Sample Problems
DO NOT SUBMIT

1. Let $x$ be a positive real number. Prove that
\[ \sqrt{\frac{[x]}{x + \{x\}}} + \sqrt{\frac{\{x\}}{x + [x]}} \geq 1, \]
where $[x]$ is the integer part of $x$ and $\{x\}$ is the fractional part.

2. A drawer has $d$ more black socks than white socks. Suppose that if two socks are selected at random then the probability that they match is $1/2$. How many socks of each color are there?

3. Prove that
\[ \log_e(e^n - 1) \log_e(e^n + 1) + \log_\pi(\pi^n - 1) \log_\pi(\pi^n + 1) < e^2 + \pi^2. \]

4. A sequence of integers is defined as follows. Starting with $n = 1$, list all the multiples of $n$ up to $n^2$. Thus, the sequence starts with the multiples of 1 up to 1, followed by the multiples of 2 up to 4, then the multiples of 3 up to 9, and so on, so that its first few terms are 1, 2, 4, 3, 6, 9, 4, 8, 12, 16. What is the 2011th term in the sequence?

5. Let $a, b, c$ be positive real numbers and let $0 < m < \frac{1}{4}$. Prove that at least one of the following equations has real roots.
\[
ax^2 + bx + cm = 0 \\
bx^2 + cx + am = 0 \\
cx^2 + ax + bm = 0.
\]

6. Let $A, B, C$ be the angles of a triangle. Prove that
\[ \sin A + \sin B \sin C \leq \frac{1 + \sqrt{5}}{2}. \]
7. Let $a, b, c$ be the length of sides opposite angles $A, B, C$ in triangle ABC. Prove that
\[
\frac{\cos^3 A}{a} + \frac{\cos^3 B}{b} + \frac{\cos^3 C}{c} < \frac{a^2 + b^2 + c^2}{2abc}.
\]

8. Let $a, b, c$ be positive real numbers satisfying $abc = 1$. Prove that
\[
a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 3 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.
\]