Second- and Higher-Order Consumption Functions: A Precautionary Tale

James Feigenbaum*
University of Iowa

November 20, 2002

Abstract

A current problem of interest in macroeconomics is the computation of consumption and saving functions for agents who experience uninsurable income shocks and engage in precautionary saving. With constant relative risk aversion utility, this decision problem cannot be solved analytically. One popular method for computing policy functions has been to use perturbation theory, Taylor-expanding around the special case of perfect certainty, where the problem is exactly solvable, and computing correction terms that are linear in the variance of the income process. Here, I show that it is straightforward to extend this procedure to compute the linear contribution of the skewness. However, in this dynamic environment where perturbations get compounded over time, for each moment of the income process there will be a lower-bound on the interest rate below which the correction terms generated by that moment diverge at large lifetimes. In the limit of large orders, this lower bound on the interest rate converges to the equilibrium interest rate for the corresponding infinite-horizon, constant-growth economy without uncertainty. Since in equilibrium, precautionary saving must lower the interest rate below that limit, perturbation theory must break down for some moment of the income process. That is, the Taylor series will not converge, so adding terms of successively higher order to the series will only improve the approximation to a point, after which adding more terms will worsen the approximation. As an example, I present a case where a second-order approximation to the consumption function, which includes variance effects but not skewness and higher-moment effects, performs substantially worse than a zeroth-order approximation, which entirely ignores the effects of uncertainty.

*I would like to thank James Bullard, Stefano Eusepi, Harry Paarsch, Chris Sleet, and Charles Whiteman for their advice and suggestions.

A problem of interest in macroeconomics is the computation of partial and general equilibria in a bond economy where agents have constant relative risk aversion (CRRA) preferences. This is a model in which there is no aggregate uncertainty, so contingent-claim assets which insure against shocks to the economy as a whole are unnecessary. Risk-free bonds are a sufficient asset to achieve intertemporal transfer needs. It has been hoped that streamlined models of this sort might help to unravel various puzzles regarding consumption and interest rates.

For the trivial case of complete markets and complete certainty, consumption and bond demand functions can be derived analytically. A more interesting problem arises when agents face an uncertain, uninsurable income stream. In this circumstance, agents wishing to hedge against negative income shocks would ideally purchase assets that are perfectly correlated with their income. However, markets are incomplete in this model, and such assets are not present. Therefore, agents must instead substitute risk-free bonds for these nonexistent assets. Although risk-free bonds offer returns uncorrelated with income, agents are, nevertheless, able to use bonds to partially alleviate the welfare loss associated with their idiosyncratic risk. One can show qualitatively that this precautionary saving motive will increase the demand for bonds (Leland (1968), Sandmo (1970)) and decrease interest rates (Aiyagari (1994), Huggett (1993)). However, consumption and bond demands cannot be computed analytically with uninsurable uncertainty if preferences are CRRA. This complicates the quantitative exercise of determining the magnitude of precautionary saving effects. While the original Lifecycle/Permanent-Income Hypothesis (LCPIH), derived under perfect foresight, is generally believed to be a poor description of empirical consumption behavior (Browning and Crossley (2001)), there is no consensus of opinion about how much precautionary saving by itself can account for observed deviations from the LCPIH.

Several researchers (a partial list would include Carroll (1997,2001a), Hall (1988), Letendre and Smith (2001), Skinner (1988), and Viceira (2001)) have investigated this consumption/saving model by perturbing consumption functions or Euler equations[1]. Although the terminology of perturbation theory is fairly new to the economics literature, the technology has been in use for as long as economics has been a mathematical science. A perturbative method is simply a method that involves the approximation of a function by its Taylor expansion. Any linearization procedure is an example of perturbation theory in its most elementary form. More generally, these methods are an integral component of the macroeconomist's toolkit. They are used regularly to express endogenous variables as approximate functions of exogenous variables. Although perturbative methods have been pitched under this name

[1] Another method for solving these kinds of models is the intrinsically numerical approach of computing consumption functions either through value-function or Euler-equation iteration (Deaton (1992)), and aggregating these computed functions to clear markets. Instead of perturbing around the known solution to a solvable problem, these methods proceed by discretizing the state space to a finite number of points and computing the exact value of the relevant function at each point.

most prominently as an adjunct to numerical methods (Judd (1999)), they are fundamentally an analytical method with the power to reveal functional dependences that can only be inferred when numerical values are plugged into exogenous parameters from the start.

This paper takes a detailed look at how perturbative methods can be used in a dynamic context and what new things can go wrong that do not arise in static problems, especially as one pushes on to higher orders. Perturbation theory expresses endogenous variables as power series in a dimensionless parameter, the perturbation parameter, and provides a prescription for computing each coefficient of the series in terms of lower-order coefficients. For small values of the perturbation parameter, the power series can then be approximated by the sum of a finite number of terms. In static problems, perturbation calculations can usually produce an entire infinite series which will converge for any value of the perturbation parameter below some radius of convergence. It then is possible to achieve an approximation to any desired accuracy as long as the perturbation parameter is below this radius. In dynamic problems, that will not always be the case, for pathologies can arise which limit the order to which a perturbation series can be computed. As a result, there can be tighter limits on how much perturbation theory can tell you than would occur in other settings. The mindset that a first or second-order calculation is an initial step in a sequence of calculations which will ultimately converge to an exact answer is overly optimistic.

Under the label of a second-order Taylor expansion, Skinner (1988) has worked out the consumption function for a finite-horizon model with income and interest-rate shocks.[2] Given the assumptions we make regarding the income process, his results are equivalent to a perturbative expansion to second order in the coefficient of variation.[3] This captures the lowest-order effects of the variance and precautionary saving. Here, we compute policy functions to third order for a model with a fixed interest rate and independently distributed income shocks. Note that there is nothing especially remarkable about the third-order contribution, which captures the lowest-order skewness effect. The main contribution here is to show that the cost of performing the third-order calculation is the same as for the second-order calculation. As such, if one is going to compute the second-order contribution, one might as well compute the third-order contribution also. The marginal computational cost only begins to increase again at fourth order, where cross terms first appear.

If we ignore cross terms and higher-order effects, it is straightforward to carry out the procedure used at second and third order to compute the lowest-order contribution of each moment of the income process to value and policy functions. In other words, we compute a part of the nth-order term in the perturbation series, notably the easiest part. Computing this part is enough to

[2] Talmain (1998) has extended these results for a general utility function in the special case where the interest rate equals the discount rate.

[3] The first-order conditions of the optimization problem guarantee that first-order effects must vanish. See Cochrane (1989). This reduction of the welfare loss from a first-order to a second-order effect is the extent to which agents can self-insure with bonds.

unearth a problem: for every n, there is a lower bound on the interest rate below which the nth-order perturbation correction will diverge for large lifetimes. This lower bound increases with n, converging to the equilibrium interest rate for a perfect-foresight infinite-horizon economy in the limit of large n–in the absence of growth, this interest rate will be the discount rate. Since the equilibrium interest rate for a model with precautionary saving must be less than the perfect-foresight interest rate, in general equilibrium the perturbation series will only exist up to a finite number of terms in the infinite horizon. Consequently, for interest rates below the discount rate, perturbative methods cannot be used to compute endogenous variables to arbitrary accuracy. On the contrary, one can find examples where a second-order approximation is substantially worse than a zeroth-order calculation. Fortunately, however, a divergence in the nth-order correction does not diminish the accuracy of lower-order perturbation calculations.

These divergences are related to the singularity in the CRRA utility function at zero consumption. This singularity causes the perturbation series to diverge for agents with low wealth, but one might not expect it to cause a problem for high-wealth agents. In an economy without growth, if interest rates are below the discount rate, then, in the absence of precautionary motives, agents will have no incentive to save. Thus, without uncertainty, all agents will eventually spend their wealth down to zero. This solution without uncertainty corresponds to the zeroth-order solution to the problem, the solution that we are expanding around. This dissipation of wealth in the solution we are expanding around takes agents out of the convergence region of the state space, and that is what causes the perturbation calculation to go wrong. The particular pathology identified in this paper does not arise for interest rates above the discount rate, where the zeroth-order solution takes agents away from zero consumption.

It should be emphasized that this dynamic pathology is entirely a failure of perturbation theory. The exact policy and value functions are well-behaved. Moreover, this failure does not alter the common wisdom that precautionary saving is a second-order effect. The pathology arises because the value function that we insert into the Bellman equation at each stage is itself the result of a perturbation calculation. If we were somehow given the exact value function for an arbitrary period, we could presumably use perturbation theory to compute the value function of the previous period without any difficulty, assuming the agent's wealth is large enough. It is the compounding of perturbation corrections over time that produces these artificial divergences.

The paper proceeds as follows. In Section 1, we introduce the model to be considered here. In Section 2, we discuss our choice of perturbation parameter and how the validity of perturbative methods depends on this choice. In Section 3, we derive the bond demand of agents to third order in the coefficient of variation. In Section 4, we consider the behavior of the lowest-order contribution of the nth moment of the income process at large lifetimes and examine how this can diverge. In Section 5, we conclude with a discussion of the generality of the pathologies found here to other dynamic models and describe other applications of perturbation theory to issues of consumption and saving.

1 The Model

Consider an economy with one consumption good in which agents live for $T+1$ periods. Although the economy will be stationary, we will allow for the possibility of constant economic growth, so behavioral quantities can depend on both absolute time and the age of an agent. Since we will solve the agent's optimization problem by backwards induction, it is convenient to measure the age of an agent in terms of the number of periods remaining in his life rather than by how many periods he has lived. Behavioral quantities like consumption will, therefore, have two time indices: a subscript that refers to absolute time and a superscript that refers to the number of periods remaining in life. For example, c_t^s is the consumption of an agent at time t who has s periods remaining.

An agent born at τ maximizes

$$U_\tau = \sum_{s=0}^{T} \beta^s u(c_{\tau+s}^{T-s}), \tag{1}$$

where the discount rate $\beta \in (0,1)$ and the utility function $u(\cdot)$ has the CRRA form. Given the coefficient of relative risk aversion $\gamma \geq 0$,

$$u(c) = \left\{ \begin{array}{cc} \ln c & \gamma = 1 \\ \frac{1}{1-\gamma} c^{1-\gamma} & \gamma \neq 1 \end{array} \right. . \tag{2}$$

(We will derive all results for the case $\gamma \neq 1$. It is easily shown that endogenous observables vary continuously with γ at $\gamma = 1$.) An agent at t with s periods left will earn a stochastic income endowment $\widetilde{y}_t^s$. We assume that $\widetilde{y}_t^s$ is nonnegative and distributed independently both across time and across agents. We will also assume that, for a given s, $\widetilde{y}_{t'}^s$ and $G^{t'-t}\widetilde{y}_t^s$ will have the same distribution for any t and t', where G is the gross growth rate of the economy. In other words, the distribution of $\widetilde{y}_t^s$ will scale as the growth factor G^t. (We will say more about the distribution of the $\widetilde{y}_t^s$ in the next section.) One intertemporal asset, a risk-free bond, is available for investment and pays a constant, exogenous gross interest rate $R > 1$. The net interest rate $r = R - 1$. Bond holdings are indexed according to when they pay off: b_t^s is purchased at $t-1$, by an agent with $s+1$ periods remaining, pays off Rb_t^s at t.

The agent's optimization problem can be expressed in terms of a recursive sequence of Bellman equations. Let $v_t^s(b,y)$ denote the value function of an agent at t with s periods left, bond holdings b, and current income realization y. Consider an agent born at τ. In his last period, he will consume any remaining wealth, so his terminal value function is

$$v_{\tau+T}^0(b,y) = u\left(y + Rb\right). \tag{3}$$

Given the value function $v_{\tau+T-s}^s(b,y)$, the value function for the agent when he has $s+1$ periods left must satisfy the Bellman equation:

$$v_{\tau+T-s-1}^{s+1}(b,y) = \max_{b',c} \left\{ u(c) + \beta E_{\tau+T-s-1}[v_{\tau+T-s}^s(b',\widetilde{y}')] \right\} \tag{4}$$

subject to

$$\begin{aligned} c + b' &= y + Rb \\ c &\geq 0. \end{aligned} \tag{5}$$

For an agent just born, $b_\tau^T = 0$ since he will have no initial bond holdings.

2 The Role of the Perturbation Parameter

Let δ be a parameter in a problem for which we know the solution if $\delta = 0$. Perturbative methods involve Taylor expansions of the solution with respect to δ. We can view δ as a bookkeeping device. Calculating the nth order effect amounts to calculating all terms in the sequence of order δ^n or less.[4] The assumption that we can ignore higher-order terms in an nth-order approximation depends on δ^{n+1} being significantly smaller than δ^n. Ideally then, researchers would like to consider situations where the value of δ is small compared to 1. In the limit of very small δ, even low-order perturbation approximations should be very accurate. Conversely, for δ on the order of 1, a researcher will need to compute solutions to a high order to achieve that same level of accuracy.

In the context of the present paper, we are typically interested in computing expectations of the form

$$E_t\left[\left(\frac{\widetilde{w}_{t+1}}{w_t}\right)^\rho\right] \tag{6}$$

for some power ρ, where w_t is total wealth (the sum of current income, expected future income, and bond holdings) at t. (We will suppress age superscripts for this discussion.) Expressions of this type arise because the Euler equation for an agent in the model of Section 1 is

$$1 = \beta R E_t\left[\left(\frac{\widetilde{c}_{t+1}}{c_t}\right)^{-\gamma}\right], \tag{7}$$

and consumption is approximately proportional to wealth.

Let $z_{t+1} = E_t[\widetilde{w}_{t+1}]$ and $\Delta\widetilde{w}_{t+1} = w_{t+1} - z_{t+1}$. Then we can write (6) as

$$\left(\frac{z_{t+1}}{w_t}\right)^\rho E_t\left[\left(1 + \frac{\Delta\widetilde{w}_{t+1}}{z_{t+1}}\right)^\rho\right]. \tag{8}$$

[4] Let $f(\delta)$ be a function of the parameter $\delta \in [0, 1)$ and $g(\delta)$ a positive function of δ. We say that $f(\delta) = O(g(\delta))$ ("$f(\delta)$ is of order $g(\delta)$") if and only if there exists a real number $M \geq 0$ such that

$$\frac{|f(\delta)|}{g(\delta)} \leq M$$

for all values of $\delta \in [0, 1)$. If $f_1(\delta) = O(g_1(\delta))$ and $f_2(\delta) = O(g_2(\delta))$, then $f_1(\delta)f_2(\delta) = O(g_1(\delta)g_2(\delta))$.

Using the Taylor expansion

$$(1+x)^\rho = 1 + \rho x + \frac{\rho(\rho-1)}{2}x^2 + \frac{\rho(\rho-1)(\rho-2)}{3!}x^3 + \cdots, \tag{9}$$

which is valid for $|x| < 1$, the expectation in (8) can be replaced by

$$\begin{aligned} 1 + \rho E_t\left[\frac{\Delta\widetilde{w}_{t+1}}{z_{t+1}}\right] + \frac{\rho(\rho-1)}{2}E_t\left[\left(\frac{\Delta\widetilde{w}_{t+1}}{z_{t+1}}\right)^2\right] \\ + \frac{\rho(\rho-1)(\rho-2)}{3!}E_t\left[\left(\frac{\Delta\widetilde{w}_{t+1}}{z_{t+1}}\right)^3\right] + \cdots, \end{aligned} \tag{10}$$

as long as the supremum of $|\Delta\widetilde{w}_{t+1}|$ is less than z_{t+1}.[5] Since current income is the only random component of $\widetilde{w}_{t+1}$,

$$\Delta\widetilde{w}_{t+1} = \widetilde{y}_{t+1} - \mu_{t+1},$$

and (10) becomes

$$1 + \frac{\rho(\rho-1)}{2}E_t\left[\left(\frac{\widetilde{y}_{t+1} - \mu_{t+1}}{z_{t+1}}\right)^2\right] + \frac{\rho(\rho-1)(\rho-2)}{3!}E_t\left[\left(\frac{\widetilde{y}_{t+1} - \mu_{t+1}}{z_{t+1}}\right)^3\right] + \cdots. \tag{11}$$

If we can set up the problem so that $E_t\left[(\Delta\widetilde{w}_{t+1}/z_{t+1})^n\right]$ is of order δ^n for some exogenous parameter δ, then we are in a position to use perturbative methods. For the purposes of this paper, we will achieve this perturbative situation by assuming the nth moment of the income process is of order δ^n.

Formally, we assume the moments of the distribution for the income of an agent at t with s periods remaining, $\widetilde{y}_t^s$, satisfy the following conditions: let $\delta \in [0, 1)$ be an exogenous, dimensionless parameter, which we will imprecisely call the "coefficient of variation", and suppose that

$$E[\widetilde{y}_t^s] = \mu_t^s, \tag{12}$$

and

$$E\left[\left(\frac{\widetilde{y}_t^s - \mu_t^s}{\mu_t^s}\right)^n\right] = O(\delta^n), \qquad n \geq 2. \tag{13}$$

[5] The power series (10) can be written $\sum_{n=0}^{\infty} E_t\left[\frac{\Gamma(\rho+1)}{n!\Gamma(\rho-n+1)}\left(\frac{\Delta\widetilde{w}_{t+1}}{z_{t+1}}\right)^n\right]$. According to 27.2 of Halmos (1974), if the power series $S = \sum_{n=0}^{\infty} E_t\left[\left|\frac{\Gamma(\rho+1)}{n!\Gamma(\rho-n+1)}\right|\left|\frac{\Delta\widetilde{w}_{t+1}}{z_{t+1}}\right|^n\right]$ is finite, then the summation and the expectation operator commute, so S equals the expectation in (**??**). Since $E_t\left[\left|\frac{\Delta\widetilde{w}_{t+1}}{z_{t+1}}\right|^n\right] \leq \left(\frac{\Delta w_{t+1}^{\text{sup}}}{z_{t+1}}\right)^n$, where $\Delta w_{t+1}^{\text{sup}}$ is the supremum of the possible values of $\Delta\widetilde{w}_{t+1}$, $S \leq \sum_{n=0}^{\infty}\left|\frac{\Gamma(\rho+1)}{n!\Gamma(\rho-n+1)}\right|\left(\frac{\Delta w_{t+1}^{\text{sup}}}{z_{t+1}}\right)^n$. Since the series in (9) has unit radius of convergence and since power series are absolutely convergent, this bound on S will be finite if $\Delta w_{t+1}^{\text{sup}} < z_{t+1}$.

(For most of the numerical examples we consider in this paper, δ will equal the ratio of the standard deviation to the mean in each period, and so the terminology that δ is the coefficient of variation would be precisely correct. That will not be true for the general model, although δ will be proportional to the coefficient of variation in the limit as $\delta \to 0$.) This δ will play the role of the perturbation parameter in our perturbation expansion. We can then define

$$\widetilde{x}_t^s = \frac{\widetilde{y}_t^s - \mu_t^s}{\delta} \tag{14}$$

and

$$k_{n,t}^s = E[(\widetilde{x}_t^s)^n] \tag{15}$$

for $n \geq 2$, where $k_{n,t}^s$ is finite as $\delta \to 0$. A distribution that satisfies

$$|(\widetilde{y}_t^s - \mu_t^s)| \leq \delta \mu_t^s$$

for all posible realizations of $\widetilde{y}_t^s$ will satisfy these moment conditions. More generally, we can view these conditions as defining a "compact" distribution in the sense that Samuelson (1977) used the term (not to be confused with the usual notion of a compact set).

As a result, under the moment conditions (13), (11) simplifies to

$$1 + \frac{\rho(\rho-1)}{2} \frac{\delta^2 k_{2,t+1}}{z_{t+1}^2} + \frac{\rho(\rho-1)(\rho-2)}{3!} \frac{\delta^3 k_{3,t+1}}{z_{t+1}^3} + \cdots . \tag{16}$$

Let $x_{t+1}^{\sup}$ be the supremum of the set of possible realizations of $|\widetilde{x}_{t+1}|$. If $\delta x_{t+1}^{\sup} < z_{t+1}$, then the series (16) will converge.

In the absence of exogenous borrowing constraints, Aiyagari (1994) showed that there will be an endogenous borrowing constraint with CRRA utility because agents will never borrow more than the minimum they can be assured they will be able to pay back. Suppose we have a constant income process independent of t. If the realization of $\widetilde{x}$ which is largest in magnitude is negative, the lowest possible income realization will be $\mu - \delta x^{\sup}$. In that case, the bond demand for an agent with an infinite lifespan must satisfy

$$b_{t+1} \geq -\frac{\mu - \delta x^{\sup}}{r},$$

and expected wealth at $t+1$, z_{t+1}, must satisfy

$$z_{t+1} = \frac{R}{r}\mu + R b_{t+1} \geq \frac{R}{r} \delta x^{\sup}. \tag{17}$$

Therefore, under these conditions, the series (16) will converge for any feasible wealth value.

Note, however, that convergence of the series does not by itself make perturbation theory useful. Truncating (16) at order n will only give a decent approximation where terms of order higher than n are negligible relative to the sum up to the nth term. Since each term is a decreasing function of z_t, there will be a lower bound on the values of z_t for which the nth order approximation achieves a given accuracy, and this lower bound will generally be larger than the bound $\delta x_t^{\sup}$.

The condition (13) is a very stringent condition to place on the moments, and it would not be satisfied by many popular income parameterizations. For example, Carroll (1997,2001b) uses an income process in which there is a small probability p that income will be zero in any period. One might think that the probability p would also be a reasonable choice of perturbation parameter since it too is a dimensionless parameter between 0 and 1. However, this is not the case. Let us consider the behavior of the moments for a probability distribution

$$\widetilde{y} = \begin{cases} Y & 1-p \\ \varepsilon Y & p \end{cases},$$

where $\varepsilon = 1 - \delta > 0$ is small. The mean of this distribution is

$$\mu = E[\widetilde{y}] = (1-p)Y + p\varepsilon Y = (1 - p(1-\varepsilon))Y.$$

The nth order moment is

$$\begin{aligned} E\left[(\widetilde{y}-\mu)^n\right] &= (1-p)\left[1-(1-p(1-\varepsilon))\right]^n Y^n + p\left[\varepsilon-(1-p(1-\varepsilon))\right]^n Y^n \\ &= (1-p)p(1-\varepsilon)^n\left[p^{n-1}+(-1)^n(1-p)^{n-1}\right]Y^n. \end{aligned}$$

For $0 < \varepsilon < 1$, these moments will be of order δ^n as we have assumed. However, if $\varepsilon = 0$, which would be the case if we want a distribution with a possibility of a zero realization, then $E[(\widetilde{y}-\mu)^n] = p + O(p^2)$ for all $n \geq 2$. There is no diminishment of the successive moments with respect to p. Thus, p would not be a good choice of a perturbation parameter if we are attempting to evaluate the moment expansion (11).

Notice, however, that while the moments do not diminish with n, they do scale as Y^n. Thus, for small Y/z_{t+1}, the terms of the series (11) should still get progressively smaller, and we can still do perturbation theory using Y/z_t as the perturbation parameter. All the perturbation calculations involving value and policy functions in this paper will remain valid for a noncompact distribution if we use $\widehat{\delta} = \mu_{t+1}/z_{t+1}$ as the perturbation parameter, where μ_{t+1} is still mean income.[6]

Unfortunately, this alternate choice of perturbation parameter has two downsides. The first is that for comparable values of wealth the perturbation parameter will be larger. If mean income is constant, average wealth will typically be on the order of μ/r, where r is the net interest rate, so for an agent of average wealth the perturbation parameter will be on the order of r. For typical values of interest rates on the order of 5%, this should be small enough to make low-order perturbation calculations accurate approximations. On the

[6] This is the approach used in Feigenbaum (2001a).

other hand, a relatively poor agent will have a larger value of $\widehat{\delta}$. For $\widehat{\delta}$ close to unity, one will have to go to very high orders of perturbation theory to obtain a decent approximation. This difficulty will plague any work involving Taylor expansions of the Euler equation (7). Moreover, these higher-order terms will be of significant magnitude for poor agents. In the context of Euler-equation estimation, if one neglects third and higher-order effects in a regression of consumption growth on state variables for a sample with a substantial proportion of low-wealth agents, estimates of structural parameters will be inconsistent because these higher-order effects are significant and depend on those structural parameters. This is one of several arguments that Carroll (2001a) has given for abandoning Euler-equation estimation of preference parameters, although this particular criticism should not present a problem if we restrict our attention to wealthy agents[7].

The second downside is that we cannot use a perturbation parameter that is specific to each agent to compute macroeconomic quantities. Consequently, if we wished to clear markets and calculate interest rates under the approach of this paper, we would have to use a compact income distribution. It may also be possible to exploit p as a perturbation parameter, although it is not clear how that could be done.

3 Value and Policy Functions

The consumption/saving decision described by the Bellman equation (4) is a recursive problem that must be solved backwards iteratively from the end of life. In this section we demonstrate how to compute the second-order perturbation correction for the first iteration, when an agent has one future period remaining in life and no existing bond holdings. The second and third-order corrections are fully worked out for an agent with T periods remaining in Appendix B.

Consider an agent who is born at $t = 0$ and lives until $t = 1$. The consumer's problem gives rise to the Lagrangian

$$\mathcal{L} = \frac{(y_0 - b_1)^{1-\gamma}}{1-\gamma} + \beta E_0 \left[\frac{1}{1-\gamma} \left(\widetilde{y}_1 + R b_1 \right)^{1-\gamma} \right]. \tag{18}$$

(We will suppress the age superscript in this discussion.) Differentiating $\mathcal{L}$ by the choice variable b_1, we obtain the first-order condition

$$(y_0 - b_1)^{-\gamma} = \beta R E_0 \left[\left(\mu_1 + \delta \widetilde{x}_1 + R b_1 \right)^{-\gamma} \right], \tag{19}$$

[7]Here, we are perturbing around a linearization of the Euler equation. The details of this discussion will differ if we proceed from a log-linearization, but the upshot should remain the same. There will be a region of the wealth space where the perturbation series converges, which will, however, be bounded both above and below. As one gets closer to the edge of this region, one will have to go to ever higher orders to maintain the same level of accuracy.

where (14) defines $\widetilde{x}_1$. Eq. (19) can be simplified to

$$y_0 - b_1 = \phi\left(\frac{\mu_1}{R} + b_1\right)\left(E_0\left[\left(1 + \frac{\delta \widetilde{x}_1}{\mu_1 + Rb_1}\right)^{-\gamma}\right]\right)^{-1/\gamma}. \tag{20}$$

The quantity

$$\phi = \left[\beta R^{1-\gamma}\right]^{-1/\gamma} \tag{21}$$

is the inverse of the marginal propensity to save in the limit as $T \to \infty$ and all marginal propensities to save and consume are functions of ϕ alone.

Using (9), we can Taylor-expand the expectation factor of (20) to obtain

$$\left(1 + \frac{\delta \widetilde{x}_1}{\mu_1 + Rb_1}\right)^{-\gamma} = 1 - \frac{\gamma \delta \widetilde{x}_1}{\mu_1 + Rb_1} + \frac{\gamma(\gamma+1)\delta^2(\widetilde{x}_1)^2}{2(\mu_1 + Rb_1)^2} + O(\delta^3). \tag{22}$$

As we discussed in the previous section, this Taylor expansion will be valid as long as

$$\left|\frac{\delta \widetilde{x}_1}{\mu_1 + Rb_1}\right| < 1 \tag{23}$$

for all possible realizations of $\widetilde{x}_1$. For sufficiently small δ, this inequality should be obeyed except for very poor agents with $\mu_1 + Rb_1 \approx 0$. Note that large positive realizations of $\widetilde{x}_1$ can be just as problematic as large negative realizations.

Given the moment assumptions (14) and (15), Eq. (22) has expectation

$$E_0\left[\left(1 + \frac{\delta \widetilde{x}_1}{\mu_1 + Rb_1}\right)^{-\gamma}\right] = 1 + \frac{\gamma(\gamma+1)\delta^2 k_{2,1}}{2(\mu_1 + Rb_1)^2} + O(\delta^3).$$

Applying (9) a second time, we get the result

$$\left(E_1\left[\left(1 + \frac{\delta \widetilde{x}_1}{\mu_1 + Rb_1}\right)^{-\gamma}\right]\right)^{-1/\gamma} = 1 - \frac{(\gamma+1)\delta^2 k_{2,1}}{2(\mu_1 + Rb_1)^2} + O(\delta^3). \tag{24}$$

Inserting (24) into the first-order condition (20) gives

$$y_0 - b_1 = \phi\left[\frac{\mu_1}{R} + b_1 - \frac{(\gamma+1)\delta^2 k_{2,1}}{2R(\mu_1 + Rb_1)}\right] + O(\delta^3). \tag{25}$$

This can be rewritten

$$b_1 = \frac{1}{1+\phi}\left\{y_0 - \phi\left[\frac{\mu_1}{R} - \frac{(\gamma+1)\delta^2 k_{2,1}}{2R(\mu_1 + Rb_1)}\right]\right\} + O(\delta^3). \tag{26}$$

Note that we have not actually solved for the bond demand b_1 since b_1 appears on the righthand side. Although we could solve exactly for b_1 here if we ignore the $O(\delta^3)$ terms, this would not be feasible if we included any higher order terms. Perturbation theory directs us to Taylor expand b_1:

$$b_1 = b_1^{(0)} + \delta b_1^{(1)} + \delta^2 b_1^{(2)} + O\left(\delta^3\right). \tag{27}$$

Let total wealth w_t^s be the sum of current income, expected future income, and current bond holdings for an agent at t when s periods remain:

$$w_t^s = y_t^s + \frac{h_{t+1}^{s-1}}{R} + Rb_t^s. \tag{28}$$

Here, h_t^s is the expected present value of the current and remaining stream of income for an agent at t with s periods remaining:

$$h_t^s = \sum_{i=0}^{s} \frac{\mu_{t+i}^{s-i}}{R^{s-i}}. \tag{29}$$

(So Eq. (28) applies for $s = 0$, we define $h_t^{-1} = 0$ for all t.) For an agent born at $t = 0$ who lives two periods, wealth at $t = 0$ is

$$w_0 = y_0 + \frac{\mu_1}{R} \tag{30}$$

since the agent will begin life with no bond holdings. We define the zeroth-order expectation of wealth at $t = 1$ as

$$z_1 = \mu_1 + Rb_1^{(0)}.$$

Then we can equate coefficients of powers of δ in Eq. (26) to determine the coefficients of the expansion (27):

$$b_1^{(0)} = \frac{1}{1+\phi}\left[y_0 - \phi\frac{\mu_1}{R}\right] \tag{31}$$

$$b_1^{(1)} = 0 \tag{32}$$

$$b_1^{(2)} = \frac{\phi}{1+\phi}\frac{(\gamma+1)k_{2,1}}{2Rz_1}$$

Since

$$z_1 = \frac{Rw_0}{1+\phi},$$

we can rewrite

$$b_1^{(2)} = \phi\frac{(\gamma+1)k_{2,1}}{2R^2 w_0}. \tag{33}$$

Notice that any first-order contribution to the bond demand vanishes. This is a fairly general result that will hold whenever the choice of the bond demand is unconstrained. Essentially, the first-order condition is a restriction that any first-order perturbations must vanish[8]. Consequently, the second-order effect of Eq. (33) is the lowest-order contribution to the precautionary saving effect. A positive variance induces agents to save more than they would if their income was certain. Because the coefficient of w_0^{-1} in (33) is positive, the bond demand function will be strictly convex. This is the flip side of Carroll and Kimball's (1996) result that the consumption function will be strictly concave.

To simplify the more general expression derived in Appendix B arising from a model where agents live for $T+1$ periods, we make use of the notation of q-deformed numbers, a concept first introduced by Heine. For $q \geq 0$ and n both real numbers, we define

$$(n)_q = \frac{1-q^n}{1-q}. \tag{34}$$

For n a positive integer, this is equivalent to the geometric series

$$(n)_q = 1 + q + \cdots + q^{n-1}.$$

Appendix A contains some useful results from q-arithmetic that help to simplify computations.

For general t and $0 \leq s \leq T$, we can write the value function that solves (4) as

$$\begin{aligned} v_t^s(b_t^s, y_t^s) &= \frac{(s+1)_{\phi^{-1}}^{\gamma}}{1-\gamma}(w_t^s)^{1-\gamma}\left[1 - \delta^2 \frac{(1-\gamma)\gamma}{2}\frac{K_{2,t}^s - k_{2,t}^s}{(w_t^s)^2}\right. \\ &\quad \left. + \delta^3 \frac{(1-\gamma^2)\gamma}{6}\frac{K_{3,t}^s - k_{3,t}^s}{(w_t^s)^3}\right] + O(\delta^4). \end{aligned} \tag{35}$$

where $K_{n,t}^0 = k_{n,t}^0$, and

$$K_{n,t}^s = k_{n,t}^s + \left(\frac{(s+1)_\phi}{(s)_\phi}\right)^{n-1}\frac{K_{n,t+1}^{s-1}}{R^n} \tag{36}$$

for $s > 0$. The bond demand at t for $0 \leq s \leq T-1$ is

$$\begin{aligned} b_{t+1}^s(b_t^{s+1}, y_t^{s+1}) &= \frac{(s+1)_\phi}{(s+2)_\phi}(y_t^{s+1} + Rb_t^{s+1}) \\ &\quad - \frac{\phi^{s+1}}{(s+2)_\phi}\left[\frac{h_{t+1}^s}{R} - \frac{\gamma+1}{2}\frac{\delta^2(K_{2,t}^{s+1} - k_{2,t}^{s+1})}{w_t^{s+1}}\right. \\ &\quad \left. + \frac{(\gamma+1)(\gamma+2)}{6}\frac{\delta^3(K_{3,t}^{s+1} - k_{3,t}^{s+1})}{(w_t^{s+1})^2}\right] + O(\delta^4). \end{aligned} \tag{37}$$

[8] In a model with binding liquidity constraints, where the first derivative of the Lagrangian is not zero, first-order perturbation corrections need not vanish.

We carry out these calculations up to third order because the only contribution to these expressions at second and third order comes from the lowest-order (i.e. linear) contribution of the variance and skewness respectively. Beginning with fourth order, we would see the effects of squares and products of variances, cross terms that complicate the algebra enough to forestall their calculation in this this paper. Cross terms do not appear at second or third order because of the cancelling of first-order corrections. If there was a nonzero first-order correction, the square of this first-order correction would appear in the equation for the second-order correction, but the first-order condition eliminates those terms. Indeed, it will generally be true that cross terms will not emerge until fourth order in perturbation calculations where first-order corrections vanish. Thus, the relative simplicity of the third-order calculation for this problem is not a special result.

The variance effect, reflected in terms involving $K_{2,t}^{s+1}$, behaves as in the two-period case. Increases in the variance of income lower utility and induce agents to increase their bond holdings. This is the precautionary saving effect. In contrast, the skewness effect, reflected in the $K_{3,t}^{s+1}$ terms, increases utility and decreases saving. Agents with CRRA preferences are risk averse but skewness loving. If we went on to work out the fourth-order corrections, these would have an ambiguous sign. Terms deriving from the fourth-order moment will lead to greater saving, but cross-terms involving products of variances will have the opposite sign. Which force wins out will depend on whether the raw kurtosis terms are greater or less than some function of the variance terms. Similar considerations would apply to fifth and higher-order moments.

By substituting (37) into the budget constraint (5), we obtain the consumption function

$$\begin{aligned} c_t^{s+1}(b_t^{s+1}, y_t^{s+1}) \quad &= \quad \frac{\phi^{s+1}}{(s+2)_\phi}\left[w_t^{s+1} - \frac{\gamma+1}{2}\frac{\delta^2(K_{2,t}^{s+1} - k_{2,t}^{s+1})}{w_t^{s+1}}\right. \\ &\left. + \frac{(\gamma+1)(\gamma+2)}{6}\frac{\delta^3(K_{3,t}^{s+1} - k_{3,t}^{s+1})}{(w_t^{s+1})^2}\right] + O(\delta^4). \quad (38) \end{aligned}$$

Let $c_{n,t}^s$ denote the nth-order perturbation approximation to the consumption function

$$c_{n,t}^s = y_t^s + Rb_t^s - \sum_{i=0}^{n} \left(b_{t+1}^{s-1}\right)^{(i)}$$

(so (38) corresponds to $c_{3,t}^{s+1}$). As an example of how much the second and third-order corrections to the consumption function, consider the case where the income process has a stationary distribution

$$\widetilde{y}_t^s = \begin{cases} 1+\delta & \text{probability } p \\ 1-\delta & \text{probability } 1-p \end{cases} \quad (39)$$

for all t and s, so the income will have variance $k_2 = 4p(1-p)\delta^2$ and skewness $8p(1-p)(1-2p)\delta^3$, which will be positive or negative depending on whether p

is less or greater than 1/2. In Fig. 1, we plot $|\Delta c^t_{n,s}|/\widehat{c}^t_s$ for $n = 0$, 2 and 3 as a function of wealth w^t_s for an agent with $\beta = 0.96$, $\gamma = 2$, $p = 0.9$, $\delta = 0.5$, and 41 periods remaining, where

$$\Delta c^t_{n,s} = c^t_{n,s} - \widehat{c}^t_s$$

and $\widehat{c}^t_s$ is a consumption function evaluated numerically using value-function iteration and Schumaker shape-preserving splines as described in Judd.(1999).[9] The chosen parameters p and δ for the income process give a variance $k_2 = 0.09$ and a negative skewness $k_3 = -0.072$. This is a situation where the agent typically receives a constant income, but has a 10% probability of receiving a temporary negative income shock corresponding to a loss of 2/3 of his usual income. For small values of w, none of the three perturbed consumption functions does a good job of approximating the numerical consumption function, which is to be expected from the discussion of Section 2. In the low-wealth regime, the consumption function is always much less than is predicted by perturbation theory. In part, this reflects additional precautionary saving induced by the Aiyagari borrowing constraint, which does not show up in low-order perturbation calculations. On the other hand, for large values of wealth, the perturbation approximation works quite well, improving with each successive order. The second-order consumption function, which begins to account for precautionary saving, improves upon the zeroth-order, certainty-equivalent consumption function by about a factor of ten in terms of the relative distance between the perturbed and numerical consumption functions. Meanwhile, the relatively large negative skewness of this example leads to additional precautionary saving, and the third-order consumption function improves upon the second-order approximation by a factor of five.

To see what happens with positive skewness, consider the reverse example of Fig. 2 where $p = 0.1$, corresponding to a situation where an agent has a ten percent chance of receiving a windfall that increases his income by a factor of 3. The relative performance of the three perturbative consumption functions is about the same for high wealth. In this case, however, the third-order approximation does worse than the zeroth- and second-order consumption functions in the low wealth regime. This is because the positive skewness, instead of adding to precautionary saving, reduces it. Since the third-order correction goes as w^{-3} while the second-order correction goes as w^{-2}, the skewness effect will actually outweigh the precautionary saving effect from the variance terms, so that third-order consumption is even greater than the certainty-equivalent consumption. However, the effects of the Aiyagari borrowing constraint, which do not appear in these perturbation calculations, will overwhelm the skewness effect, reducing the actual consumption function far below even the second-order approximation.

To close this section, we can use third-order consumption function (38) to derive an expression for the expected rate of consumption growth (see Ap-

[9] For more details on this procedure, see Feigenbaum (2001a).

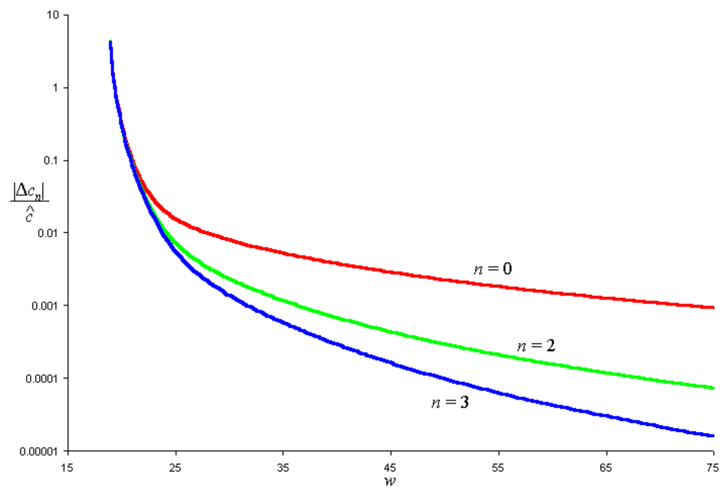

Figure 1: Relative distance between the nth-order consumption function c_n and the numerically computed consumption function $\widehat{c}$ for an agent with 41 periods remaining, given $R = \beta^{-1} = 1.04167$, $\gamma = 2$, and the income distribution (39) with $p = .9$ and $\delta = 0.5$, so the variance is $k_2 = 0.09$ and the skewness is $k_3 = -.072$.

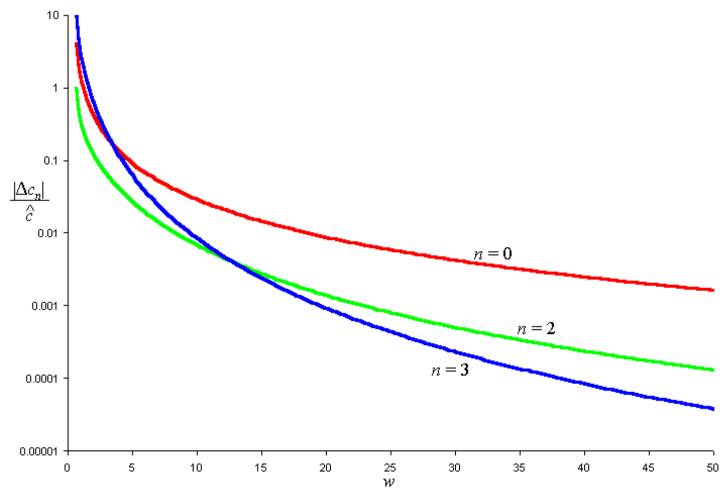

Figure 2: Relative distance between the nth-order consumption function c_n and the numerically computed consumption function $\widehat{c}$ for an agent with 41 periods remaining, given $R = \beta^{-1} = 1.04167$, $\gamma = 2$, and the income distribution (39) with $p = .1$ and $\delta = 0.5$, so the variance is $k_2 = 0.09$ and the skewness is $k_3 = .072$.

pendix C):

$$\frac{E_t[\widetilde{c}^s_{t+1}]}{c^{s+1}_t} = (\beta R)^{1/\gamma}\left[1 + \frac{\gamma+1}{2}\frac{\delta^2 k^{s+1}_{2,t}}{E_t[\widetilde{w}^{s+1}_t]^2} - \frac{(\gamma+1)(\gamma+2)}{6}\frac{\delta^3 k^{s+1}_{3,t}}{E_t[\widetilde{w}^{s+1}_t]^3}\right] + O(\delta^4). \tag{40}$$

Notice that only the moments of income at period t, $\widetilde{y}_t$, enter this relative expression. Unlike the absolute consumption function, consumption growth does not depend on the moments of income at later periods, so the cumulative moments $K_{n,t}$ do not appear in (40). If we restrict our attention to second order, we can obtain a similar expression for the actual rate of consumption growth and not just its expectation[10].

$$\frac{\widetilde{c}^s_{t+1}}{c^{s+1}_t} = (\beta R)^{1/\gamma}\frac{\widetilde{w}^s_{t+1}}{w^{s+1}_t}\left[1 + \frac{\gamma+1}{2}\frac{\delta^2 k^{s+1}_{2,t}}{E_t[\widetilde{w}^{s+1}_t]^2}\right] + O(\delta^3). \tag{41}$$

Given the assumptions we have made regarding the income process, this is equivalent to the expression for the rate of consumption growth obtained by Skinner (1988).

4 Large Lifetime Limit of Policy and Value Functions

In Appendix B we compute the lowest-order contribution of the nth moment to both the value function and the bond demand function for $n \geq 2$:

$$v^{(n)}_{t,s+1} = (-1)^{n+1}\frac{\Gamma(\gamma+n-1)}{n!\Gamma(\gamma)}\delta^n (K^{s+1}_{n,t} - k^{s+1}_{n,t})(s+2)^{\gamma}_{\phi^{-1}}(w^{s+1}_t)^{1-\gamma-n} + O(\delta^{n+1}) \tag{42}$$

$$b^{(n)}_{t+1,s} = (-1)^n \frac{\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)}\frac{\phi^{s+1}}{(s+2)_\phi}\frac{\delta^n (K^{s+1}_{n,t} - k^{s+1}_{n,t})}{w^{n-1}_t} + O(\delta^{n+1}), \tag{43}$$

where the K satisfy the difference equation (36).

Let us focus on the case where the only variation in income over time comes from the presence of economic growth–i.e. the distribution of $\widetilde{x}^s_t$ is independent of s. By assumption then, $\widetilde{x}^{s'}_{t'}$ and $G^{t'-t}\widetilde{x}^s_t$ will have the same distribution for all s, s', t, and t'. This will imply that

$$k^{s'}_{n,t'} = G^{n(t'-t)}k^s_{n,t}$$

[10]At third order, cross terms involving products of $\Sigma^2_{t,s+1}$ and $\widetilde{x}^{s+1}_t$ will occur. These vanish upon taking expectations. However, the actual consumption growth rate will depend on $\Sigma^2_{t,s+1}$ and therefore moments of income at later periods.

for all s, s', t, and t' and $n \geq 2$. If we define

$$\widehat{K}_n^s = G^{-nt} K_{n,t}^s$$

and

$$\widehat{k}_n = G^{-nt} k_{n,t}^s,$$

we can rewrite (36) as

$$\widehat{K}_n^s = \frac{(s+1)_\phi^{n-1}}{(s)_\phi^{n-1}} \left(\frac{G}{R}\right)^n \widehat{K}_n^{s-1} + \widehat{k}_n. \tag{44}$$

Now consider what happens to $\widehat{K}_n^s$ in the limit as $s \to \infty$, which corresponds to the limit of a large lifetime. We will assume that $R > G$ so that net interest rates are positive and the present value of future income is finite. We can also assume that $\phi > 1$ because this is the usual condition that must be satisfied in order to guarantee that lifetime utility, defined by (1)-(2), is finite for all feasible consumption paths. In that case, the limit

$$\lim_{s\to\infty} \frac{\phi^s}{(s+1)_\phi} = \lim_{s\to\infty} (s)_{\phi^{-1}}^{-1} = \lim_{s\to\infty} (\phi - 1) \frac{\phi^s}{\phi^{s+1} - 1} = \frac{\phi - 1}{\phi}$$

exists. Consequently, the finiteness of (42)-(43) depends entirely on the behavior of the coefficients $K_{n,t}^s$ (and thereby $\widehat{K}_n^s$).

For large s,

$$\lim_{s\to\infty} \frac{(s+1)_\phi}{(s)_\phi} = \lim_{s\to\infty} \frac{\phi^{s+1} - 1}{\phi^s - 1} = \phi,$$

so Eq. (44) approximates to

$$\widehat{K}_n^s \approx \phi^{n-1} \left(\frac{G}{R}\right)^n \widehat{K}_n^{s-1} + \widehat{k}_n \tag{45}$$

in that limit. The solution to (45) will be finite at large t if and only if

$$\phi^{n-1} \left(\frac{G}{R}\right)^n < 1. \tag{46}$$

Let

$$R_G = \beta^{-1} G^\gamma. \tag{47}$$

This is the interest rate for which consumption will grow at the constant rate G in an economy without uncertainty, and it is the unique equilibrium interest rate for an infinite-horizon deterministic economy where income grows at the rate G. This also corresponds to the interest rate which satisfies $\phi G = R$. If $R \geq R_G$, then we will have $\phi G \leq R$. Since we have also assumed $G < R$, the condition (46) must hold in that case. So for $R \geq R_G$, the expressions (42)-(43) will be well-defined for all n.

On the other hand, if we solve the inequality (46) to isolate R, we find the condition holds only for

$$R > R_n = \left[\beta^{\frac{1-n}{\gamma}} G^n\right]^{\frac{\gamma}{\gamma+n-1}} .$$

The threshold interest rate for order n, R_n, is an increasing function of n and converges to R_G as $n \to \infty$.[11] This is a problem because when we introduce precautionary saving into an infinite-horizon economy, the increased demand for bonds will push the interest rate below the interest rate, R_G, for the deterministic economy. If $R < R_G$, then there must be some n for which $R > R_n$. That is, for some n, the nth-order value and policy functions must diverge.

We have to assume that $R > G$, which means the economy must be dynamically efficient, in order for the zeroth-order policy and value functions that we perturb around to exist. As long as $\phi > 1$, which is also needed for the zeroth-order functions to exist, the nth-order bound will be a stronger bound than the bound required by dynamic efficiency since if $G \geq R$ and $\phi > 1$ then the condition (46) will be violated.[12] Moreover, since ϕ is strictly greater than 1, there will be a neighborhood of G such that all R in that neighborhood will be below the second-order threshold R_2. Thus, we are not simply replicating the trivial result that the economy must be dynamically efficient in order for perturbation theory to work because, otherwise, the solution we are expanding around does not exist.

What do these divergences in the perturbation calculations mean for the practical application of perturbative methods? First, we should note that these divergences are not real. If we solve for value and policy functions numerically, we do not see any significant change when the interest rate crosses any of the thresholds specified by (46). In Fig. 3, we compare the second-order perturbation calculation of the consumption function c_2 to a numerical calculation $\widehat{c}$ for a case where 41 periods remain and $\phi G^2/R^2 = 1.16$. Notice that the numerical consumption function is well-behaved, deviating only slightly from a straight line, and everywhere positive. The effect of the divergence is better seen in Fig. 4, which compares the relative distances between the nth-order consumption function c_n and the numerical consumption function $\widehat{c}$ for $n = 0$ and $n = 2$. Notice that the zeroth-order consumption function is everywhere a better approximation than the second-order consumption function. If we adjust the risk aversion coefficient γ from 1.1 to 0.5, $\phi G^2/R^2$ increases to 1.43, and we

[11] Because these interest rate thresholds are scale-invariant, it is not possible to escape or alleviate this pathology by changing the time scale. Suppose the length of a period is τ. We define $R = e^{q\tau}$, $G = e^{g\tau}$, and $\beta = e^{-\rho\tau}$, where q, g, and ρ are the the continuous-time interest, growth, and discount rates respectively. The condition $\phi^{n-1}(G/R)^n < 1$ holds if and only if $q > (\frac{n-1}{\gamma}\rho + ng)/(1 + \frac{n-1}{\gamma})$, independent of τ.

[12] A sufficient condition for $\phi(R) > 1$ for $R \in [G, R_G]$ is that $G < R_G$ since that will guarantee both $\phi(G)$ and $\phi(R_G) > 1$ and since $\phi(R)$ is monotonic.

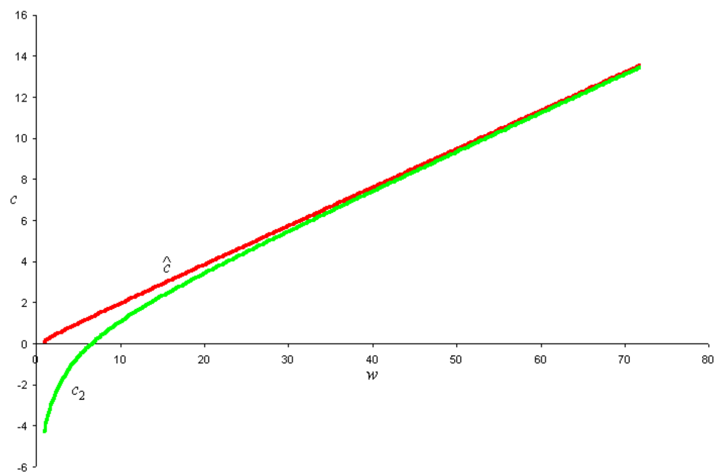

Figure 3: Second-order consumption function c_2 and numerical consumption function $\widehat{c}$ as a function of wealth w for $\beta = 0.8$, $\gamma = 1.1$, $R = 1.03$, and an income distribution given by (39) with $p = 0.5$ and $\delta = 0.1$.

get an even more spectacular failure of perturbation theory as seen in Fig. 5.[13] Clearly, however, since $\widehat{c}$ is well-behaved in this case also, the divergence arises because perturbative methods have broken down, not because there is anything divergent about the exact problem.

What happens if $\phi G^2/R^2 < 1$? Although the second-order correction will not blow up, we know that if $R < R_G$ then the nth-order consumption function must diverge at large lifetimes for some n. Does this interfere with the approximation ability of lower-order consumption functions? Consider Fig. 6, in which we compare the zeroth-order consumption function to the second-order consumption function for a case where 249 periods remain, $R = 1.03 < \beta^{-1} = 1.04167$, and $\phi G^2/R^2 = 0.98$. At such a large lifetime, if the second-order approximation is going to be thrown off by divergences at higher orders, we should see it here. However, the second-order calculation is always better than the zeroth-order calculation and usually is significantly better. Thus, even though perturbation calculations must diverge at some finite order if $R < \beta^{-1}$, this does not invalidate perturbation calculations at lower orders.

Notice that $K^s_{n,t}$ will diverge if (46) is violated no matter how large δ is. The coefficient of variation could be less than one part in a billion, and $K^s_{n,t}$

[13] Actually, Fig. 5 somewhat exaggerates the difference between the second-order consumption function and the actual consumption function because we have not enforced the non-negativity constraint on consumption in the perturbation calculations, and the second-order consumption function is negative for all pictured wealth values.

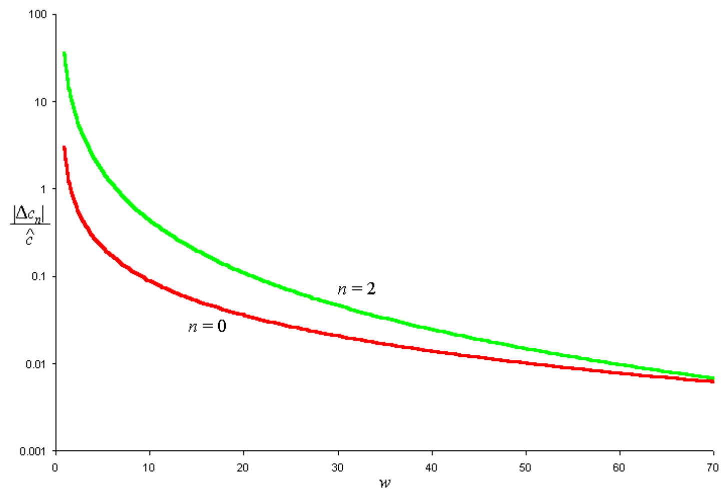

Figure 4: Relative distance between the nth-order consumption function c_n and the numerically computed consumption function $\widehat{c}$ for an agent with 41 periods remaining, given $\beta = 0.8$, $\gamma = 1.1$, $R = 1.03$ and the income distribution (39) with $p = .5$ and $\delta = 0.1$. In this case, $\phi G^2/R^2 = 1.16$, so the second-order consumption function diverges at large lifetimes.

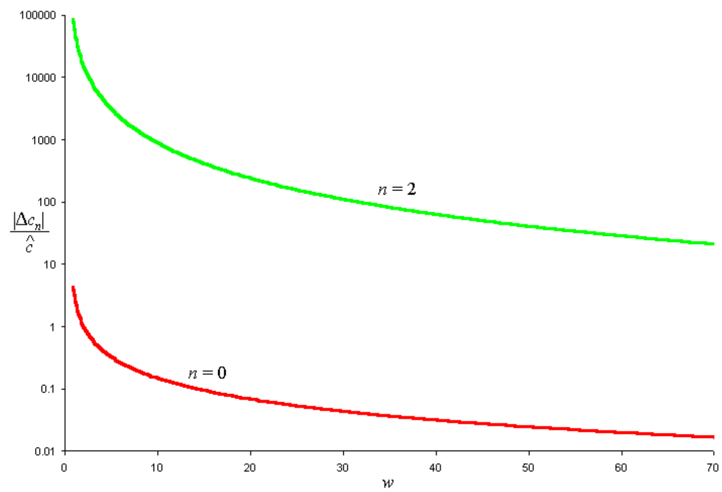

Figure 5: Relative distance between the nth-order consumption function c_n and the numerically computed consumption function $\widehat{c}$ for an agent with 41 periods remaining, given $\beta = 0.8$, $\gamma = 0.5$, $R = 1.03$ and the income distribution (39) with $p = .5$ and $\delta = 0.1$. In this case, $\phi G^2/R^2 = 1.43$.

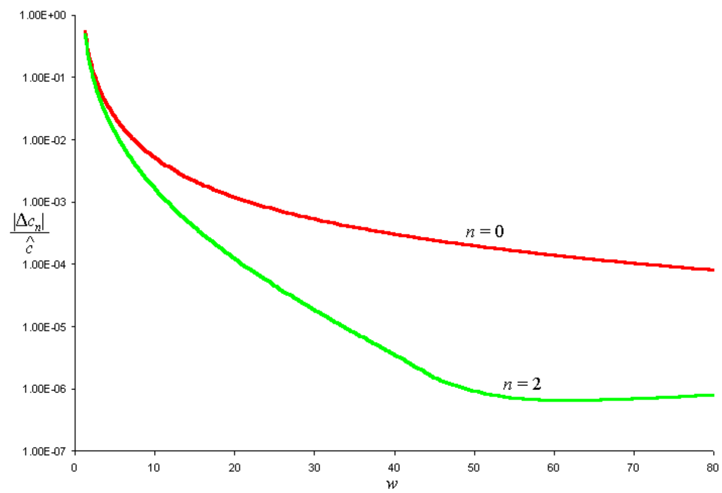

Figure 6: Relative distance between the nth-order consumption function c_n and the numerically computed consumption function $\widehat{c}$ for an agent with 249 periods remaining, given $R = 1.03 < \beta^{-1} = 1.04167$, $\gamma = 1.1$, and the income distribution (39) with $p = .5$ and $\delta = 0.1$. In this case, $\phi G^2/R^2 = 0.98$.

will still grow without bound, although this divergence might only affect the value and policy functions at gigantic values of the remaining lifetime s. What causes this divergence? As we discussed in Section 2, the perturbation series of expected utility will only converge for values of wealth w_t^s above some cutoff related to the variance of income. Given a constant income process, this cutoff should not vary with time. Yet here we find that for any value of w_t^s, there will be some s at which perturbation corrections explode *if* the interest rate is less than R_G.

The dependence of this effect on the interest rate suggests it is related to how the long-term trend in consumption growth depends on the interest rate. Consider the case without growth.[14] From Eq. (40), we see that in the absence of uncertainty, consumption will decay exponentially if the interest rate R is less than the discount rate β^{-1}. In this case, consumers will spend their wealth down to its minimum value, zero, which is less than the minimum value allowed if there is uncertainty (on account of the Aiyagari (1994) endogenous borrowing constraint). Conversely, if $R > \beta^{-1}$, consumption will increase without bound. Consumption will remain stable over time only if $R = \beta^{-1}$. When uncertainty is turned on, consumption will still grow without bound almost surely if $R > \beta^{-1}$ or if $R = \beta^{-1}$ and income is sufficiently stochastic (Chamberlain and Wilson (2000)). If $R < \beta^{-1}$, then the convexity of the saving function will prevent agents from spending their wealth down to its minimum. Instead, there will be a unique steady state in the mapping of current wealth to expected next-period wealth (Carroll (1997, 2001b)), and consumers will be attracted to this value of wealth, known as the buffer-stock wealth.

Perturbative methods cannot properly represent the utility deriving from a future eventuality where the agent has near-zero wealth. Nevertheless, if the buffer-stock wealth is high enough, one might hope that it would be extremely unlikely for an agent at or above the buffer stock to approach the minimum wealth and that the contribution to the value function from this possible eventuality would, thus, be negligible. However, the buffer stock is not a property of the zeroth-order solution that we are expanding around. Evidently, the zeroth-order behavior of spending wealth down to zero gets reflected in perturbation corrections. As R approaches β^{-1} from below, the rate at which agents spend down their wealth in the zeroth-order solution will decrease to nil, but a low income shock can speed up this process. Since information about low-income shocks is conveyed by the higher-order moments of the income process, it makes sense that the contribution of each successive moment should become pathological at a higher threshold interest rate.

While these divergences arise in expressions of absolute consumption and absolute wealth, it is worth noting that they cancel out of some relative expressions. For example, the expressions for the rate of consumption growth given by Eqs. (40)-(41) depend on $k_{2,t}^s$ and $k_{3,t}^s$ but not on the cumulative moments $K_{2,t}^s$ and $K_{3,t}^s$ where the divergences arise. This may be a special property

[14] The case with growth will essentially be the same if we substitute the consumption to income ratio and the wealth to income ratio for consumption and wealth respectively.

that only holds to third order though. It is not known whether it will persist at fourth and higher orders where cross terms appear. Nevertheless, these divergences are one perturbation pathology that may not beset Euler-equation regressions of consumption growth. The finding above that the economy must be dynamically efficient for perturbation theory to work may, however, relate to another perturbation pathology involving Euler equations. Carroll (2001a) has found in simulated economies that the estimation of second-order approximations to the Euler equation gives inconsistent estimates for the known preference parameters of these artificial economies. However, all of the agents in his examples experience an interest rate less than or equal to their income growth rate. Conceivably, this may account for his finding that the second-order Euler equation is such a poor approximation to the true Euler equation.

5 Concluding Remarks

Perturbative methods can be a very powerful tool. In physics, they are commonplace, and theorists have used them to accurately predict the magnetic moment of the electron to a stunning nine decimal places. Nevertheless, perturbative methods are not foolproof. They need to be used with some care. Applied blindly, they can produce nonsense just like any other method.

Unfortunately, the theory underlying perturbative methods is limited. The Taylor theorem provides the foundational basis for perturbative methods but offers little practical guidance regarding their usage. Given a function of the perturbation parameter which is $\mathcal{C}^{\infty}$ at a chosen point, it establishes that a Taylor series for this function will converge within an open ball of that point, and the radius of that ball will depend on the analyticity properties of the function. However, this does not help much if our knowledge of the function is limited to a finite number of terms in its Taylor expansion or if higher-order terms do not exist. In the rare situation where we know an entire perturbation series, we can probably solve the relevant problem analytically without any need for perturbative methods. If a radius of convergence exists, for values of the perturbation parameter near but less than this radius, Taylor approximations will only be accurate when computed to large orders. For values of the perturbation parameter above the radius, Taylor approximations are completely uninformative. Consequently, it is of tantamount importance that any research involving perturbative methods clearly specify what the perturbation parameter is.

Similar considerations apply if a radius of convergence does not exist. As the example of this paper demonstrates, that may often be the case in a dynamic model. The singularity in the CRRA utility function implies that perturbation expansions of expected utility will only be valid at wealth levels above some lower bound. This restriction reflects the above discussion and will hold both in a static model where agents make decisions involving only a single period of uncertainty and in a dynamic model where agents must make a sequence of decisions involving multiple periods of uncertainty. In the second case, however, the potential exists for the agent to eventually leave the convergent region of the state space, and this can cause individual terms in the perturbation series to diverge as the number of future periods increases no matter how small the perturbation parameter is. The possibility of such dynamic pathologies will have to be considered in any model involving CRRA preferences and stochastic histories. In fact, they are a property of most of the hyperbolic absolute risk aversion (HARA) class of utility functions (Feigenbaum (2001b)). (Quadratic and CARA utility are immune to the specific pathology examined here.) Most likely, these pathologies will show up in any dynamic model involving perturbative approximations where the zeroth-order dynamics imply the state variables should inevitably leave the convergence region for the perturbation series.

That said, we have only considered temporary income shocks in this model. Viceira (2001) is another perturbative treatment of a consumption/saving problem in which there are only permanent income shocks. This is a somewhat more tractable problem under CRRA utility since it has more of the structure of a standard portfolio-allocation problem (Koo (1999)). Viceira does not consider the possibility of dynamic pathologies, so it remains to be seen whether dynamic pathologies will arise in the absence of temporary income shocks.

I should emphasize here that while we established a lower bound on the set of interest rates at which the perturbation series to a given order will remain finite at large lifetimes, we only considered divergences that arise from the lowest-order contribution of each moment of the income process. Higher-order contributions may induce tighter bounds. For example, at fourth order, the cross terms arising from products of variances may produce a tighter bound on the interest rate than the bound we computed which comes from the pure kurtosis terms.

Despite these words of caution, I do not wish to overly discourage researchers from exploiting perturbative methods. In Feigenbaum (2001a), I close the economy in the present paper by introducing an overlapping generations (OLG) structure. If we continue with the assumption that the income process has a "compact" distribution, then we can substitute our perturbed bond demands into the market-clearing equation and solve for perturbation corrections to the interest rate. The result is an approximate expression for the interest rate that is a function of exogenous parameters, and, in fact, the second-order expression expression is quite simple in the limit of large lifetimes[15]. I find that, at least for short lifetimes, this second-order prediction compares favorably to numerical results.

As a numerical method, hybrid perturbative methods may be superior to value-function iteration for the purpose of computing interest rates in an economy without borrowing constraints if they can more efficiently approximate bond demand functions. Assuming only a small measure of agents fall outside the region of the state space where perturbative methods are inaccurate, one

[15]For realistic values of the exogenous parameters, the bond demand should not diverge at large lifetimes to second order. However, even if they do, these divergences will not show up in the interest rate until fourth order.

can aggregate these perturbed demand functions and solve for the equilibrating interest rate with a numerical equation solver. I would not advocate using perturbative methods to compute interest rates directly as a numerical method for two reasons. First, this would be inefficient since a much greater effort is needed to compute perturbation corrections than is required to input functions into an equation solver. Second, direct computation of interest rates would only be possible if the income distribution is compact. However, as we have seen, a compact income distribution is not required to accurately perturb policy functions for agents at high enough levels of wealth.

Throughout this paper, we have maintained the assumption of no exogenous borrowing constraints. This assumption was made because even the zeroth-order problem without uncertainty becomes much more complicated when borrowing constraints are introduced. Nevertheless, this problem is still analytically solvable for finite lifetimes. With borrowing constraints, value and policy functions will be piecewise smooth functions. The number of intervals required to define these functions will grow as the lifetime increases, and for that reason the problem quickly loses tractability at large lifetimes. Yet for small lifetimes, the zeroth-order problem will be tractable and can be perturbed around. In Feigenbaum (2002), I use perturbative methods to study how borrowing constraints and precautionary saving interact in a simple general-equilibrium model.

As a theoretical tool, I believe this last manner of application is where perturbative methods will be most beneficial. They can be used quite effectively to analyze models which are simple enough to afford insight about the workings of more complicated models and yet which are complex enough themselves to be unsolvable by exact methods.

A A Brief Introduction to q-Arithmetic

For real numbers $q \geq 0$ and n, we define the q-analog of n as

$$(n)_q = \frac{1 - q^n}{1 - q}.$$

In the limit as $q \to 1$, we can apply l'Hôpital's rule to evaluate $(n)_q$:

$$\lim_{q \to 1} (n)_q = \lim_{q \to 1} \frac{-nq^{n-1}}{-1} = n.$$

Thus as q deviates from 1, $(n)_q$ is a "deformation" of n.

Note that $(0)_q = 0$ and $(1)_q = 1$ for all values of q. For n a positive integer,

$$(n)_q = \sum_{i=0}^{n-1} q^i = 1 + q + \ldots + q^{n-1}.$$

Thus q-deformed numbers often arise in environments where geometric series appear.

We can express $(n)_q$ in terms of $(n)_{q^{-1}}$ as follows.

$$\begin{aligned}(n)_{q^{-1}} &= \frac{1-q^{-n}}{1-q^{-1}} \\ &= \frac{q^{-n}}{q^{-1}}\frac{q^n-1}{q-1}\end{aligned}$$

Thus,

$$(n)_{q^{-1}} = q^{1-n}(n)_q. \tag{48}$$

A useful result from q-arithmetic is

$$\begin{aligned}(n)_q + q^n(m)_q &= \frac{1-q^n}{1-q} + q^n\frac{1-q^m}{1-q} \\ &= \frac{1-q^{n+m}}{1-q}\end{aligned}$$

$$(n)_q + q^n(m)_q = (n+m)_q. \tag{49}$$

B Lowest-Order Contribution of nth-Order Moment to Policy and Value Functions

We will consider the lowest-order contribution of the nth-order moment to the solution of the perturbed problem (**??**). This term can be computed separately from all other terms except the zeroth-order term because any interaction between the nth-order moment and other moments (including itself) will occur as higher-order cross terms, which we neglect here.

Recall that the Euler gamma function satisfies the property

$$\Gamma(t+1) = t\Gamma(t).$$

Thus, we can write the product of a sequence of factors differing by 1 as

$$\gamma(\gamma+1)\cdots(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}.$$

(This ratio of gamma functions will be well-defined even if the individual gamma functions are not.)

Let us suppose that the value function for an agent t with s periods remaining has the partial form

$$\begin{aligned} v_t^s(w_t^s) &= \frac{1}{1-\gamma}(s+1)_{\phi^{-1}}^{\gamma}(w_t^s)^{1-\gamma} \\ &\times \left[1+(-1)^n \frac{\Gamma(\gamma+n-1)}{n!\Gamma(\gamma-1)} \frac{(s+1)_\phi^{n-1}}{(s)_\phi^{n-1}} \frac{\delta^n K_{n,t+1}^{s-1}}{R^n (w_t^s)^n}\right], \end{aligned} \tag{50}$$

where we include only the lowest-order contribution of the nth-order moment and the zeroth-order term. The variable w_t^s is the wealth of the individual, defined by Eq. (28), and $K_{n,t}^s$ is a sequence of constants in s satisfying $K_{n,t}^{-1} = 0$ for all t. The determination of the $K_{n,t}^s$ is the key to computing the value and policy functions.

Notice that for $s = 0$,

$$v_t^0(w) = \frac{(w_t^0)^{1-\gamma}}{1-\gamma},$$

the utility obtained from consuming all wealth, which will be known with certainty in the last period of life.

Next assume that (50) is correct for $s \geq 0$. Then the Bellman equation (**??**) generates the Lagrangian

$$\mathcal{L}_t^{s+1} = \frac{1}{1-\gamma}\left[y_t^{s+1} + R b_t^{s+1} - b_{t+1}^s\right]^{1-\gamma} + \beta E_t[v_{t+1}^s(b_{t+1}^s, \widetilde{y}_{t+1}^s)]. \tag{51}$$

Inserting (50) into the Lagrangian, we obtain

$$\begin{aligned} \mathcal{L}_t^{s+1} &= \frac{1}{1-\gamma}\left(y_t^{s+1} + R b_t^{s+1} - b_{t+1}^s\right)^{1-\gamma} \\ &+ \frac{1}{1-\gamma}(s+1)_{\phi^{-1}}^{\gamma} \\ &\times E_t\left[\left(\widetilde{w}_{t+1}^s\right)^{1-\gamma}\left(1+(-1)^n \frac{\Gamma(\gamma+n-1)}{n!\Gamma(\gamma-1)} \frac{(s+1)_\phi^{n-1}}{(s)_\phi^{n-1}} \frac{\delta^n K_{n,t+2}^{s-1}}{R^n \left(\widetilde{w}_{t+1}^s\right)^n}\right)\right] \end{aligned} \tag{52}$$

Note that the $K_{n,t+2}^{s-1}$ factor is indexed to time $t+2$. This represents the effects of uncertainty for all periods after $t+1$, starting with $t+2$. Since $d\left(\widetilde{w}_{t+1}^s\right)/db_{t+1}^s = R$, differentiating $\mathcal{L}_t^{s+1}$ by the choice variable b_{t+1}^s gives the first-order condition

$$\begin{aligned} &\left(y_t^{s+1} + R b_t^{s+1} - b_{t+1}^s\right)^{-\gamma} \\ = \;& \beta R (s+1)_{\phi^{-1}}^{\gamma} E_t\left[\left(\widetilde{w}_{t+1}^s\right)^{-\gamma}\left(1+(-1)^n \frac{\Gamma(\gamma+n-1)}{n!\Gamma(\gamma-1)} \frac{(s+1)_\phi^{n-1}}{(s)_\phi^{n-1}} \frac{\delta^n K_{n,t+2}^{s-1}}{R^n \left(\widetilde{w}_{t+1}^s\right)^n}\right)\right. \\ &\left. - \left(\widetilde{w}_{t+1}^s\right)^{1-\gamma}(-1)^n \frac{n}{1-\gamma} \frac{\Gamma(\gamma+n-1)}{n!\Gamma(\gamma-1)} \frac{(s+1)_\phi^{n-1}}{(s)_\phi^{n-1}} \frac{\delta^n K_{n,t+2}^{s-1}}{R^n \left(\widetilde{w}_{t+1}^s\right)^{n+1}}\right]. \end{aligned} \tag{53}$$

Since

$$\begin{aligned}\left(1-\frac{n}{1-\gamma}\right)\frac{\Gamma(\gamma+n-1)}{\Gamma(\gamma-1)} &= \frac{1-\gamma-n}{1-\gamma}\frac{\Gamma(\gamma+n-1)}{\Gamma(\gamma-1)} \\ &= \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)},\end{aligned}$$

we have

$$\left(y_t^{s+1}+Rb_t^{s+1}-b_{t+1}^s\right)^{-\gamma}=\beta R(s+1)_{\phi^{-1}}^{\gamma} \tag{54}$$
$$\times E_t\left[\left(\widetilde{w}_{t+1}^s\right)^{-\gamma}\left(1+(-1)^n\frac{\Gamma(\gamma+n)}{n!\Gamma(\gamma)}\frac{(s+1)_\phi^{n-1}}{(s)_\phi^{n-1}}\frac{\delta^n K_{n,t+2}^{s-1}}{R^n\left(\widetilde{w}_{t+1}^s\right)^n}\right)\right].$$

Let us define

$$z_{t+1}^s=h_{t+1}^s+R\left(b_{t+1}^s\right)^{(0)} \tag{55}$$

and

$$B_{t+1}^s=b_{t+1}^s-\left(b_{t+1}^s\right)^{(0)},$$

where $\left(b_{t+1}^s\right)^{(0)}$ is the zeroth-order contribution to the bond demand function–i.e. the bond demand function in the absence of uncertainty. Then

$$\widetilde{w}_{t+1}^s=z_{t+1}^s+RB_{t+1}^s+\delta\widetilde{x}_{t+1}^s. \tag{56}$$

Notice that any deviation of $\left(\widetilde{w}_{t+1}^s\right)^{-n}$ from $\left(z_{t+1}^s\right)^{-n}$ will be of order δ, so we can ignore such deviations since $(\delta/\widetilde{w}_{t+1}^s)^n$ is already of order δ^n. Therefore, we can factor out the terms in parentheses from the expectation.since they will be known with certainty to order δ^n. The remaining factor inside the expectation will then be

$$E_{t+1}\left[\widetilde{w}_t^{-\gamma}\right]=\left(z_t+RB_t\right)^{-\gamma}E_{t+1}\left[\left(1+\frac{\delta\widetilde{x}_t}{z_t+RB_t}\right)^{-\gamma}\right].$$

The Taylor expansion of

$$(1+x)^{-\gamma}=1-\gamma x+\frac{\gamma(\gamma+1)}{2}x^2+\cdots+(-1)^n\frac{\Gamma(\gamma+n)}{n!\Gamma(\gamma)}x^n+O(x^{n+1}).$$

Since we are only concerned with terms of order δ^n arising from the nth-order moment,

$$E_t\left[\left(\widetilde{w}_{t+1}^s\right)^{-\gamma}\right]=\left(z_{t+1}^s+RB_{t+1}^s\right)^{-\gamma}\left[1+(-1)^n\frac{\Gamma(\gamma+n)}{n!\Gamma(\gamma)}\frac{\delta^n k_{n,t+1}^s}{\left(z_{t+1}^s\right)^n}\right].$$

(Notice that RB_{t+1}^s will be of order δ, so any contribution of the RB_{t+1}^s in the denominator will also be of order greater than δ^n.) We assume here that $z_{t+1}^s+RB_{t+1}^s$ is larger than any possible realization of $\delta\widetilde{x}_{t+1}^s$.

Thus, we can write (54) as

$$\begin{aligned}
&\left(y_t^{s+1} + Rb_t^{s+1} - b_{t+1}^s\right)^{-\gamma} \\
= \quad & \beta R (s+1)_{\phi^{-1}}^{\gamma} \left(z_{t+1}^s + RB_{t+1}^s\right)^{-\gamma} \left[1 + (-1)^n \frac{\Gamma(\gamma+n)}{n!\Gamma(\gamma)} \frac{\delta^n D_{n,t+1}^s}{\left(z_{t+1}^s\right)^n}\right], \quad (57)
\end{aligned}$$

where

$$D_{n,t+1}^s = \frac{(s+1)_\phi^{n-1}}{(s)_\phi^{n-1}} \frac{K_{n,t+2}^{s-1}}{R^n} + k_{n,t+1}^s. \tag{58}$$

Since

$$(1+x)^{-1/\gamma} = 1 - \frac{x}{\gamma} + O(x^2),$$

we can raise (57) to the power of $-1/\gamma$:

$$\begin{aligned}
& y_t^{s+1} + Rb_t^{s+1} - b_{t+1}^s \\
= \quad & \frac{(\beta R)^{-1/\gamma}}{(s+1)_{\phi^{-1}}} \left[h_{t+1}^s + Rb_{t+1}^s - (-1)^n \frac{\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)} \frac{\delta^n D_{n,s+1}^t}{\left(z_{t+1}^s\right)^{n-1}}\right],
\end{aligned}$$

where we have used the identity

$$h_{t+1}^s + Rb_{t+1}^s = z_{t+1}^s + RB_{t+1}^s.$$

Multiplying and dividing the righthand side by R gives

$$\begin{aligned}
& y_t^{s+1} + Rb_t^{s+1} - b_{t+1}^s \\
= \quad & \frac{\phi^{t+1}}{(s+1)_\phi} \left[\frac{h_{t+1}^s}{R} + b_{t+1}^s - (-1)^n \frac{\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)} \frac{\delta^n D_{n,t+1}^s}{R\left(z_{t+1}^s\right)^{n-1}}\right],
\end{aligned}$$

where we have used (48).

Using (49), we can solve for b_t:

$$\begin{aligned}
b_{t+1}^s \quad = \quad & \frac{(s+1)_\phi}{(s+2)_\phi}(y_t^{s+1} + Rb_t^{s+1}) \\
& - \frac{\phi^{s+1}}{(s+2)_\phi} \left[\frac{h_{t+1}^s}{R} - (-1)^n \frac{\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)} \frac{\delta^n D_{n,t+1}^s}{R\left(z_{t+1}^s\right)^{n-1}}\right]. \quad (59)
\end{aligned}$$

Thus, we can read off that the zeroth-order contribution to the bond demand is the familiar result that

$$\left(b_{t+1}^s\right)^{(0)} = \frac{1}{(s+2)_\phi} \left[(s+1)_\phi (y_t^{s+1} + Rb_t^{s+1}) - \phi^{s+1} \frac{h_{t+1}^s}{R}\right] \tag{60}$$

while the lowest-order contribution of the nth moment is

$$(-1)^n \frac{\phi^{s+1}}{(s+2)_\phi} \frac{\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)} \frac{\delta^n D^s_{n,t+1}}{R\left(z^s_{t+1}\right)^{n-1}}. \tag{61}$$

Notice that the sign of this contribution will alternate with n. The precautionary saving effect is positive, the skewness effect is negative, the direct kurtosis effect is positive again, etc.

Substituting (60) into (55) and using (28), we can determine how the expected wealth at $t+1$ depends on current wealth at t.

$$z^s_{t+1} = \frac{(s+1)_\phi}{(s+2)_\phi} R w^{s+1}_t. \tag{62}$$

Next, we consider consumption. The budget constraint gives

$$c^{s+1}_t = y^{s+1}_t + R b^{s+1}_t - b^s_{t+1}. \tag{63}$$

Plugging (59) into (63) and using (**??**) to further simplify, we get

$$c^{s+1}_t = \frac{\phi^{s+1}}{(s+2)_\phi} \left[w^{s+1}_t - (-1)^n \frac{\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)} \frac{\delta^n D^s_{n,t+1}}{R\left(z^s_{t+1}\right)^{n-1}} \right].$$

Next we factor out w^{s+1}_t and make use of (48) and (62) to get

$$c^{s+1}_t = \frac{w^{s+1}_t}{(s+2)_{\phi^{-1}}} \left[1 - (-1)^n \frac{\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)} \frac{(s+1)_\phi}{(s+2)_\phi} \frac{\delta^n D^s_{n,t+1}}{\left(z^s_{t+1}\right)^{n}} \right]. \tag{64}$$

Raising c^{s+1}_t to the power of $1-\gamma$ as in the utility function,

$$\left(c^{s+1}_t\right)^{1-\gamma} = \left(\frac{w^{s+1}_t}{(s+2)_{\phi^{-1}}} \right)^{1-\gamma} \left[1 - (-1)^n \frac{(1-\gamma)\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)} \frac{(s+1)_\phi}{(s+2)_\phi} \frac{\delta^n D^s_{n,t+1}}{\left(z^s_{t+1}\right)^{n}} \right]. \tag{65}$$

We also need to work out the expected value function, which depends on $\widetilde{w}^s_{t+1}$. Plugging the lowest-order contribution of the nth moment, as expressed in (61) into (56),

$$\widetilde{w}^s_{t+1} = z^s_{t+1} + (-1)^n \frac{\phi^{s+1}}{(s+2)_\phi} \frac{\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)} \frac{\delta^n D^s_{n,t+1}}{\left(z^s_{t+1}\right)^{n-1}} + \delta \widetilde{x}^s_{t+1}. \tag{66}$$

Inserting this into the value function (50), we obtain

$$\begin{aligned} v^s_{t+1}(\widetilde{w}^s_{t+1}) &= \frac{1}{1-\gamma} (s+1)^\gamma_{\phi^{-1}} \left(z^s_{t+1}\right)^{1-\gamma} \\ &\quad \times \left[1 + (-1)^n \frac{\phi^{s+1}}{(s+2)_\phi} \frac{\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)} \frac{\delta^n D^s_{n,t+1}}{\left(z^s_{t+1}\right)^{n}} + \frac{\delta \widetilde{x}^s_{t+1}}{z^s_{t+1}} \right]^{1-\gamma} \\ &\quad \times \left[1 + (-1)^n \frac{\Gamma(\gamma+n-1)}{n!\Gamma(\gamma-1)} \frac{(s+1)^{n-1}_\phi}{(s)^{n-1}_\phi} \frac{\delta^n K^{s-1}_{n,t}}{R^n \left(\widetilde{w}^s_{t+1}\right)^{n}} \right] \end{aligned}$$

Using the Taylor expansion (9), the expectation of the value function is (focusing only on the lowest-order contribution of the nth moment)

$$
\begin{aligned}
E_t\left[v^s_{t+1}(\widetilde{w}^s_{t+1})\right] &= \frac{1}{1-\gamma}(s+1)^{\gamma}_{\phi^{-1}}\left(z^s_{t+1}\right)^{1-\gamma} \\
&\times\left[1+(-1)^n\frac{\phi^{s+1}}{(s+2)_\phi}\frac{(1-\gamma)\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)}\frac{\delta^n D^s_{n,t+1}}{\left(z^s_{t+1}\right)^n}\right. \\
&+(-1)^n\frac{\Gamma(\gamma+n-1)}{n!\Gamma(\gamma-1)}\frac{\delta^n k^s_{n,t+1}}{\left(z^s_{t+1}\right)^n} \\
&\left.+(-1)^n\frac{\Gamma(\gamma+n-1)}{n!\Gamma(\gamma-1)}\frac{(s+1)^{n-1}_\phi}{(s)^{n-1}_\phi}\frac{\delta^n K^{s-1}_{n,t+2}}{R^n\left(z^s_{t+1}\right)^n}\right].
\end{aligned}
\tag{67}
$$

Notice in the last term that we can replace $\widetilde{w}^s_{t+1}$ by z^s_{t+1} since any deviations from z^s_{t+1} will be of order δ. Making use of (58) and (62), we can write (67) as

$$
\begin{aligned}
E_t\left[v^s_{t+1}(\widetilde{w}^s_{t+1})\right] &= \frac{1}{1-\gamma}(s+1)^{\gamma}_{\phi^{-1}}\left(\frac{(s+1)_\phi}{(s+2)_\phi}Rw^{s+1}_t\right)^{1-\gamma} \\
&\times\left[1+(-1)^n\frac{\phi^{s+1}}{(s+2)_\phi}\frac{(1-\gamma)\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)}\frac{\delta^n D^s_{n,t+1}}{\left(z^s_{t+1}\right)^n}\right. \\
&\left.+(-1)^n\frac{\Gamma(\gamma+n-1)}{n!\Gamma(\gamma-1)}\frac{\delta^n D^s_{n,t+1}}{\left(z^s_{t+1}\right)^n}\right].
\end{aligned}
\tag{68}
$$

Using (48) once again,

$$
\begin{aligned}
E_t\left[v^s_{t+1}(\widetilde{w}^s_{t+1})\right] &= \frac{1}{1-\gamma}(s+1)_{\phi^{-1}}\left(\frac{w^{s+1}_t}{(s+2)_{\phi^{-1}}}\frac{R}{\phi}\right)^{1-\gamma} \\
&\times\left[1+(-1)^n\frac{\phi^{s+1}}{(s+2)_\phi}\frac{(1-\gamma)\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)}\frac{\delta^n D^s_{n,t+1}}{\left(z^s_{t+1}\right)^n}\right. \\
&\left.+(-1)^n\frac{\Gamma(\gamma+n-1)}{n!\Gamma(\gamma-1)}\frac{\delta^n D^s_{n,t+1}}{\left(z^s_{t+1}\right)^n}\right].
\end{aligned}
\tag{69}
$$

Finally, we substitute (65) and (69) into the Bellman equation (**??**):

$$
\begin{aligned}
v_t^{s+1} &= \frac{1}{1-\gamma}\left(\frac{w_{t+1}^s}{(s+2)_{\phi^{-1}}}\right)^{1-\gamma} \\
&\quad \times\left[1-(-1)^n\frac{(1-\gamma)\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)}\frac{(s+1)_\phi}{(s+2)_\phi}\frac{\delta^n D_{n,t+1}^s}{\left(z_{t+1}^s\right)^n}\right] \\
&\quad +\frac{1}{1-\gamma}(s+1)_{\phi^{-1}}\frac{\beta R^{1-\gamma}}{\phi^{1-\gamma}}\left(\frac{w_t^{s+1}}{(s+2)_{\phi^{-1}}}\right)^{1-\gamma} \\
&\quad \times\left[1+(-1)^n\frac{\phi^{s+1}}{(s+2)_\phi}\frac{(1-\gamma)\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)}\frac{\delta^n D_{n,t+1}^s}{\left(z_{t+1}^s\right)^n}\right. \\
&\quad \left.+(-1)^n\frac{\Gamma(\gamma+n-1)}{n!\Gamma(\gamma-1)}\frac{\delta^n D_{n,t+1}^s}{\left(z_{t+1}^s\right)^n}\right].
\end{aligned}
$$

Using (21) and combining terms, we obtain

$$
\begin{aligned}
v_t^{s+1} &= \frac{1}{1-\gamma}\left(\frac{w_t^{s+1}}{(s+2)_{\phi^{-1}}}\right)^{1-\gamma}\left\{1-(-1)^n\frac{(1-\gamma)\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)}\frac{(s+1)_\phi}{(s+2)_\phi}\frac{\delta^n D_{n,t+1}^s}{\left(z_{t+1}^s\right)^n}\right. \\
&\quad +(s+1)_{\phi^{-1}}\phi^{-1}\left[1+(-1)^n\frac{\phi^{s+1}}{(s+2)_\phi}\frac{(1-\gamma)\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)}\frac{\delta^n D_{n,t+1}^s}{\left(z_{t+1}^s\right)^n}\right. \\
&\quad \left.\left.+(-1)^n\frac{\Gamma(\gamma+n-1)}{n!\Gamma(\gamma-1)}\frac{\delta^n D_{n,t+1}^s}{\left(z_{t+1}^s\right)^n}\right]\right\}.
\end{aligned}
$$

The identities (48) and (49) imply

$$
\begin{aligned}
v_t^{s+1} &= \frac{1}{1-\gamma}\left(\frac{w_t^{s+1}}{(s+2)_{\phi^{-1}}}\right)^{1-\gamma} \\
&\quad \times\left\{(s+2)_{\phi^{-1}}-(-1)^n\frac{(1-\gamma)\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)}\frac{(s+1)_\phi}{(s+2)_\phi}\frac{\delta^n D_{n,t+1}^s}{\left(z_{t+1}^s\right)^n}\right. \\
&\quad +(-1)^n\frac{(s+1)_\phi}{(s+2)_\phi}\frac{(1-\gamma)\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)}\frac{\delta^n D_{n,t+1}^s}{\left(z_{t+1}^s\right)^n} \\
&\quad \left.+(-1)^n(s+1)_{\phi^{-1}}\phi^{-1}\frac{\Gamma(\gamma+n-1)}{n!\Gamma(\gamma-1)}\frac{\delta^n D_{n,t+1}^s}{\left(z_{t+1}^s\right)^n}\right\}.
\end{aligned}
$$

Cancelling the second and third terms and factoring out $(s+2)_{\phi^{-1}}$, the value function becomes

$$
\begin{aligned}
v_t^{s+1} &= \frac{(s+2)_{\phi^{-1}}^\gamma}{1-\gamma}\left(w_t^{s+1}\right)^{1-\gamma} \\
&\quad \times\left\{1+(-1)^n\frac{(s+1)_{\phi^{-1}}\phi^{-1}}{(s+2)_{\phi^{-1}}}\frac{\Gamma(\gamma+n-1)}{n!\Gamma(\gamma-1)}\frac{\delta^n D_{n,t+1}^s}{\left(z_{t+1}^s\right)^n}\right\}.
\end{aligned}
$$

Making use of (48) and (62), we get the expression

$$\begin{aligned} v_t^{s+1} &= \frac{(s+2)_{\phi^{-1}}^{\gamma}}{1-\gamma}\left(w_t^{s+1}\right)^{1-\gamma} \\ &\quad \times\left\{1+(-1)^n \frac{\Gamma(\gamma+n-1)}{n!\Gamma(\gamma-1)}\left(\frac{(s+2)_\phi}{(s+1)_\phi}\right)^{n-1} \frac{\delta^n D_{n,t+1}^s}{R^n\left(w_t^{s+1}\right)^n}\right\}. \quad (70) \end{aligned}$$

Compare this to our posited value function (35). We see that the expressions are the same if

$$K_{n,t+1}^s = D_{n,t+1}^s.$$

Thus by Eq. (58), $K_{n,t+1}^s$ satisfies the difference equation

$$K_{n,t+1}^s = \frac{(s+1)_\phi^{n-1}}{(s)_\phi^{n-1}} \frac{K_{n,t+2}^{s-1}}{R^n} + k_{n,t+1}^s. \quad (71)$$

Making use of Eqs. (62) and (71), we can rewrite (59) and (70) as

$$\begin{aligned} v_t^{s+1} &= \frac{(s+2)_{\phi^{-1}}^{\gamma}}{1-\gamma}\left(w_t^{s+1}\right)^{1-\gamma} \\ &\quad \times\left\{1+(-1)^n \frac{\Gamma(\gamma+n-1)}{n!\Gamma(\gamma-1)} \frac{\delta^n (K_{n,t}^{s+1}-k_{n,t}^{s+1})}{\left(w_t^{s+1}\right)^n}\right\}. \quad (72) \end{aligned}$$

$$\begin{aligned} b_{t+1}^s &= \frac{(s+1)_\phi}{(s+2)_\phi}(y_t^{s+1}+Rb_t^{s+1}) \\ &\quad - \frac{\phi^{s+1}}{(s+2)_\phi}\left[\frac{h_{t+1}^s}{R} - (-1)^n \frac{\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)} \frac{\delta^n (K_{n,t}^{s+1}-k_{n,t}^{s+1})}{\left(w_t^{s+1}\right)^{n-1}}\right]. \quad (73) \end{aligned}$$

Now, let us collect the second and third-order contributions expressed in Eqs. (72) and (73). We stop at third order because cross terms will begin to appear at fourth order. In order to compute the entire fourth-order correction, we would have to include terms involving products of variances in addition to the lowest order contribution of the fourth-order moment computed above. The algebra of these cross terms is messier than the algebra for the lowest-order contributions, so we leave that to future work. We obtain

$$\begin{aligned} v_t^{s+1}(b_t^{s+1}, y_t^{s+1}) &= \frac{(s+2)_{\phi^{-1}}^{\gamma}}{1-\gamma}\left(w_t^{s+1}\right)^{1-\gamma}\left[1-\delta^2 \frac{(1-\gamma)\gamma}{2} \frac{K_{2,t}^{s+1}-k_{2,t}^{s+1}}{\left(w_t^{s+1}\right)^2}\right. \\ &\quad \left. +\delta^3 \frac{(1-\gamma^2)\gamma}{6} \frac{K_{3,t}^{s+1}-k_{3,t}^{s+1}}{\left(w_t^{s+1}\right)^3}\right] + O(\delta^4) \end{aligned}$$

and

$$
\begin{aligned}
b_{t+1}^{s}(b_t^{s+1}, y_t^{s+1}) &= \frac{(s+1)_\phi}{(s+2)_\phi}(y_t^{s+1} + Rb_t^{s+1}) \\
&\quad - \frac{\phi^{s+1}}{(s+2)_\phi}\left[\frac{h_{t+1}^s}{R} - \frac{\gamma+1}{2}\frac{\delta^2(K_{2,t}^{s+1} - k_{2,t}^{s+1})}{w_t^{s+1}}\right. \\
&\quad \left. + \frac{(\gamma+1)(\gamma+2)}{6}\frac{\delta^3(K_{3,t}^{s+1} - k_{3,t}^{s+1})}{\left(w_t^{s+1}\right)^2}\right] + O(\delta^4).
\end{aligned}
$$

These are the results reported in Section 3.

C Consumption Growth Equation

The consumption function to third order is given by Eq. (38):

$$
\begin{aligned}
c_t^{s+1} &= \frac{\phi^{s+1}}{(s+2)_\phi}\left[w_t^{s+1} - \frac{\gamma+1}{2}\frac{\delta^2(K_{2,t}^{s+1} - k_{2,t}^{s+1})}{w_t^{s+1}}\right. \\
&\quad \left. + \frac{(\gamma+1)(\gamma+2)}{6}\frac{\delta^3(K_{3,t}^{s+1} - k_{3,t}^{s+1})}{\left(w_t^{s+1}\right)^2}\right] + O(\delta^4).
\end{aligned}
$$

If we make use of Eqs. (62) and (71), we can rewrite this as

$$
c_t^{s+1} = \frac{\phi^{s+1}}{(s+2)_\phi}\left[w_t^{s+1} - \frac{\gamma+1}{2}\frac{K_{2,t+1}^s}{Rz_{t+1}^s} + \frac{(\gamma+1)(\gamma+2)}{6}\frac{K_{3,t+1}^s}{R\left(z_{t+1}^s\right)^2}\right] + O(\delta^4). \tag{74}
$$

Expected wealth next period is

$$
E_t[\widetilde{w}_{t+1}^s] = h_{t+1}^s + Rb_{t+1}^s.
$$

Substituting in Eq. (37), we obtain

$$
\begin{aligned}
E_t[\widetilde{w}_{t+1}^s] &= \frac{R(s+1)_\phi}{(s+2)_\phi}w_t^{s+1} \\
&\quad + \frac{\phi^{s+1}}{(s+2)_\phi}\left[\frac{\gamma+1}{2}\frac{\delta^2 K_{2,t+1}^s}{z_{t+1}^s} - \frac{(\gamma+1)(\gamma+2)}{6}\frac{\delta^3 K_{3,t+1}^s}{\left(z_{t+1}^s\right)^2}\right] + O\left(\delta^4\right).
\end{aligned}
$$

Solving for w_t^{s+1}, we obtain

$$
\begin{aligned}
w_t^{s+1} &= \frac{(s+2)_\phi}{R(s+1)_\phi}E[\widetilde{w}_{t+1}^s] \\
&\quad - \frac{\phi^{s+1}}{R(s+1)_\phi}\left[\frac{\gamma+1}{2}\frac{\delta^2 K_{2,t+1}^s}{z_{t+1}^s} - \frac{(\gamma+1)(\gamma+2)}{6}\frac{\delta^3 K_{3,t+1}^s}{\left(z_{t+1}^s\right)^2}\right] + O\left(\delta^4\right)
\end{aligned}
$$

This can be put into Eq. (74). Using (49), we then obtain

$$c_t^{s+1} = \frac{\phi^{s+1}}{R(s+1)_\phi}\left[E[\widetilde{w}_{t+1}^s] - \frac{\gamma+1}{2}\frac{K_{2,t+1}^s}{z_{t+1}^s} + \frac{(\gamma+1)(\gamma+2)}{6}\frac{K_{3,t+1}^s}{\left(z_{t+1}^s\right)^2}\right] + O(\delta^4).$$

Since

$$E[\widetilde{w}_{t+1}^s] = z_{t+1}^s + O(\delta^2),$$

this can be simplified to

$$c_t^{s+1} = \frac{\phi^{s+1}}{R(s+1)_\phi}E[\widetilde{w}_{t+1}^s]\left[1 - \frac{\gamma+1}{2}\frac{\delta^2 K_{2,t+1}^s}{E[\widetilde{w}_{t+1}^s]^2} + \frac{(\gamma+1)(\gamma+2)}{6}\frac{\delta^3 K_{3,t+1}^s}{E[\widetilde{w}_{t+1}^s]^3}\right] + O\left(\delta^4\right). \tag{75}$$

Meanwhile if we update (38) to $t+1$, we obtain

$$\begin{aligned}
\widetilde{c}_{t+1}^s &= \frac{\phi^s}{(s+1)_\phi}\left[E[\widetilde{w}_{t+1}^s] + \delta\widetilde{x}_{t+1}^s - \frac{\gamma+1}{2}\frac{\delta^2(K_{2,t+1}^s - k_{2,t+1}^s)}{E[\widetilde{w}_{t+1}^s] + \delta\widetilde{x}_{t+1}^s}\right. \\
&\quad \left. + \frac{(\gamma+1)(\gamma+2)}{6}\frac{\delta^3(K_{3,t+1}^s - k_{3,t+1}^s)}{(E[\widetilde{w}_{t+1}^s] + \delta\widetilde{x}_{t+1}^s)^2}\right] + O(\delta^4) \\
&= \frac{\phi^s}{(s+1)_\phi}\left[E[\widetilde{w}_{t+1}^s] + \delta\widetilde{x}_{t+1}^s - \frac{\gamma+1}{2}\frac{\delta^2(K_{2,t+1}^s - k_{2,t+1}^s)}{E[\widetilde{w}_{t+1}^s]}\left(1 - \frac{\delta\widetilde{x}_{t+1}^s}{E[\widetilde{w}_{t+1}^s]}\right)\right. \\
&\quad \left. + \frac{(\gamma+1)(\gamma+2)}{6}\frac{\delta^3(K_{3,t+1}^s - k_{3,t+1}^3)}{E[\widetilde{w}_{t+1}^s]^2}\right] + O(\delta^4).
\end{aligned}$$

Taking the expectation,

$$\begin{aligned}
E_t[\widetilde{c}_{t+1}^s] &= \frac{\phi^s}{(s+1)_\phi}E[\widetilde{w}_{t+1}^s]\left[1 - \frac{\gamma+1}{2}\frac{\delta^2(K_{2,t+1}^s - k_{2,t+1}^s)}{E[\widetilde{w}_{t+1}^s]^2}\right. \\
&\quad \left. + \frac{(\gamma+1)(\gamma+2)}{6}\frac{\delta^3(K_{3,t+1}^s - k_{3,t+1}^s)}{E[\widetilde{w}_{t+1}^s]^3}\right] + O(\delta^4).
\end{aligned} \tag{76}$$

Dividing (76) by (75), we obtain the expected rate of consumption growth

$$\frac{E_t[\widetilde{c}_{t+1}^s]}{c_t^{s+1}} = \frac{R}{\phi}\left[1 + \frac{\gamma+1}{2}\frac{\delta^2 k_{2,t+1}^s}{E[\widetilde{w}_{t+1}^s]^2} - \frac{(\gamma+1)(\gamma+2)}{6}\frac{\delta^3 k_{3,t+1}^s}{E[\widetilde{w}_{t+1}^s]^3}\right] + O(\delta^4).$$

References

[1] Aiyagari, S. Rao, (1994), "Uninsured Idiosyncratic Risk and Aggregate Saving," *Quarterly Journal of Economics* 109: 659-684.

[2] Browning, Martin and Thomas F. Crossley, (2001), "The Life-Cycle Model of Consumption and Saving," *Journal of Economic Perspectives* 15(3): 3-22.

[3] Carroll, Christopher D., (1997), "Buffer-Stock Saving and the Life-Cycle/Permanent Income Hypothesis," *Quarterly Journal of Economics* 112: 1-55.

[4] Carroll, Christopher D., (2001), "Death to the Log-Linearized Consumption Euler Equation! (And Very Poor Health to the Second-Order Approximation," *Advances in Macroeconomics* 1: Article 6.

[5] Carroll, Christopher D., (2001), "Theoretical Foundations of Buffer Stock Saving," John Hopkins Working Paper.

[6] Carroll, Christopher D. and Miles S. Kimball, (1996), "On the Concavity of the Consumption Function," *Econometrica* 64: 981-992.

[7] Chamberlain, Gary and Charles A. Wilson, (2000), "Optimal Intertemporal Consumption Under Uncertainty," *Review of Economic Dynamics* 3: 365-395.

[8] Cochrane, John H., (1989), "The Sensitivity of Tests of the Intertemporal Allocation of Consumption to Near-Rational Alternatives," *American Economic Review* 79(3): 319-337.

[9] Deaton, Angus, (1991), "Saving and Liquidity Constraints," *Econometrica* 59: 1221-1248.

[10] Feigenbaum, James, (2001), "Perturbative and Numerical Computations of Interest Rates in an OLG Bond Economy," Working Paper.

[11] Feigenbaum, James, (2001), "Symmetry in the HARA Class of Utility Functions," Working Paper.

[12] Feigenbaum, James, (2002), "Anticipatory and Precautionary Saving in an Economy Without Capital," Working Paper.

[13] Hall, Robert E., (1988), "Intertemporal Substitution in Consumption," *Journal of Political Economy* 96: 339-357.

[14] Halmos, Paul R., (1974), *Measure Theory* (Springer-Verlag: New York).

[15] Huggett, Mark, (1993), "The Risk-Free Rate in Heterogeneous-Agent Incomplete-Insurance Economies," *Journal of Economic Dynamics and Control* 17: 953-969.

[16] Judd, Kenneth L., (1999), *Numerical Methods in Economics* (MIT Press: Cambridge, MA).

[17] Koo, Hyeung Keun, (1999), "Consumption and Portfolio Selection with Labor Income: A Discrete-Time Approach," *Mathematical Methods of Operations Research* 50: 219-243.

[18] Leland, Hayne E., (1968), "Saving and Uncertainty: The Precautionary Demand for Saving," *Quarterly Journal of Economics* 82: 465-473.

[19] Letendre, Marc-André and Gregor W. Smith, (2001), "Precautionary Saving and Portfolio Allocation: DP by GMM," *Journal of Monetary Economics* 48: 197-215.

[20] Samuelson, Paul A., (1970), "The Fundamental Approximation Theorem of Portfolio Analysis in terms of Means, Variances, and Higher Moments," *Review of Economic Studies* 37: 537-542.

[21] Sandmo, A., (1970), "The Effect of Uncertainty on Saving Decisions," *Review of Economic Studies* 37: 353-360.

[22] Skinner, Jonathan, (1988), "Risky Income, Life Cycle Consumption, and Precautionary Savings," *Journal of Monetary Economics* 22: 237-255.

[23] Talmain, Gabriel, (1998), "An Analytical Approximate Solution to the Problem of Precautionary Savings," *Journal of Economic Dynamics and Control* 21: 113-124.

[24] Viceira, Luis M., (2001), "Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income," *Journal of Finance* 56: 433-470.