Summer Institute for Mathematics at the University of Washington

# Sample solutions to SIMUW problems

The problems that are part of the SIMUW application may be different from those you have encountered. The desired solutions may also be different from solutions that might be acceptable in a different context. We want to see that you understand how to attack a non-standard problem and that you can write up a careful solution that addresses logical issues that may arise. Therefore, you need to pay careful attention to both the mathematics and your exposition. Checking a particular case of a problem or performing a numerical experiment can be a good way to guess what the answer to the problem is, but they are generally not sufficient to show that the answer is correct. This document will give some examples of incomplete or unsatisfactory solutions as well as correct solutions. There are often many approaches to a correct solution, and still more ways to write a careful exposition of the solution, so that the correct solutions provided below should be regarded merely as model examples.

**1.** For which positive integer $n$ is $\sqrt[n]{n}$ the largest possible?

(a) *A Proposed Solution*: We compute a few values: $\sqrt[1]{1} = 1, \sqrt[2]{2} = 1.414, \sqrt[3]{3} = 1.442, \sqrt[4]{4} = 1.414, \sqrt[5]{5} = 1.380$. From this we see that the maximum occurs for $n = 3$.

(b) *Analysis of the Proposed Solution*: To say that the maximum occurs for $n = 3$ is to claim that an infinite sequence of statements is true – that $\sqrt[3]{3} \geq \sqrt[n]{3}$ for $n = 1, 2, 3, 4, 5, \cdots$ . The proposed solution only verifies this inequality for the first five values of $n$. This sheds no light on what happens for $n \geq 6$. Therefore this solution is incomplete, not convincing, and possibly wrong. As it turns out, the guessed answer is correct, but the argument given is insufficient.

(c) *A Better Solution*: We compute a few values: $\sqrt[1]{1} = 1, \sqrt[2]{2} = 1.414, \sqrt[3]{3} = 1.442, \sqrt[4]{4} = 1.414, \sqrt[5]{5} = 1.380$. From this we make the guess that the maximum occurs for $n = 3$. Let's prove that this guess is correct. It would suffice to show for $n > 3$ that $\sqrt[n]{n}$ is decreasing; that is, if $n > 3$, then

$$\sqrt[n]{n} > \sqrt[n+1]{n+1}.$$

Raising both sides to the $n(n+1)$ power, we see that this inequality is equivalent to the inequality

$$n^{n+1} > (n+1)^n,$$

and dividing both sides by $n^n$, we see that this is equivalent to the inequality

$$n > (1 + \frac{1}{n})^n.$$

Since $n > 3$, if we can show $3 > (1 + \frac{1}{n})^n$, we are done. To do so, we can use the binomial theorem:

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n =& 1 + n \cdot \frac{1}{n} + \binom{n}{2} \cdot \frac{1}{n^2} + \cdots + \frac{1}{n^n} \\ <& 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots + \frac{1}{1\cdots n} \\ <& 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \\ <& 1 + 2\end{aligned}$$

**2.** Let the sequence $L_n$ be defined recursively by $L_1 = 1, L_2 = 3, L_n = L_{n-1} + L_{n-2},$ if $n \geq 3$. Compute $L_n^2 - L_{n-1}L_{n+1}$.

(a) *A Proposed Solution*: We evaluate $L_n$ through $n = 8$ and find: $L_3 = 4,\ L_4 = 7,\ L_5 = 11,\ L_6 = 18,\ L_7 = 29,\ L_8 = 47$. Then we compute $L_n^2 - L_{n-1}L_{n+1}$ from $n = 2$ through $n = 7$ and obtain, in succession, the values $5,\ -5,\ 5,\ -5,\ 5,\ -5$. From this we conclude for all integers $n \geq 2$ that

$$L_n^2 - L_{n-1}L_{n+1} = 5 \cdot (-1)^n.$$

(b) *Analysis of the Proposed Solution*: Once again, this solution is incomplete, because it tests only a finite number of values of $n$, whereas we must compute $L_n^2 - L_{n-1}L_{n+1}$ for infinitely many values of $n$. Guessing a pattern for $L_n^2 - L_{n-1}L_{n+1}$ is a good first step, but it is not a solution.

(c) *A Better Solution*: We evaluate $L_n$ through $n = 8$ and find: $L_3 = 4,\ L_4 = 7,\ L_5 = 11,\ L_6 = 18,\ L_7 = 29,\ L_8 = 47$. Then we compute $L_n^2 - L_{n-1}L_{n+1}$ from $n = 2$ through $n = 7$ and obtain, in succession, the values $5,\ -5,\ 5,\ -5,\ 5,\ -5$. From this we make the guess that for any integer $n \geq 2$ the equality

$$L_n^2 - L_{n-1}L_{n+1} = 5 \cdot (-1)^n$$

holds.

How might we verify this? Notice that since we know that $L_2^2 - L_1L_3 = 5$, it suffices to show for each $n \geq 2$ that the terms $L_{n+1}^2 - L_nL_{n+2}$ alternate in sign and have the same magnitude. Equivalently, it suffices to show for $n \geq 2$ that

$$\left(L_{n+1}^2 - L_nL_{n+2}\right) + \left(L_n^2 - L_{n-1}L_{n+1}\right) = 0.$$

By re-arranging terms and factoring, we obtain

$$\begin{aligned}&\left(L_{n+1}^2 - L_nL_{n+2}\right) + \left(L_n^2 - L_{n-1}L_{n+1}\right) \\ &\qquad = L_{n+1}\left(L_{n+1} - L_{n-1}\right) + L_n\left(L_n - L_{n+2}\right).\end{aligned}$$

Using the recursive formula that defines the sequence of $L_k$'s, we obtain

$$L_{n+1}\left(L_{n+1} - L_{n-1}\right) + L_n\left(L_n - L_{n+2}\right) = L_{n+1}L_n + L_n\left(-L_{n+1}\right) = 0.$$

This proves as desired that

$$\left(L_{n+1}^2 - L_n L_{n+2}\right) + \left(L_n^2 - L_{n-1}L_{n+1}\right) = 0,$$

allowing us to conclude that the terms $L_n^2 - L_{n-1}L_{n+1}$ alternate between the values $5$ and $-5$, as desired.