations of the original Balvers and Cosimano model yields an optimal experimentation problem in which there are two control variables, eleven state variables including lagged dependent variables, and two signals used to estimate four parameters. Yet, the solution can be approximated quickly on a standard computer.

The bank sees the demand for loans

$$L_t = l_{0,t} + l_1 L_{t-1} - l_{2,t}\left[r^L_{t,i} - r^L_t\right] + \epsilon_{t,1}.$$

and the supply of deposits

$$D_t = d_{0,t} + d^1_1 D_{t-1} + d_{2,t}\left[r^D_{t,i} - r^D_t\right] + \epsilon_{t,2}.$$

Here define L_t as the demand for loans by the *ith* bank at time t; $\epsilon_{t,1}$ is the random change

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[21] See Varian (1980) for a model of monopolistically competitive market in which a distribution in price represents an equilibrium strategy. The model only represents the decision problem of an individual bank for simplicity. McGrattan (1994) could be used to introduce strategic considerations but this would be beyond the scope of this paper.

in the demand for loans for the *ith* bank at time t;[22] $r^L_{t,i}$ is the *ith* bank's loan rate at time t; $r^L_t \equiv \frac{1}{N-1}\sum_{j=1,j\neq i}^{N} r^L_{t,j}$ is the average loan rate in the bank's market at time t excluding this institution, where N is the number of competitor banks; D_t represents the supply of deposits to the *ith* bank at time t; $r^D_{t,i}$ is the *ith* bank's deposit rate at time t; $\epsilon_{t,2}$ is the random change in the supply of deposit for the *ith* bank at time t; $r^D_t \equiv \frac{1}{N-1}\sum_{j=1,j\neq i}^{N} r^D_{t,j}$ is the average deposit rate in the bank's market at time t excluding this institution, where N is the number of banks in the market;

The bank observes the quantity of loans and deposits but does not know the true slope and intercepts for the demand for loans and supply of deposits. The intercepts are autoregressive to represent the consumers who are not sensitive to changes in interest rates. As a result the bank sees the two signals

$$s_1 = l_{0,t} - \tau_L \epsilon_{t,5}\left(r^L_{t,i} - r^L_t\right) + \epsilon_{t,1} \text{ and } s_2 = d_{0,t} - \tau_D \epsilon_{t,6}\left(r^D_{t,i} - r^D_t\right) + \epsilon_{t,2}.$$

The bank choose loan and deposit rates which maximizing profits

$$\left(r^L_{t,i} - r_t - C^L_t\right) L_t + \left(r_t(1-\alpha) - r^D_{t,i} - C^D_t\right) D_t$$

subject to the demand for loans and the supply of deposits. The r_t is the treasury bill rate; α is the reserve ratio and C^L_t, C^D_t are the marginal resource cost of loans and deposits, respectively. The control vector is $(\, r^L_{t,i} \quad r^D_{t,i}\,)'$, the endogenous state vector is $(\, C^L_t \quad C^D_t \quad L_{t-1} \quad D_{t-1} \quad l_{0,t} \quad d_{0,t}\,)'$ and the exogenous state vector is $z_t \equiv (\, r^L_t \quad r^D_t \quad r_t \quad 1\,)'$. The matrices in the augmented linear regulator problem are

$$R \equiv \begin{pmatrix} l_2 & 0 \\ 0 & d_2 \end{pmatrix}; W_z \equiv \begin{pmatrix} -l_2 & 0 & -l_2 & 0 \\ 0 & -d_2 & -d_2(1-\alpha) & 0 \end{pmatrix}; W_y \equiv \begin{pmatrix} -l_2 & 0 & -l_1 & 0 & -1 & 0 \\ 0 & d_2 & 0 & d_1 & 0 & 1 \end{pmatrix};$$

Q_{yy} is zero; A_{zz} has roots less than one;

$$Q_{yy} \equiv \frac{1}{2}\begin{pmatrix} 0 & 0 & l_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & d_1 & 0 & 1 \\ l_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad Q_{zz} \equiv \frac{1}{2}\begin{pmatrix} 0 & 0 & l_2 & 0 \\ 0 & 0 & d_2(1-\alpha) & 0 \\ l_2 & d_2(1-\alpha) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

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[22] The normality assumption cannot be strictly true since the demand for loans could be negative. To avoid this possibility the normality assumption could be dropped, as long as, the bank cares about the mean and variance of the state variables. See Ljung qvist and Sargent (2000) for the derivation of the Kalman Filter under this case.

$$Q_{yz} \equiv \frac{1}{2}\begin{pmatrix} l_2 & 0 & 0 & 0 \\ 0 & -d_2 & 0 & 0 \\ 0 & 0 & l_1 & 0 \\ 0 & 0 & -d_1(1-\alpha) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -(1-\alpha) & 0 \end{pmatrix}; \quad A_{yy} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & l_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & d_1 & 0 & 1 \\ 0 & 0 & 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{22} \end{pmatrix};$$

$$A_{yz} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ l_2 & 0 & 0 & 0 \\ 0 & -d_2 & 0 & 0 \\ 0 & 0 & 0 & l_0 \\ 0 & 0 & 0 & d_0 \end{pmatrix}; \quad B_y \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -l_2 & 0 \\ 0 & d_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad G_{yy} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_1 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & \sigma_2 & 0 & \sigma_4 \\ 0 & 0 & 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_4 \end{pmatrix};$$[23]

$C_{sy} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$; C_{sz} and D are zero. $H \equiv \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$, $V_3 = V_4 \equiv \begin{pmatrix} \sigma_5^2 & 0 \\ 0 & \sigma_6^2 \end{pmatrix}$, $V_5' \equiv \begin{pmatrix} \sigma_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & 0 & 0 & 0 & 0 \end{pmatrix}$.[24]

The parameter values for this model are listed in Table 1. The parameter estimates are based on monthly data from 1993-1999 which was taken from a financial institution in a large metropolitan area.[25] Loan commitments are used for the loan demand and savings accounts are used for deposits. Both accounts are highly persistent with the expected dependence on the spread between bank rates and market rates for the metropolitan area. The exogenous state variable, z_t, is represented by the VAR model in Tables 2 and 3. This state vector includes the market rates for loan commitments, savings deposit, interest bearing checking accounts and treasury bills. One month lag in each of the interest rate variables was sufficient to generate white noise errors.

The first step of the simulation procedure is to implement the doubling algorithm of Hansen and Sargent (1998) and Anderson *et. al.* (1996). This procedure generates the augmented linear regulator solution. The behavior of the bank's loan and deposit rates are portrayed by the squares in Figures 1 and 2, respectively. For these simulations 11 random draws from the normal distribution each month were used to generate the movement in the state vector, x_t from month to month. The first month is an initial value for the state vector which approximates a steady state. In Table 4 the state vectors for the first and sixth month are recorded for the

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[23] ${}^{2}{}_{t,3}$ and ${}^{2}{}_{t,4}$ are random shocks to the intercepts $\mathsf{l}_{0,t}$ and $\mathsf{d}_{0,t}$, respectively.

[24] The stochastic specification is slightly different since there is a variance-covariance matrix, V_5, between w_{1t+1} and w_{2t}. This causes the Kalman filter to change by replacing $\mathsf{A}\Sigma_t\mathsf{C}'$ with $\mathsf{A}\Sigma_t\mathsf{C}' + \mathsf{V}_5$

[25] Rich Sheehan provided these estimates.

linear regulator problem in columns 2 and 4. The optimal loan and deposit rate for the bank are listed in Table 5. The loan rate is about 24 basis points below the market average in the sixth month, while the deposit rate is about .87% above the market average.

In Table 6 the Kalman gain starts at about .5 for both rates and after six months drops to about .3 and .08 for the loan and deposit rate, respectively. The initial value of the Kalman gain can be manipulated by changing the variance of the constant relative to the variance in the regression. By lowering the variance of the constant the conditional variance of the constant decreases relative to the variance of the regression which leads to a decrease in the Kalman gain. As the conditional variance of the constant for loans decreases over the six months, as seen in Table 7 row 2, the Kalman gain decreases which increases the convergence of this conditional variance.

In Table 8 the marginal value of the state variable, V_i is listed. These results are consistent with intuition. An increase in cost reduces the value of the bank, while an increase in demand increases its value. In summary the linear regulator solution behaves in a consistent and intuitive way.

The conditional variance has a positive impact on the value of the bank in Table 9. Balvers and Cosimano (1990) show that the bank has an increasing return to uncertainty in the intercept. As a result, Jensen's inequality implies that higher uncertainty leads to an increase in the value of the bank.

We can now examine the impact of optimal experimentation on the behavior of the bank. The simulation starts at the same initial values so that the first occurrence of experimenting is in month 2. This behavior is pictured in Figures 1 and 2 for the loan and deposit rates. The rhombus represents the optimal experimentation. The loan rate initially goes below the benchmark linear regulator solution by 35 basis points and goes about 7 basis points above by the sixth month. This increases the uncertainty in the regression which lowers the Kalman Gain to .08 in the second month and converges to the linear regulator value by the sixth month as seen in Table 6 row 1. As a result the conditional variance for the loan intercept converges faster to a higher value which can be seen in Table 7 column 3. The same experimentation occurs for the deposit rate except that the deposit rate is higher since it is a source of cost rather than revenue.

In Table 10 we can see the effect of the conditional uncertainty on the optimal decisions of the bank. An increase in the conditional variance of the demand for loan intercept tends to increase the loan rate while the increase in the condition variance for deposits lowers the deposit rate. This result works through the impact of the control variable on the state vector, F^i_α, in equation (10). An increase in the bank's loan rate decreases demand while the deposit rate increases supply. These effects follow from the marginal value of the state vector, V_i in Table 8, interacting with, F^j_{Ik}. The sign of these partial derivatives determines the negative influence of V_{Ik} through equation (8), which in turn by equation (10) implies the partial derivative of the control variable with respect to its conditional variance.

The behavior of the loan rate in Figure 1 can now be understood. The conditional variance of the demand for loan intercept under experimentation is initially below the linear regulator case so that the loan rate is lower. By the sixth month this conditional variance is higher under optimal experimentation and its marginal impact on the optimal loan rate has also fallen. As a result the loan rate is now slightly above the linear regulator case. This continues till the 30^{th} month when the loan rate is only 1 basis point above the linear regulator case.

The same basic pattern occurs for the deposit rate. The deposit rate first goes above the linear regulator case by 65 basis points. By the sixth month the spread is down to 7 basis points and goes below the linear regulator deposit rate by 3 basis points in the 8^{th} month. The conditional variance of the supply of deposit intercept converges quicker to a smaller value.

These simulations confirm most of the qualitative results of Balvers and Cosimano (1990). The main new result is that after experimenting for a short period of time, the optimal loan rate goes above the loan rate under optimal experimenting. This new insight occurs since it is now possible to analyze optimal experimentation problems in a setting with changing state variables as well as more complex dynamic linear regulator problems. In addition, these simulations take less than a minute on a standard PC with a Pentium II 400 MHz chip. The estimation of the parameters of these models would involve the repeated solution of the algorithm as the parameters are changed to optimize a likelihood function. Thus, it is now feasible to estimate complex models of optimal experimentation.

6 Conclusion

This paper has developed a procedure for approximating optimal experimentation problems for the class of augmented linear regulator problems. This procedure uses the perturbation method of Judd and Gaspar (1997), Judd (1998), and Jin and Judd (2002). The optimal learning problem within the context of the linear regulator problem is modified by introducing parameters into the conditional variance-covariance matrix. These parameters introduce the possibility that either the control variables or the exogenous state vector can influence this variance-covariance matrix. This parameterization of the optimal experimentation problem includes all the examples seen in the literature such as Wieland (2000b, 2002), as well as more complex problems. When these parameters are zero, the optimal experimentation problem reduces to the optimal learning problem which has a well-defined solution. Thus, the perturbation procedure can be used to find the first order approximation of the optimal decision of the agents and the second order approximation of the value function.[26]

The optimal decision under experimentation, (4), is a linear combination of the usual solution found by iterating on the Riccati equation and a term that captures the effect of uncertainty on the value function of the agent, (10). This second term uses four matrices as inputs which consist of derivatives of the equations of motion for the state vector and the conditional variance-covariance matrix from the Kalman Filter. The formula's for these matrices are provided in the appendix. As a result optimal experimentation can be analyzed for any augmented linear regulator problem. To implement this program: first define the matrices for your particular problem as in Hansen and Sargent (1998). Second, apply the formulas given in the appendix for the four matrices in equations (8) and (10). Third, iterate on the first order difference equation, (8), to measure the impact of the conditional variance-covariance matrix

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[26] This procedure could be used to find higher order approximations, however most empirical problems would only find second moments to be significant.

on the marginal value of each state variable. The Final step implements equation (10), which yields the effect of optimal experimentation on the optimal decision of the agent.

Implementation of this algorithm allows for the empirical evaluation of optimal experimentation on the optimal decisions of agents. Once the optimal decision for a particular problem is known, such as the optimal loan and deposit rate decisions found in section 5, the estimation procedure of Anderson *et. al.* (1996) and Hansen and Sargent (1998) can be modified by replacing the linear regulator solution with the optimal experimentation solution. The estimates of the underlying parameters are found by optimizing the likelihood function built on this algorithm. It is feasible to estimate the effect of experimentation since each iteration on this algorithm takes less than a minute on a standard PC. Thus, the impact of optimal experimentation on optimal decisions of agents can be accomplished for a large class of applied economic problems.

7 Appendix

7.1 Derivatives of F.

Let u_t have dimension $px1$, y_t have dimension $qx1$, z_t have dimension $rx1$, and a_t have dimension $sx1$. The matrices have the dimensions so that product is well defined.

$$\frac{\partial F}{\partial u_t} = vec\left(\left[B_y', 0'\right]\right) + vec\left(\left[\frac{\partial K_t a_t}{\partial u_t}, 0'\right]\right).$$[27]

$K_t a_t$ is dependent on V_t. As a result, look at

$$\frac{\partial V_t}{\partial u_t} = \frac{\partial \tau_1 u_t'}{\partial u_t} V_3 u_t \tau_1' + \left(\tau_1 u_t' \otimes I_p\right)\left(V_3 \otimes I_p\right) \frac{\partial u_t \tau_1'}{\partial u_t}$$

which is zero for $\tau_1 = 0$.[28] Thus, $\frac{\partial K_t a_t}{\partial u_t} = 0$, so that $\frac{\partial F}{\partial u_t} = vec\left(\left[B_y', 0'\right]\right)$.

$$\frac{\partial F}{\partial \hat{y}_t} = vec\left(\left[A_{yy}', 0'\right]\right) + vec\left(\left[\frac{\partial K_t a_t}{\partial \hat{y}_t}, 0'\right]\right)$$

$$\frac{\partial \begin{pmatrix} K_t a_t \\ 0 \end{pmatrix}}{\partial \hat{y}_t} = -\left(\begin{pmatrix} K_t \\ 0 \end{pmatrix} \otimes I_q\right)(C_{sy} \otimes I_q) Vec(I_q).$$[29]

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[27] See Theorem 6.3 p.43 and item 4 p.50 of Rogers (1980).

[28] See Theorem 6.4 from Rogers (1980, p. 43).

[29] See item 16 Rogers (1980,p.53).

Here 0 has dimension rxs. Thus,

$$\frac{\partial F}{\partial \hat{y}_t} = vec\left(\left[A'_{yy}, 0'\right]\right) - \left(\begin{pmatrix} K_t \\ 0 \end{pmatrix} \otimes I_q\right)(C_{sy} \otimes I_q) vec\left(I_q\right).$$

$$\frac{\partial F}{\partial z_t} = vec\left(\left[A'_{yz}, A'_{zz}\right]\right) + vec\left(\left[\frac{\partial K_t a_t}{\partial z_t}, 0'\right]\right).$$

$K_t a_t$ is dependent on z_t through V_t, as a result look at

$$\frac{\partial V_t}{\partial z_t} = \frac{\partial \tau_2 z'_t}{\partial z_t} V_4 z_t \tau'_2 + \left(\tau_2 z'_t \otimes I_r\right)\left(V_4 \otimes I_r\right) \frac{\partial z_t \tau'_2}{\partial z_t}$$

which is zero for $\tau_2 = 0$. Thus, $\frac{\partial K_t a_t}{\partial z_t} = 0$, so that $\frac{\partial F}{\partial z_t} = vec\left(\left[A'_{yz}, A'_{zz}\right]\right)$.

$$\frac{\partial F}{\partial \Sigma_t} = \begin{pmatrix} \frac{\partial K_t a_t}{\partial \Sigma_t} \\ 0 \end{pmatrix}.$$

where 0 has dimension $qrxq$. $K_t a_t$ is the product of three matrices $X \equiv A_{yy}\Sigma_t C'_{sy}$, $Y \equiv \left(C_{sy}\Sigma_t C'_{sy} + HH'\right)^{-1}$ and $Z \equiv a_t$. By the product rule for differentiation of matrices[30]

$$\frac{\partial K_t a_t}{\partial \Sigma_t} = \frac{\partial X}{\partial \Sigma_t}\left(Y \otimes I_q\right)\left(Z \otimes I_q\right) + \left(X \otimes I_q\right)\frac{\partial Y}{\partial \Sigma_t}\left(Z \otimes I_q\right).$$

Next,

$$\frac{\partial X}{\partial \Sigma_t} = \frac{\partial A_{yy}\Sigma_t C'_{sy}}{\partial \Sigma_t} = vec(A'_{yy})vec(C'_{sy})',^{31}$$

which has dimension q^2xqs.

$$\frac{\partial Y}{\partial \Sigma_t} = -\left(\left(C_{sy}\Sigma_t C'_{sy} + HH'\right)^{-1} \otimes I_q\right) vec(C'_{sy})vec(C_{sy})'\left(\left(C_{sy}\Sigma_t C'_{sy} + HH'\right)^{-1} \otimes I_q\right),^{32}$$

which has dimension $sqxsq$.

$$\begin{aligned}\frac{\partial K_t a_t}{\partial \Sigma_t} = {} & vec(A'_{yy})vec(C'_{sy})'\left(\left(C_{sy}\Sigma_t C'_{sy} + HH'\right)^{-1} \otimes I_q\right)\left(a_t \otimes I_q\right) \\ & - \left(A_{yy}\Sigma_t C'_{sy} \otimes I_q\right)\left(\left(C_{sy}\Sigma_t C'_{sy} + HH'\right)^{-1} \otimes I_q\right) \\ & vec(C'_{sy})vec(C_{sy})'\left(\left(C_{sy}\Sigma_t C'_{sy} + HH'\right)^{-1} \otimes I_q\right)\left(a_t \otimes I_q\right).\end{aligned}$$

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[30] a_t is not dependent on Σ_t.

[31] See item 13, Rogers (1980, p.52).

[32] See item 4 Rogers (1980, p.50).

Next look at the effect of the perturbation vector

$$\frac{\partial F}{\partial \tau_1} = \begin{pmatrix} \frac{\partial K_t a_t}{\partial \tau_1} \\ 0 \end{pmatrix}.$$

$K_t a_t$ is dependent on τ_1 through V_t, as a result look at

$$\frac{\partial V_t}{\partial \tau_1} = \frac{\partial \tau_1 u_t'}{\partial \tau_1} V_3 u_t \tau_1' + \left(\tau_1 u_t' \otimes I_p\right)\left(V_3 \otimes I_p\right) \frac{\partial u_t \tau_1'}{\partial u_t}$$

which is zero for $\tau_1 = 0$. Thus, $\frac{\partial K_t a_t}{\partial \tau_1} = 0$, so that $\frac{\partial F}{\partial \tau_1} = 0$. It also follows immediatly that $\frac{\partial F}{\partial \tau_2} = 0$

Now turn to the second order derivatives.

$$\frac{\partial^2 F}{\partial u_t \partial \Sigma_t} = \begin{pmatrix} \frac{\partial^2 K_t a_t}{\partial u_t \partial \Sigma_t} \\ 0 \end{pmatrix}.$$

$\frac{\partial V_t}{\partial u_t}$ is independent of Σ_t so that all the terms in this second order derivatives are dependent on this derivative. Thus, $\frac{\partial^2 F}{\partial u_t \partial \Sigma_t} = 0$.

$$\frac{\partial^2 F}{\partial y_t \partial \Sigma_t} = -\frac{\partial \begin{pmatrix} K_t \otimes I_q \\ 0 \otimes I_q \end{pmatrix}}{\partial \Sigma_t} \left[(C_{sy} \otimes I_q) Vec(I_q) \otimes I_s\right] =$$

$$-\begin{pmatrix} \left(I_{(q,q)} \otimes I_q\right)\left(I_q \otimes \frac{\partial K_t}{\partial \Sigma_t}\right)\left(I_{(s,q)} \otimes I_q\right) \\ 0 \end{pmatrix} \left[(C_{sy} \otimes I_q) Vec(I_q) \otimes I_q\right],^{33}$$

where $I_{(s,q)}$ is the commutation matrix and 0 has dimension $q^2 r \times q^2$. The partial derivative is

$$\frac{\partial K_t}{\partial \Sigma_t} = \left[vec(A_{yy}') - (K_t \otimes I_s)\, vec(C_{sy}')\right] vec(C_{sy})' \left(\left(C_{sy} \Sigma_t C_{sy}' + HH'\right)^{-1} \otimes I_s\right)$$

$$= \left[vec(A_{yy}') - vec(C_{sy} K_t')\right] vec(C_{sy})' \left(\left(C_{sy} \Sigma_t C_{sy}' + HH'\right)^{-1} \otimes I_s\right).^{34}$$

The final second order derivative is

$$\frac{\partial^2 F}{\partial \Sigma_t^2} = \begin{pmatrix} \frac{\partial^2 K_t a_t}{\partial \Sigma_t^2} \\ 0 \end{pmatrix}$$

where 0 has dimension $q^2 r x q^2$.

---

[33] See Theorem 6.6 of Rogers (1980, p. 45).

[34] By Theorem 4.1 of Rogers(1980, p. 21), $\mathsf{vec}(XYZ) = (Z' \otimes X)\mathsf{vec}(Y)$.

Let $S \equiv \left(I_{(q,s)} \otimes I_q\right)\left(I_q \otimes \frac{\partial\left(C_{sy}\Sigma_t C'_{sy}+HH'\right)^{-1}}{\partial \Sigma_t}\right)\left(I_{(s,q)} \otimes I_q\right)$ and $T \equiv \left(I_{(q,q)} \otimes I_q\right) \times \left(I_q \otimes vec(A'_{yy})vec(C'_{sy})'\right)\left(I_{(s,q)} \otimes I_q\right)$ so that

$$\frac{\partial^2 K_t a_t}{\partial \Sigma_t^2} = \left[vec(A'_{yy})vec(C'_{sy})' \otimes I_q\right] S\left[(a_t \otimes I_q) \otimes I_q\right]$$

$$-T\left[\left(\left(C_{sy}\Sigma_t C'_{sy} + HH'\right)^{-1} \otimes I_q\right) \otimes I_q\right]$$

$$\times\left[\left(vec(C'_{sy})vec(C'_{sy})'\left(\left(C_{sy}\Sigma_t C'_{sy} + HH'\right)^{-1} a_t \otimes I_q\right)\right) \otimes I_q\right]$$

$$-\left[\left(A_{yy}\Sigma_t C'_{sy} \otimes I_q\right) \otimes I_q\right] S$$

$$\times\left[\left(vec(C'_{sy})vec(C'_{sy})'\left(\left(C_{sy}\Sigma_t C'_{sy} + HH'\right)^{-1} a_t \otimes I_q\right)\right) \otimes I_q\right]$$

$$-\left[\left(A_{yy}\Sigma_t C'_{sy} \otimes I_q\right)\left(\left(C_{sy}\Sigma_t C'_{sy} + HH'\right)^{-1} \otimes I_q\right) \otimes I_q\right]$$

$$\left[\left[vec(C'_{sy})vec(C'_{sy})' \otimes I_q\right] S\left[(a_t \otimes I_q) \otimes I_q\right]\right],$$

where the partial derivative is calculated in $\frac{\partial Y}{\partial \Sigma_t}$.

7.2 Derivatives of G

u_t, z_t, and τ only effect G through the Kalman Gain which in turn is influenced by V_t. As a result $\frac{\partial G}{\partial u_t}$, $\frac{\partial G}{\partial z_t}$ and $\frac{\partial G}{\partial \tau}$ are all zero. Next,

$$\frac{\partial G}{\partial \Sigma_t} = vec(A'_{yy})vec(A'_{yy})'$$

$$-vec(A'_{yy})vec(C'_{sy})'\left[\left(C_{sy}\Sigma_t C'_{sy} + HH'\right)^{-1} \otimes I_q\right]\left[C_{sy}\Sigma_t A'_{yy} \otimes I_q\right]$$

$$+\left[A_{yy}\Sigma_t C'_{sy} \otimes I_q\right]\left(\left(C_{sy}\Sigma_t C'_{sy} + HH'\right)^{-1} \otimes I_q\right) vec(C'_{sy})vec(C_{sy})'$$

$$\times\left(\left(C_{sy}\Sigma_t C'_{sy} + HH'\right)^{-1} \otimes I_q\right)\left[C_{sy}\Sigma_t A'_{yy} \otimes I_q\right]$$

$$-\left[A_{yy}\Sigma_t C'_{sy} \otimes I_q\right]\left[\left(C_{sy}\Sigma_t C'_{sy} + HH'\right)^{-1} \otimes I_q\right] vec(C'_{sy})vec(A'_{yy})'.$$

Let $U \equiv \left(I_{(q,s)} \otimes I_q\right)\left(I_q \otimes vec(C'_{sy})vec(A'_{yy})'\right)\left(I_{(q,q)} \otimes I_q\right)$ so that the second order partial derivative is

$$\frac{\partial^2 G}{\partial \Sigma_t^2} = -\left[vec(A'_{yy})vec(C'_{sy})' \otimes I_q\right] S\left[\left(C_{sy}\Sigma_t A'_{yy} \otimes I_q\right) \otimes I_q\right]$$

$$
\begin{gathered}
-\Big[\Big(vec(A'_{yy})vec(C'_{sy})'\Big[\Big(C_{sy}\Sigma_t C'_{sy}+HH'\Big)^{-1}\otimes I_q\Big]\Big)\otimes I_q\Big]U \\
-T\Big[\Big(\Big(\Big(C_{sy}\Sigma_t C'_{sy}+HH'\Big)^{-1}\otimes I_s\Big)vec(C'_{sy})vec(C_{sy})' \\
\times\Big(\Big(C_{sy}\Sigma_t C'_{sy}+HH'\Big)^{-1}\otimes I_s\Big)\Big[C_{sy}\Sigma_t A'_{yy}\otimes I_q\Big]\Big)\otimes I_q\Big] \\
+\Big[\Big[A_{yy}\Sigma_t C'_{sy}\otimes I_q\Big]\otimes I_q\Big] \\
\times\Big\{S\Big[\Big(vec(C'_{sy})vec(C_{sy})'\Big(\Big(C_{sy}\Sigma_t C'_{sy}+HH'\Big)^{-1}\otimes I_q\Big)\Big)\otimes I_q\Big] \\
+\Big[\Big(\Big(\Big(C_{sy}\Sigma_t C'_{sy}+HH'\Big)^{-1}\otimes I_q\Big)vec(C'_{sy})vec(C_{sy})'\Big)\otimes I_q\Big]S\Big\} \\
\times\Big[\Big[C_{sy}\Sigma_t A'_{yy}\otimes I_q\Big]\otimes I_q\Big] \\
+\Big\{\Big(\Big[A_{yy}\Sigma_t C'_{sy}\otimes I_q\Big]\Big(\Big(C_{sy}\Sigma_t C'_{sy}+HH'\Big)^{-1}\otimes I_q\Big)vec(C'_{sy})vec(C_{sy})' \\
\times\Big(\Big(C_{sy}\Sigma_t C'_{sy}+HH'\Big)^{-1}\otimes I_q\Big)\Big)\otimes I_q\Big\}U \\
-T\Big[\Big(\Big[\Big(C_{sy}\Sigma_t C'_{sy}+HH'\Big)^{-1}\otimes I_q\Big]vec(C'_{sy})vec(A'_{yy})'\Big)\otimes I_q\Big] \\
-\Big[\Big[A_{yy}\Sigma_t C'_{sy}\otimes I_q\Big]\otimes I_q\Big]S\Big[vec(C'_{sy})vec(A'_{yy})'\otimes I_s\Big]
\end{gathered}
$$

7.3 Derivation of (7)

The second order effects on the value function are calculated by taking the total differentiation of (6) with respect to the $q+r$ state variables to yield $q+r$ difference equations for each variance-covariance term

$$
\begin{gathered}
V_{Ik}\left[\hat{y}_t, z_t, \Sigma_t, \tau\right] = \\
E_t\Big[\beta V_{jl}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right]\Big(F^l_k\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + F^l_\alpha\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right]u^\alpha_k\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]\Big)F^j_I\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + \\
\beta V_{jL}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right]\Big(G^L_k\left[u_t, z_t, \Sigma_t, \tau\right] + G^L_\alpha\left[u_t, z_t, \Sigma_t, \tau\right]u^\alpha_k\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]\Big)F^j_I\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + \\
\beta V_{Jl}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right]\Big(F^l_k\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + F^l_\alpha\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right]u^\alpha_k\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]\Big)G^J_I\left[u_t, z_t, \Sigma_t, \tau\right] + \\
\beta V_{JL}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right]\Big(G^L_k\left[u_t, z_t, \Sigma_t, \tau\right] + G^L_\alpha\left[u_t, z_t, \Sigma_t, \tau\right]u^\alpha_k\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]\Big)G^J_I\left[u_t, z_t, \Sigma_t, \tau\right] + \\
\beta V_j\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right]\Big(F^j_{Ik}\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + F^j_{I\alpha}\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right]u^\alpha_k\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]\Big) + \\
\beta V_J\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right]\Big(G^J_{Ik}\left[u_t, z_t, \Sigma_t, \tau\right] + G^J_{I\alpha}\left[u_t, z_t, \Sigma_t, \tau\right]u^\alpha_k\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]\Big)\Big].
\end{gathered} \tag{9}
$$

If the perturbation vector is set equal to zero, then by Lemma 1 equation (9) becomes (7). In these calculations, I use the result that $E_t[F^j_I]$ is zero, since $E_t[a_t]=0$.

7.4 Derivation of V_{IJ}

The second order partial derivatives of the value function with respect to the elements of the variance-covariance matrix satisfies

$$\begin{gathered}
V_{IJ}\left[\hat{y}_t, z_t, \Sigma_t, \tau\right] = \\
E_t \Big[\beta V_{jl}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] \Big(F^l_J\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + F^l_\alpha\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] u^\alpha_J\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]\Big) F^j_I\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + \\
\beta V_{jL}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] \Big(G^L_J\left[u_t, z_t, \Sigma_t, \tau\right] + G^L_\alpha\left[u_t, z_t, \Sigma_t, \tau\right] u^\alpha_J\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]\Big) F^j_I\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + \\
\beta V_j\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] \Big(F^j_{IJ}\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + F^j_{I\alpha}\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] u^\alpha_J\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]\Big) + \\
\beta V_{Kl}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] \Big(F^l_J\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + F^l_\alpha\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] u^\alpha_J\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]\Big) G^K_I\left[u_t, z_t, \Sigma_t, \tau\right] + \\
\beta V_{KL}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] \Big(G^L_J\left[u_t, z_t, \Sigma_t, \tau\right] + G^L_\alpha\left[u_t, z_t, \Sigma_t, \tau\right] u^\alpha_J\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]\Big) G^K_I\left[u_t, z_t, \Sigma_t, \tau\right] + \\
\beta V_K\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] \Big(G^K_{IJ}\left[u_t, z_t, \Sigma_t, \tau\right] + G^K_{I\alpha}\left[u_t, z_t, \Sigma_t, \tau\right] u^\alpha_J\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]\Big)\Big].
\end{gathered} \tag{10}$$

These second order partial derivatives are dependent on $u^\alpha_J\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]$, however, it will turn out that these partial derivatives can be calculated independent of (10) when the perturbation vector is zero. Once this is complete, (10) may be stacked into a $\frac{q^2(q-1)^2}{4}$ vector to yield a first order linear difference equation in $V_{IJ}\left[\hat{y}^{LR}_t, z_t, \Sigma^{LR}_t, 0\right]$.

7.5 Derivation of (8)

To find (8) first take the total differentiation of the Euler condition with respect to Σ_t for each control variable α

$$\begin{gathered}
E_t \big[\Pi_{\alpha,\gamma}\left[u_t, y_t, z_t, \tau\right] u^\gamma_J\left[\hat{y}_t, z_t, \Sigma_t, \tau\right] + \\
\beta V_{ik}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] F^i_\alpha\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] \Big(F^k_J\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + F^k_\gamma\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] u^\gamma_J\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]\Big) + \\
\beta V_{iK}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] F^i_\alpha\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] \Big(G^K_J\left[u_t, z_t, \Sigma_t, \tau\right] + G^K_\gamma\left[u_t, z_t, \Sigma_t, \tau\right] u^\gamma_J\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]\Big) + \\
\beta V_i\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] \Big(F^i_{\alpha J}\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + F^i_{\alpha\gamma}\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] u^\gamma_J\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]\Big) + \\
\beta V_{Ik}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] G^I_\alpha\left[u_t, z_t, \Sigma_t, \tau\right] \Big(F^k_J\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + F^k_\gamma\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] u^\gamma_J\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]\Big) + \\
\beta V_{IK}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] G^I_\alpha\left[u_t, z_t, \Sigma_t, \tau\right] \Big(G^K_J\left[u_t, z_t, \Sigma_t, \tau\right] + G^K_\gamma\left[u_t, z_t, \Sigma_t, \tau\right] u^\gamma_J\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]\Big) + \\
\beta V_I\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] \Big(G^I_{\alpha J}\left[u_t, z_t, \Sigma_t, \tau\right] + G^I_{\alpha\gamma}\left[u_t, z_t, \Sigma_t, \tau\right] u^\gamma_J\left[\hat{y}_t, z_t, \Sigma_t, \tau\right]\Big)\Big].
\end{gathered} \tag{11}$$

(11) can be solved for u^{α}_{J} to yield

$$\begin{gathered}
u^{\alpha}_{J}\left[\hat{y}_t, z_t, \Sigma_t, \tau\right] = -\left\{E_t(\Pi_{\alpha,\gamma}\left[u_t, y_t, z_t, \tau\right] + \right. \\
\beta V_{ik}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] F^{i}_{\alpha}\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] F^{k}_{\gamma}\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + \\
\beta V_{iK}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] F^{i}_{\alpha}\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] G^{K}_{\gamma}\left[u_t, z_t, \Sigma_t, \tau\right] + \\
\beta V_{i}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] F^{i}_{\alpha\gamma}\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + \\
\beta V_{Ik}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] G^{I}_{\alpha}\left[u_t, z_t, \Sigma_t, \tau\right] F^{k}_{\gamma}\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + \\
\beta V_{IK}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] G^{I}_{\alpha}\left[u_t, z_t, \Sigma_t, \tau\right] G^{K}_{\gamma}\left[u_t, z_t, \Sigma_t, \tau\right] + \\
\left. \beta V_{I}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] G^{I}_{\alpha\gamma}\left[u_t, z_t, \Sigma_t, \tau\right]\right\}^{-1} X \\
\Big\{E_t\Big[\beta V_{ik}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] F^{i}_{\alpha}\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] F^{k}_{J}\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + \\
\beta V_{iK}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] F^{i}_{\alpha}\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] G^{K}_{J}\left[u_t, z_t, \Sigma_t, \tau\right] + \\
\beta V_{i}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] F^{i}_{\alpha J}\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + \\
\beta V_{Ik}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] G^{I}_{\alpha}\left[u_t, z_t, \Sigma_t, \tau\right] F^{k}_{J}\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] + \\
\beta V_{IK}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] G^{I}_{\alpha}\left[u_t, z_t, \Sigma_t, \tau\right] G^{K}_{J}\left[u_t, z_t, \Sigma_t, \tau\right] + \\
\beta V_{I}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] G^{I}_{\alpha J}\left[u_t, z_t, \Sigma_t, \tau\right]\Big]\Big\}.
\end{gathered} \tag{12}$$

Equation (8) in the text is found by using Lemma 1 to evaluate the partial derivatives at the linear regulator solution.

7.6 Proof of $V_{\mathcal{I}} = 0$ and $u^{\gamma}_{\mathcal{I}} = 0$.

For each of the perturbation parameters in the vector, τ,

$$\begin{gathered}
V_{\mathcal{I}}\left[\hat{y}_t, z_t, \Sigma_t, \tau\right] = E_t\Big[\Pi_{\mathcal{I}}\left[u_t, y_t, z_t, \tau\right] + \beta V_{j}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] F^{j}_{\mathcal{I}}\left[u_t, \hat{y}_t, z_t, \Sigma_t, \tau\right] \\
+\beta V_{J}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right] G^{J}_{\mathcal{I}}\left[u_t, z_t, \Sigma_t, \tau\right] + \beta V_{\mathcal{I}}\left[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau\right]\Big].
\end{gathered} \tag{13}$$

The partial derivatives, $F^{j}_{\mathcal{I}}$ and $G^{J}_{\mathcal{I}}$, are zero by Lemma 1 when they are evaluated at the linear regulator solution. The partial derivatives of the variance-covariance terms HH' in the Kalman Filter are zero when the perturbation parameters are zero. It follows that $u^{\gamma}_{\mathcal{I}}(\hat{y}^{LR}_{t}, z_t, \Sigma^{LR}_{t}, 0) = 0$ since it is dependent on the partial derivatives $F^{j}_{\mathcal{I}}$ and $G^{J}_{\mathcal{I}}$.

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Table 1: Parameters

Variable	Value	standard deviation	Value
l_0	$4.8601*10^4$	σ_1	$1.4436*10^6$
a_{11}	.9000	σ_2	$1.2017*10^7$
d_0	$2.1029*10^5$	σ_3	$5.9955*10^5$
a_{22}	.9000	σ_4	$7.4062*10^5$
l_1	.9946	σ_5	$1.8423*10^8$
d_1	.9245	σ_6	$2.1667*10^9$
l_2	$9.0943*10^8$		
d_2	$4.3335*10^9$		
α	0		

Table 2: VAR for Exogenous State Vector

State Variable	r^L	r^{D1}	r^{D2}	r	constant
r^L	.5512	-.5415	.5000	.2785	.0470
r^{D1}	.0172	.6793	.4354	.0367	-.00218
r^{D2}	-.01459	.0527	.9075	.0400	-.0006
r	.0352	-.9238	1.2885	.8467	.0094

Table 3: Variance-covariance Matrix for VAR

State Variable	r^L	r^{D1}	r^{D2}	r
r^L	$2.0177*10^{-5}$	$5.5679*10^{-7}$	$1.1708*10^{-7}$	$4.4902*10^{-9}$
r^{D1}		$1.2331*10^{-7}$	$-1.1915*10^{-8}$	$-5.5679*10^{-10}$
r^{D2}			$8.2302*10^{-7}$	$1.0038*10^{-9}$
r				$3.27*10^{-6}$

Table 4: State Vector for 1^{st} and 6^{th} Month

State Variable	x_1	x_1^{LR}	x_6	x_6^{LR}
C^L	0.0600	0.0600	0.0600	0.0600
C^D	0.0100	0.0100	0.0100	0.0100
L	$4.4424*10^7$	$4.4424*10^7$	$9.0006 \ast 10^7$	$8.9132*10^7$
D	$4.4062*10^7$	$4.4062*10^7$	$1.7462 \ast 10^8$	$1.1577*10^8$
$l_{0,t}$	$1.3108*10^6$	$1.3108*10^6$	$2.8550*10^6$	$1.5322*10^6$
$d_{0,t}$	$-5.9325*10^7$	$-5.9325*10^7$	$-3.3591*10^7$	$-3.3592*10^7$
r^L	0.1240	0.1240	0.1216	0.1216
r^{D1}	0.0260	0.0260	0.0247	0.0247
r^{D2}	0.0147	0.0147	0.0134	0.0134
r	0.0503	0.0503	0.0516	0.0516
constant	1	1	1	1

Table 5: Control Vector for 1^{st} and 6^{th} Month

Control Variable	u_1	u_1^{LR}	u_6	u_6^{LR}
r_i^L	0.12212	0.12212	0.11983	0.11917
r_i^{D1}	0.02542	0.02542	0.04131	0.04065

Table 6: Kalman Gain for 1^{st} and 6^{th} Month

Kalman Gain	K_1	K_1^{LR}	K_6	K_6^{LR}
r_i^L	0.21628	0.49558	0.28141	0.29107
r_i^{D1}	0.006741	0.44867	0.0015415	0.077448

Table 7: Variance for 1^{st} and 6^{th} Month

	Σ_1^L	Σ_1^{D1}	Σ_6^L	Σ_6^{D1}
Optimal	$6.0988*10^{11}$	$9.9114*10^{12}$	$8.8885*10^{11}$	$2.21946*10^{12}$
LR	$2.0474*10^{12}$	$1.1751*10^{13}$	$8.5563*10^{11}$	$1.2122*10^{13}$

Table 8: Partial Derivative of Value Function with Respect to State Vector

State Variable	V_i
C^L	$-1.3149 *10^9$
C^D	$-5.5258*10^9$
L	0.1366
D	0.1107
$l_{0,t}$	0.7419
$d_{0,t}$	0.6506
r^L	9.075210^7
r^{D1}	$-4.2130*10^9$
r^{D2}	$-1.0731*10^9$
r	$1.3256*10^9$
constant	$2.4822*10^8$

Table 9: V_I for 1^{st} and 6^{th} Month

	1^{st} Month	6^{th} Month
Σ_1^L	$1.2730*10^{-9}$	$7.5818*10^{-10}$
Σ_1^{D1}	$4.9437*10^{-10}$	$4.9505*10^{-10}$

Table 10: u_J^γ for 1^{st} and 6^{th} Month

	Σ_1^L	Σ_1^{D1}	Σ_6^L	Σ_6^{D1}
r_i^L	$3.6500*10^{-14}$	0	$1.2785*10^{-14}$	0
r_i^{D1}	0	$-4.2285*10^{-17}$	0	$-4.7765*10^{-16}$

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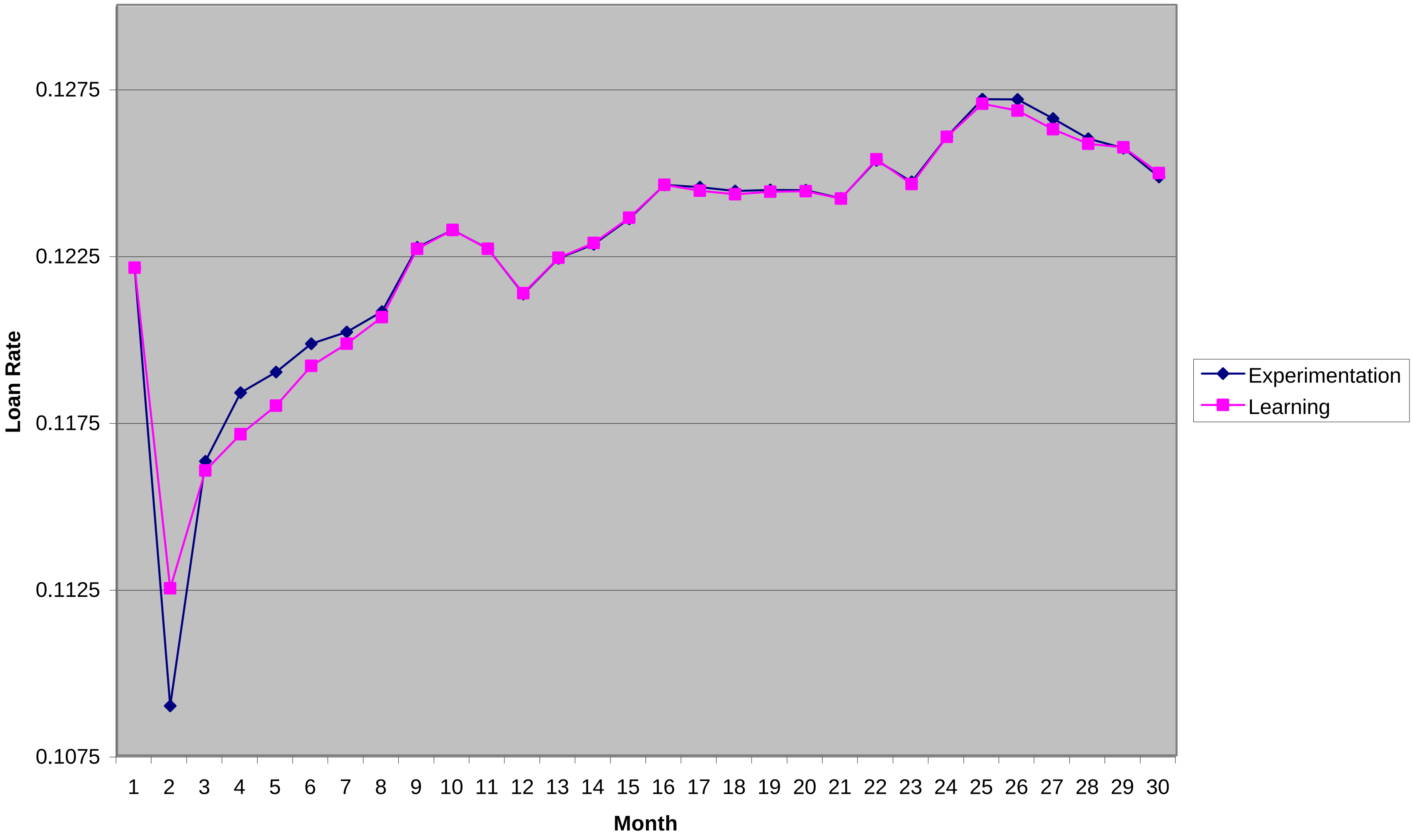
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0.1075
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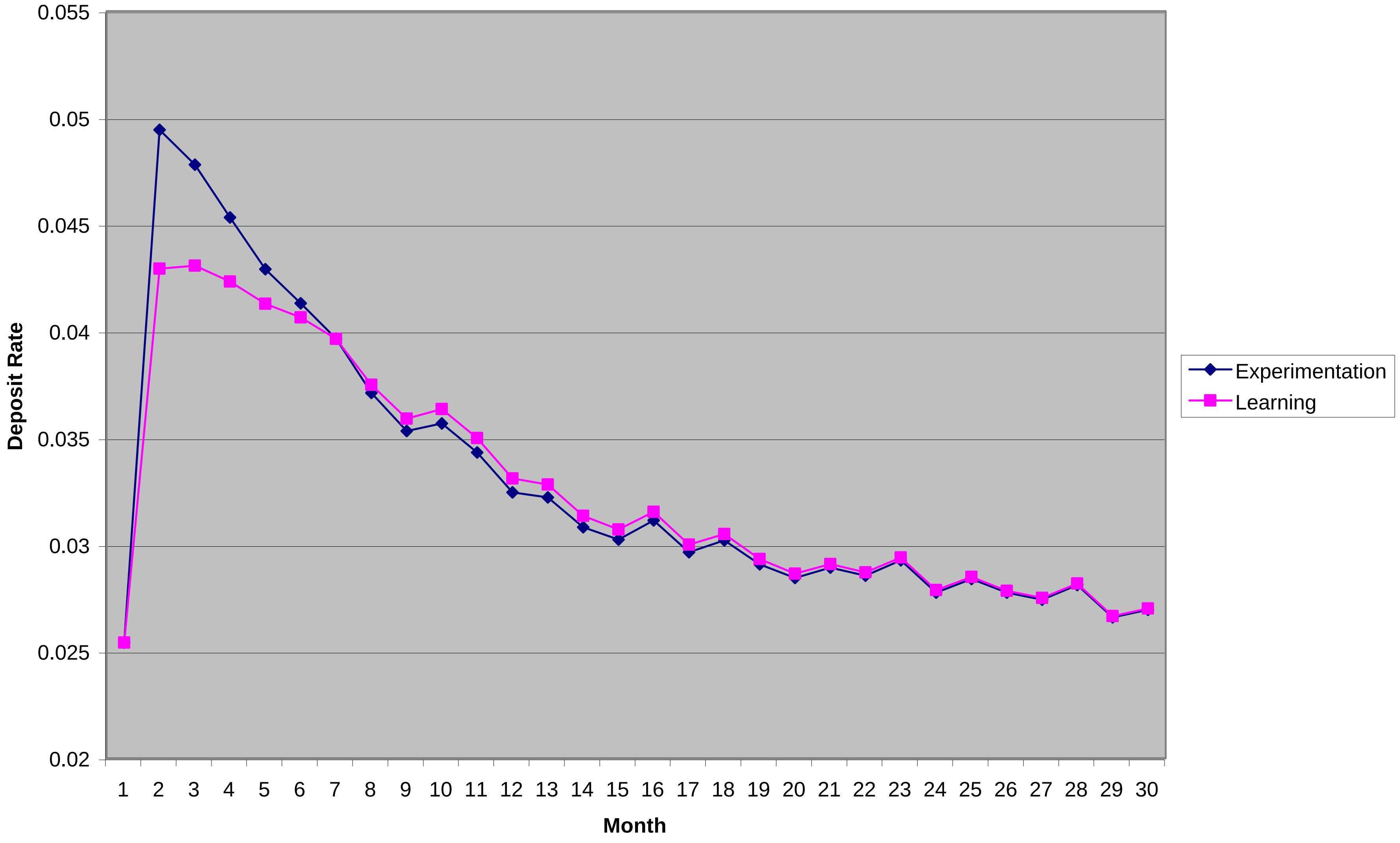
Month
Experimentation
Learning

Figure 2.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0.03
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Deposit Rate
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Month
Experimentation
Learning