Problem # 4 Pg. 28 Show that the Poisson distribution ...

Poisson distribution $\quad P_n = \frac{a^n}{n!} e^{-a} \quad$ with $\mu_1 = a, \ \sigma^2 = a$

Since a is very large, we will look at P_n when n is very large.

$$\log P_n = n \log a - a - \log n!$$

Using Stirling approximation for large n

$$n! = n^n e^{-n} \sqrt{2\pi n}$$

$$\therefore \log P_n = n \log a - a - \log[n^n e^{-n} \sqrt{2\pi n}]$$
$$= n \log a - a - n \log n + n - \log \sqrt{2\pi n}$$
$$= n - a - n \log \frac{n}{a} - \log \sqrt{2\pi n}$$

Let $m = n - a \quad , \quad m \ll a$

$$\therefore \ n = m + a$$

$$\therefore \log P_n = m - (m+a) \log[1 + \frac{m}{a}] - \log \sqrt{2\pi n}$$

Taylor's expansion: $\quad \log[1 + \frac{m}{a}] = \frac{m}{a} - \frac{1}{2}\left(\frac{m}{a}\right)^2 + \frac{1}{3}\left(\frac{m}{a}\right)^3 - \frac{1}{4}\left(\frac{m}{a}\right)^4 + \cdots$

$$\log P_n = m - (m+a)\left[\frac{m}{a} - \frac{1}{2}\left(\frac{m}{a}\right)^2 + \cdots\right] - \log \sqrt{2\pi n}$$

For very large a and $m \ll a$, we ignore terms higher than 2nd order Taylor's expansion of $\log[1 + \frac{m}{a}]$ and let $n \approx a$

Then $\quad \log P_n \approx \cancel{m} - \frac{m^2}{a} - \cancel{m} + \frac{1}{2}\frac{m^2}{a} - \log \sqrt{2\pi a}$

$$= -\frac{1}{2}\frac{m^2}{a} - \log \sqrt{2\pi a}$$

Thus $\quad P_n = \frac{1}{\sqrt{2\pi a}} \exp\left[\frac{1}{2}\frac{(n-a)^2}{a}\right] \quad$ Gussian distribution with $\mu_1 = a, \ \sigma^2 = a$