probability q.

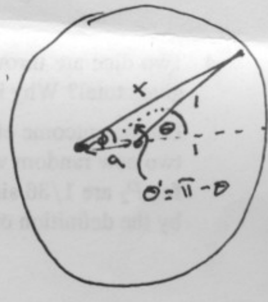
7. A point lies on a circle at a random position with uniform distribution along the circle. When viewed by an eccentric observer, what is the probability distribution of the azimuth?

Since this mapping is bijective the transformation formula directly applies, without having to sum over multiple solutions of $\phi = f(\theta)$. Let θ be the original random variable, which is uniformly distributed from $-\pi$ to π along the circle and let ϕ be the azimuth for the eccentric observer. Assume without loss of generality that the circle has radius 1 and also that the eccentricity a (displacement from the center) is less than 1. The transformation function f and its derivative are

$$\phi = f(\theta) = \tan^{-1}\left(\frac{\sin(\theta)}{\cos(\theta)+a}\right), \qquad \frac{d\phi}{d\theta} = f' = \frac{1+a\cos(\theta)}{1+a^2+2acos(\theta)}$$

The trickier part is to get f^{-1} in order to express f' as a function of ϕ. First apply the law of cosines and solve the resulting quadratic equation to get the missing side x of the triangle.

$$x = \frac{2a\cos(\phi) \pm \sqrt{4a^2\cos^2(\phi) - 4(a^2-1)}}{2}$$



Then we can apply the law of cosines again, this time solving for $\theta' = \pi - \theta$

$$\theta(\phi) = f^{-1} = \operatorname{sgn}(\phi)\left(\pi - \cos^{-1}\frac{a^1+1-x^2}{2a}\right)$$

Some sample plots of $P_\phi = \frac{P_\theta(f^{-1}(\phi))}{|f'(f^{-1}(\phi))|}$ for $a = 0.01, 0.2, 0.4, 0.6, 0.8, 0.99$ are shown in Figure 1.

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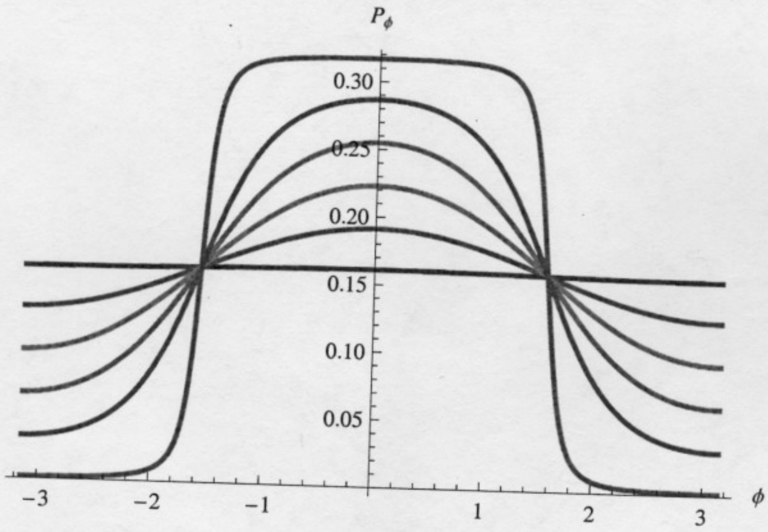

Figure 1: P_ϕ for $a = 0.01, 0.2, 0.4, 0.6, 0.8, 0.99$ in problem 7