Summer Institute for Mathematics at the University of Washington

2021 Problems

1. Prove that the angles of a triangle all have rational cosines if and only if the triangle is similar to one with rational sides.

2. Given a prime number $p \geq 7$, the fraction $1/p$ has a periodic decimal expansion of the form

$$1/p = .a_1 a_2 \ldots a_k a_1 a_2 \ldots a_k a_1 \ldots,$$

where k is the length of the period. Let N be the positive integer whose expression in decimal notation is $a_1 a_2 \ldots a_k$. Prove that N is divisible by 9.

3. Find the last two digits of 3^{1000}.

4. Suppose that x, y, and z are real numbers satisfying $xyz = (1-x)(1-y)(1-z)$ and $0 \leq x, y, z \leq 1$. Prove that

$$x(1-z) + y(1-x) + z(1-y) \geq 3/4.$$

5. Determine all triples (x, y, z) of integers satisying

$$x^3 + y^3 + z^3 = (x+y+z)^3.$$

6. Suppose that n is an integer greater than or equal to 3 and that $a_0, a_1, \dots, a_{n-3}$ are real numbers. Prove that the zeros of the polynomial

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-3}x^3 + x^2 + x + 1$$

are not all real.

7. Prove that the equation $x^n + y^n = z^n$, where n is an integer greater than 1, has no solution in integers x, y, z with x and y satisfying $0 < x \le n$ and $0 < y \le n$. You may not quote Fermat's Last Theorem.

8. Let A, B, C be consecutive points on a circle such that the length of the arc from A to B is strictly greater than the length of the arc from B to C. Let D be the midpoint of the arc from A to C. Let the line through D perpendicular to the line through A and B intersect the segment AB at E. Prove that

$$AE = EB + BC,$$

where AE, EB, BC denote the lengths of the corresponding segments.