Stochastic Processes EE546 - Homework 2 — Solution 3 & 5

3. Apply the central limit theorem to the random walk in 4, comp. 4.7. Compare the results with an explicit calculation as above.

   Since the total displacement is a sum of identical, independent steps we can apply the central limit theorem to the displacement after r steps. Let X_i represent the ith step, for all X_i we have $\langle X_i \rangle = 0$ and $\langle X_i^2 \rangle = 1$. Let $Y = \sum_{i=1}^{r} X_i$ and $Z = Y/\sqrt{r}$. Then by the CLT

   $$P_Z \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}},$$

   which means that we can approximate $p_n(r) = 1/2^r \binom{r}{\frac{r-n}{2}}$ by

   $$P_Y \approx \frac{1}{\sqrt{2\pi r}} e^{-\frac{y^2}{2r}}.$$

   This approximation allows for displacements y larger than the number of steps, which is not admissible by the construction of the random walk. As r grows large we need to show that this probability,

   $$P_Y(|y| > r) = \frac{2}{\sqrt{2\pi r}} \int_r^\infty e^{-\frac{x^2}{2r}},$$

   vanishes. By examining that this expression note that this is equivalent to asking the probability of being $r/\sqrt{r} = \sqrt{r}$ standard deviations away from the mean, which vanishes as r grows.

5. A random set of dots on $(0, \infty)$ is constructed according to the following recipe. The probability for the first dot to lie in $(\tau_1 + \tau_1 + d\tau_1)$ is $w(\tau_1)d\tau_1$, where w is a given non-negative function with

   $$\int_0^\infty w(\tau) d\tau = \frac{1}{\gamma} < 1.$$

   The probability density for the second dot is $w(\tau_2 - \tau_1)$ and ro on. Calculate Q_s.

   The probability of the first event NOT happening is $Q_0 = 1 - \frac{1}{\gamma}$. The probability of only one event happening is the same as the second event not happening and thus $Q_1(\tau_1) = w(\tau_1)(1 - \frac{1}{\gamma}$. Following this line of reasoning and making use of the fact that joint PDFs for independent random variables are simply the products of the individual PDFs we get,

   $$Q_s(\tau_1, \tau_2, ..., \tau_s) = (1 - \frac{1}{\gamma}) \prod_{i=1}^{s} w(\tau_i - \tau_{i-1}),$$

   where $\tau_0 = 0$. It remains to show that 1.2 holds and that Q is indeed a probability density function for random dots and sums to 1. The construction already assumes that $\tau_1 < \tau_2 < \tau_3$ ..., so

   $$Q_0 + \int_{-\infty}^{\infty} Q_1(\tau_1) d\tau_1 + \int_{-\infty}^{\infty}\int_{\tau_1}^{\infty} Q_2(\tau_1, \tau_2) d\tau_2 d\tau_1 + \int_{-\infty}^{\infty}\int_{\tau_1}^{\infty}\int_{\tau_2}^{\infty} Q_3(\tau_1, \tau_2, \tau_3) d\tau_3 \tau_2 d\tau_1 + ...$$

   $$= (1 - \frac{1}{\gamma}) \left(1 + \int_{-\infty}^{\infty} w(\tau_1) d\tau_1 + \int_{-\infty}^{\infty}\int_{\tau_1}^{\infty} w(\tau_1) w(\tau_2 - \tau_1) d\tau_2 d\tau_1 + \int_{-\infty}^{\infty}\int_{\tau_1}^{\infty}\int_{\tau_2}^{\infty} w(\tau_1) w(\tau_2 - \tau_1) w(\tau_3 - \tau_2) d\tau_3 \tau_2 d\tau_1 + ... \right)$$

   $$= (1 - \frac{1}{\gamma}) \sum_{n=0}^{\infty} \left(\frac{1}{\gamma}\right)^n = (1 - \frac{1}{\gamma}) \frac{1}{1 - \frac{1}{\gamma}} = 1,$$

   as required.