Notes on Jordan Canonical Form

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1 Jordan blocks and Jordan form

A **Jordan Block** of size m and value λ is a matrix $J_m(\lambda)$ having the value λ repeated along the main diagonal, ones along the superdiagonal and zeros everywhere else. For example:

$$J_2(4) = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix} \qquad \text{and} \qquad J_4(-\frac{1}{2}) = \begin{pmatrix} -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

A **Jordan Form** is a block diagonal matrix consisting of several Jordan blocks. For example, we can form a Jordan Form from two copies of $J_2(4)$ and one copy of $J_4(-\frac{1}{2})$ as follows.

$$J = \begin{pmatrix} J_2(4) & 0 & 0 \\ 0 & J_2(4) & 0 \\ 0 & 0 & J_4(-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

Notice that the eigenvalues and eigenvectors of a matrix in Jordan Form can be read off without your having to do any work.

1. The eigenvalues are along the main diagonal (this is true of any upper-triangular matrix).

2. Eigenvectors can be found on the columns at the beginning of each block.

For example, in the above form J, we have the eigenvalues $\lambda = 1$ with multiplicity 4 and $\lambda = \frac{1}{2}$ with multiplicity 4. Furthermore, there are two eigenvectors associated with $\lambda = 1$, namely

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and one eigenvector associated with the eigenvalue $\lambda = -\frac{1}{2}$, namely

$$v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Note that any two basis vectors for $\text{span}(v_1, v_2)$ can be chosen as the eigenvectors for $\lambda = 1$ and any non-zero scalar multiple of v_3 can be chosen as the eigenvector for $\lambda = -\frac{1}{2}$.

2 Any matrix can be put in Jordan Form

Theorem: Let $A \in \mathbb{R}^{n \times n}$ and suppose that A has $k \leq n$ linearly independent eigenvectors $v_1, ..., v_k$ associated with (not necessarily distinct) eigenvalues $\lambda_1, ..., \lambda_k$. Then A is similar to a Jordan form with blocks

$$J_{m_1}(\lambda_1), ..., J_{m_k}(\lambda_k)$$

where $m_1 + ..., m_k = n$.

Note that there may be several choices that need to be evaluated for what the m_is are. These are discussed below. It will be useful to define the following notions.

1. The *algebraic multiplicity* $am(\lambda)$ of an eigenvalue λ is the number of times it shows up in the characteristic polynomial $|\lambda I - A|$.

2. The *geometric multiplicity* $gm(\lambda)$ of an eigenvalue λ is the dimension of the eigenspace associated with λ.

2.1 The geometric multiplicity equals algebraic multiplicity

In this case, there are as many blocks as eigenvectors for λ, and each has size 1. For example, take the identity matrix $I \in \mathrm{n} \times \mathrm{n}$. There is one eigenvalue $\lambda = 1$ and it has n eigenvectors (the standard basis $e_1, .., en$ will do). So $am(\lambda) = gm(\lambda) = n$ and I is similar to (and equal to) the Jordan form

$$J = \begin{pmatrix} J_1(1) & 0 & ... & 0 \\ 0 & J_1(1) & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & J_1(1) \end{pmatrix}$$

2.2 The geomestric multiplicity equals 1

In this case, there is one block for the eigenvalue and its size is $m_j = am(\lambda_j)$ – that is, the block is the size of the algebraic multiplicity. For example, say

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This system has a signle eigenvalue $\lambda = 1$ with algebraic multiplicity 4 (which is clear since A is upper diagonal). Solving the equation

$$Av = \lambda v$$

gives

$$v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

as an eigenvector and geometric multiplicity of $\lambda = 1$ is therefore 1. Thus, A is similar to the matrix

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

To find the transformation M so that $AM = MJ$ we have to solve for M, which we do as follows. First, write out the columns of M in the similarity statement.

$$A(m_2\ m_2\ m_3\ m_4) = (m_2\ m_2\ m_3\ m_4)\begin{pmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Then note that

$$\begin{aligned}Am_1 &= m_1\\ Am_2 &= m_1 + m_2\\ Am_3 &= m_2 + m_3\\ Am_4 &= m_3 + m_4.\end{aligned}$$

Clearly, the vector m_1 is an eigenvector for $\lambda = 1$. The rest of the columns are called generalized eigenvectors. To find them, we put $m_1 = v$ in the second equation to get

$$m_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

for example (any nonzero multiple of this will v work, although it will change what follows). Then we put m_2 in the third equation and solve for m_3 to get

$$m_3 = \begin{pmatrix}0\\-1\\\frac{1}{2}\\0\end{pmatrix}.$$

And finally, putting m_3 into the last equation gives

$$m_4 = \begin{pmatrix}0\\0\\0\\\frac{1}{2}\end{pmatrix}.$$

The resulting transformation from A to J is then

$$M = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -1 & \frac{1}{2}\\ 0 & 0 & \frac{1}{2} & -\frac{1}{2}\\ 0 & 0 & 0 & \frac{1}{2}\end{pmatrix}.$$

2.3 The geometric multiplcity not 1 but is less than the algebraic multiplicity

In this case, there is one Jordan block for each dimension of the eigenspace (as usual). However, the sizes of those blocks are not determined by the theorem above. For example, it may be that $gm(\lambda) = 2 < 4 = am(\lambda)$. Then there are either two blocks of size 2, or there are one two blocks of size one and three.

Here is a more detailed example. Suppose that

$$A = \begin{pmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Once again, you can see by inspection that there is one eigenvalue $\lambda = 1$ with algebraic multiplicity 4. Finding the eigenvectors by solving $Av = v$ for v gives a two dimensional eigenspace spanned by the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

for example (once again, any basis will do).

Now we have a choice to make. Either there are two blocks of size two, or one block of size one and one block of size three. Suppose it is the latter, then we have

$$A(m_2\ m_2\ m_3\ m_4) = (m_2\ m_2\ m_3\ m_4) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

and

$$\begin{aligned} Am_1 &= m_1 \\ Am_2 &= m_2 \\ Am_3 &= m_2 + m_3 \\ Am_4 &= m_3 + m_4. \end{aligned}$$

The first two equations here are just eigenvalue equations, but we don't know which eigenvalue is which. Once again, we have choices. Either $m_2 = v_1$ or $m_2 = v_2$. Let's try both:

1. Possibility 1: $m_2 = v_1$. In this case, we get

$$Am_3 = v_1 + m_3$$

and

$$Am_4 = m_3 + m_4.$$

You can check (do this) that a solution to the first equation is $m_3 = (0\ 1\ 0\ 0)^T$ but that putting this solution into the second equation yields no solution!

2. Possibility 2: $m_2 = v_2$. In this case you can also check (do this) that there is no solution for m_3 and m_4.

Therefore, our initial guess that there was one block of size 1 and one block of size three **is wrong**.

Now: You check that you can solve

$$A(m_2\ m_2\ m_3\ m_4) = (m_2\ m_2\ m_3\ m_4)\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

There are more sophisticated wasy to find the Jordan form of a matrix than just guessing. Try looking some up on the web. Unfortunately, they take a bit more machinery than we have time to cover in this class, so for now we will just use this method.