Martingale transforms

The heart of the matter

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The Kai Lai Chung Lecture
UCSD

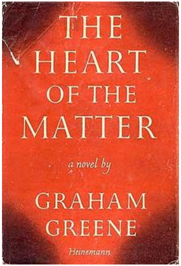
Kai Lai Chung 1917-2009

Marc Yor: 1949-2014

Donald Lyman Burkholder

January 19, 1927, Octavia, Nebraska–April 14, Urbana, Illinois, 2013

A word about the title,. . . , in case you noticed



Not from the 1948 Novel

But from "Singular Integrals, the heart of the matter"

Title of §2.2 in E.M. Stein's famous (AMS Steele Prize winner) book "Singular Integrals and differentiability properties of functions"

The title borrowed from Stein with his (written) permission.

The Haar System on $[0, 1]$

$$\psi(x) = \begin{cases} 1, & 0 \leqslant x < \frac{1}{2} \\ \\ -1, & \frac{1}{2} < x \leqslant 1 \end{cases}$$

and

$$\psi_{jk}(x) = \psi(2^j x - k)$$

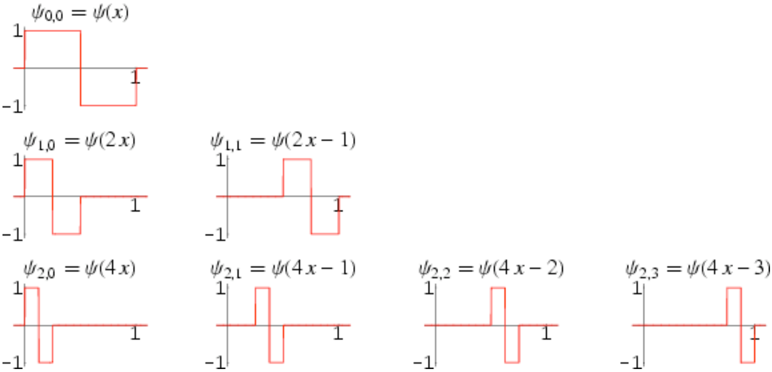
for j nonnegative and $0 \leqslant k \leqslant 2^j - 1$

$$\int_0^1 \psi_{jk}(x)\psi_{lm}(x)dx = 0, \ (j,k) \neq (l,m)$$

Any f can be written as (Schauder 1928)

$$f(x) = c_0 + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} c_{jk}\psi_{jk}(x)$$

$$
\begin{aligned}
\psi_{00} &= \psi(x) \\
\psi_{10} &= \psi(2x), \quad \psi_{11} = \psi(2x-1) \\
\psi_{20} &= \psi(4x), \quad \psi_{21} = \psi(4x-1), \\
\psi_{22} &= \psi(4x-2), \quad \psi_{23} = \psi(4x-3)
\end{aligned}
$$



The Haar Martingales

$$h_0(x) = 1, \quad h_1(x) = \psi_{00}(x), \quad h_2(x) = \psi_{10}(x), \quad h_3(x) = \psi_{11}(x), \ldots$$

$$f_n(x) = \sum_{k=0}^{n} a_k h_k(x), \quad a_k \in \mathbb{R}$$

The sequence $\{f_n\}$ is a martingale on the probability space $[0,1]$.

$\varphi(h_1, \ldots, h_k)$ is constant whenever h_{k+1} is not zero, any function φ. So,

$$\int_0^1 h_{k+1}(x)\varphi(h_1, \ldots, h_k)dx = C\int_0^1 h_{k+1}(x)dx = 0$$

Theorem (R.E.A.C. Paley 1932, Marcinkiewicz 1937)

For all $a_k \in \mathbb{R}$ real numbers and $\varepsilon_k \in \{1, -1\}$, $1 < p < \infty$ there is a constant C_p depending only p such that

$$\Big\| \sum_{k=1}^{n} \varepsilon_k a_k h_k \Big\|_p \leqslant C_p \Big\| \sum_{k=1}^{n} a_k h_k \Big\|_p$$

1. The unconditional constant β_p for a basis $\{e_k\}$ in L^p is the least least extended real number β_p with the property that for any n and any $a_k \in \mathbb{R}$ with $\Big\| \sum_{k=1}^{n} a_k e_k \Big\|_p = 1$, then for any choice of signs $\varepsilon \in \{1, -1\}$

$$\Big\| \sum_{k=1}^{n} \varepsilon_k a_k e_k \Big\|_p \leqslant \beta_p$$

General Martingale Transforms

f_n a martingale with difference sequence d_n, $f_n = \sum_{k=0}^{n} d_k$. $\{v_k\}$ be predicable (measurable to $\mathcal{F}_{k-1}$) taking values in $[-1,1]$ for all k (symmetric multiplier).

$$g_n = \sum_{k=0}^{n} v_k d_k, \quad \textbf{Martingale transform of } f_n$$

Theorem (Burkholder 1966: Paley–Marcinkiewicz holds for arbitrary martingales)

$$\|g_n\|_p \leqslant M_p \|f_n\|_p, \quad 1 < p < \infty$$

Theorem (Burkholder 1984–18 years later)

$$\|g_n\|_p \leqslant (p^* - 1)\|f_n\|_p,$$

$$p^* - 1 = \begin{cases} p-1, & 2 \leqslant p < \infty \\ \frac{1}{p-1}, & 1 < p \leqslant 2 \end{cases}$$

and this constant $(p^* - 1)$ cannot be improved!
(The unconditional constant for the Haar system is $(p^* - 1)$)

Three steps in the 1966 paper, all rather simple (even for the time):

Step 1: Boundedness on L^2

Straight from martingale from orthogonality of martingale difference sequence:

$$\|g_n\|_2^2 = \mathbb{E}\sum_{k=0}^{n}|v_k d_k|^2 \leqslant \mathbb{E}\sum_{k=0}^{n}|d_k|^2 = \|f_n\|_2^2$$

Step 2: Weak-type bound on L^1

$$\mathbb{P}\{|g_n| > \lambda\} \leqslant \frac{C}{\lambda}\mathbb{E}|f_n|, \text{ for all } \lambda > 0,$$

Step 3: Marcinkiewicz interpolation and duality

The martingale square function

$$S_n(f) = \left(\sum_{k=0}^{n} |d_k|^2 \right)^{1/2}$$

Note: With our assumption that $|v_k| \leqslant 1$,

$$S_n(g) = \left(\sum_{k=0}^{n} |v_k d_k|^2 \right)^{1/2} \leqslant \left(\sum_{k=0}^{n} |d_k|^2 \right)^{1/2} = S_n(f)$$

If we knew the following "Square Function" inequality:

$$(*) \quad a_p \|f^*\|_p \leqslant \|S(f)\|_p \leqslant b_p \|f^*\|_p, \quad 1 < p < \infty \quad \text{(Note: trivial case } p = 2\text{)}$$

with a_p and b_p depending only on p, Burkholder's 1966 inequality would follow.

Burkholder proved this inequality as a consequence of the boundedness of martingale transforms.

Many questions arose from $a_p\|f^*\|_p \leqslant \|S(f)\|_p \leqslant b_p\|f^*\|_p, \quad 1 < p < \infty$

Theorem

- *Inequality does not hold for $0 < p < 1$ but there is an LlogL inequality (Burkholder)*
- *Burgess Davis (1970) Inequality holds for $p = 1$.*
- *Burkholder-Gundy (1970) Inequality holds for all $0 < p < \infty$ provided the martingales are "regular", in particular for all martingales X_t, $t \geqslant 0$, indexed by continuous time for which the function*

$$t \rightarrow X_t, \quad \textit{is continuous a.s. (continuous trajectories!)}$$

$B_t = (B_t^1, B_t^2, \ldots, B_t^n)$ Brownian motion on $\mathbb{R}^n$

$$X_t = \int_0^t H_s \cdot dB_s$$

Max Function $X_t^* = \sup_{0\leqslant s\leqslant t} |X_t|$. "Square Function" $\langle X\rangle_t^{1/2} = \left(\int_0^t |H_s|^2 ds\right)^{1/2}$

"Burkholder-Gundy good-λ principle" (widely used even now in norm comparison problems).

Lemma (Burkholder-Gundy 1970 (on any measure space))

Suppose f and g non-negative satisfy: For $\varepsilon > 0$ and $\lambda > 0$,

$$\mu\{f > 2\lambda,\ g \leqslant \varepsilon\lambda\} \leqslant C\varepsilon^2\mu\{f > \lambda\}.$$

Then

$$\|f\|_p \leqslant C_p\|g\|_p, \quad 0 < p < \infty$$

Where C_p is a constant depending on C and p.

More: There are "Φ"–Inequalities.

$\Phi(0) = 0$, increasing, $\Phi(2x) \leqslant C\Phi(x) \Rightarrow \mathbb{E}\Phi(f)| \leqslant C\mathbb{E}\Phi(g)$.

$$\frac{1}{2^p}\|f\|_p^p = \left\|\frac{f}{2}\right\|_p^p = p\int_0^\infty \lambda^{p-1}\mu\{f > 2\lambda\}d\lambda$$

$$\leqslant \quad p\int_0^\infty \lambda^{p-1}\mu\{f > 2\lambda, g \leqslant \varepsilon\lambda\}d\lambda + p\int_0^\infty \lambda^{p-1}\mu\{g > \varepsilon\lambda\}d\lambda$$

$$\leqslant \quad C\varepsilon^2 p\int_0^\infty \lambda^{p-1}\mu\{f > \lambda\}d\lambda + p\int_0^\infty \lambda^{p-1}\mu\{g > \varepsilon\lambda\}d\lambda$$

$$\leqslant \quad C\varepsilon^2\|f\|_p^p + \frac{1}{\varepsilon^p}\|g\|_p^p$$

Theorem (Burkholder-Gundy, Acta Math (1970))

The pairs $(X_t^*, \langle X\rangle_t^{1/2})$, $(\langle X\rangle_t^{1/2}, X_t^*)$ satisfy the good-λ principle.

A major achievement of the these inequalities in analysis was the Burkholder, Gundy and Silverstein solution in 1971 of a 1930 problem of Hardy and Littlewood: The Hardy spaces H^p are characterize by the integrability of the maximal function of its real part. The birth of huge activity in analysis which lasted many years.

The sharper the good-λ, the better its applications.

Sharp good-λ: R.B. 1987 but known to Burkholder earlier

For all $\varepsilon > 0$, $\lambda > 0$ and $1 < \delta$,

$$\mathbb{P}\{X_t^* > \delta\lambda,\ \langle X\rangle_t^{1/2} \leqslant \varepsilon\lambda\} \leqslant 2\exp\left(-\frac{(\delta-1)^2}{2\varepsilon^2}\right)\mathbb{P}\{X_t^* > \lambda\}$$

$$\mathbb{P}\{\langle X\rangle_t^{1/2} > \delta\lambda,\ X_t^* \leqslant \varepsilon\lambda\} \leqslant C\exp\left(-\frac{\pi^2}{8}\frac{(\delta^2-1)}{\varepsilon^2}\right)\mathbb{P}\{\langle X\rangle_t^{1/2} > \lambda\}$$

and these are best possible.

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Corollary: There are positive constants C_1, C_2, C_3, C_4 such that

$$(\textit{Classic})\quad C_1 \leqslant \limsup_{t\to\infty} \frac{X_t^*}{\sqrt{\langle X\rangle_t \log\log(\langle X\rangle_t)}} \leqslant C_2, \quad \textit{a.s. on } \{\langle X\rangle = \infty\}$$

$$(\textit{Chung})\quad C_3 \leqslant \liminf_{t\to\infty} \left(\frac{\log\log(\langle X\rangle_t)}{\langle X\rangle_t}\right)^{1/2} X_t^* \leqslant C_4, \quad \textit{a.s. on } \{\langle X\rangle = \infty\}$$

u harmonic functions in upper-half space of $\mathbb{R}^d$

$$\mathbb{R}^{d+1}_+ = \{(x,y) : x \in \mathbb{R}^n, y > 0\}$$

$$\Gamma^1_\alpha(x) = \{(\bar{x},y) \in \mathbb{R}^{d+1}_+ : |x-\bar{x}| < \alpha y, 0 < y < 1\},$$

$$N^1_\alpha u(x) = \sup_{(\bar{x},y)\in\Gamma^1_\alpha(x)} |u(\bar{x},y)|, \quad \text{non-tangential maximal function}$$

$$A^1_\alpha u(x) = \left(\int_{\Gamma^1_\alpha(x)} y^{1-n}|\nabla u(\bar{x},y)|^2 d\bar{x}\,dy\right)^{1/2}, \text{ Lusin square (area) function}$$

Theorem (Privalov (1916), Marcinkiewicz–Zygmund (1938), Spencer (1943), Calderón (1950, 1951), Stein (1961): Except for sets of Lebesgue measure zero)

$$\begin{aligned}\{x \in \mathbb{R}^d : A^1_\alpha u(x) < \infty\} &= \{x \in \mathbb{R}^d : N^1_\alpha u(x) < \infty\} \\ &= \{x \in \mathbb{R}^d : \lim_{\substack{(y,t)\to(x,0)\\(y,t)\in\Gamma^1_\alpha(x)}} u(y,t) \textit{ exists and is finite}\}.\end{aligned}$$

Burkholder-Gundy 1972

- The pairs (A, N) and (N, A) have the good-λ principle.
- The good-λ inequalities imply Privalov et al ...

Question (Richard Gundy (1970's))

Does the LIL hold for harmonic functions, X^* replace by N, $\langle X\rangle^{1/2}$ replaced by A?

Burkholder-Gundy 1972

- The pairs (A, N) and (N, A) have the good-λ principle.
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Does the LIL hold for harmonic functions, X^* replace by N, $\langle X\rangle^{1/2}$ replaced by A?

R.B, C. Moore and I. Klemeš, 1986–1994: The answer is yes

1. Kolmogorov-type LIL holds
2. Chung-type LIL holds
3. Kensten-type LIL holds

$limsup's$, $liminf's$ between constants a.e. on the complement of Privalov et al set.

Need sharp (Gaussian) good-λ's (like the martingale above). More: need them on Lipschitz domains. Proved via an "invariance principle" for harmonic functions.

The 1984 paper: The story began in 1981.

Burkholder, Ann of Prob. 1981. For a Banach space B TFAE

1. B is ζ-convex: There exist a biconvex function $\zeta : B \times B \to \mathbb{R}$ such that $\zeta(0,0) > 0$ and $\zeta(x,y) \leqslant |x+y|$, if $|x| = 1 = |y|$.
2. $B \in UMD$: Martingale transforms of B valued martingales are bounded on L^p, $1 < p < \infty$.

(1) Burkholder (1981): Boundedness of martingale transforms $\Rightarrow$ boundedness of Hilbert transform.
(2) Bourgain (1983): Boundedness of Hilbert transform $\Rightarrow$ boundedness of martingale transforms.

A geometric characterization of such Banach spaces had been pondered by many

- Maurey, Pisier mid 70's introduced the acronym UMD (unconditionality of martingale difference sequences) and raised the question of characterizing the Banach spaces with this property. Aldous, Lindenstrauss, Pelczyński, . . .
- Hilbert transform. S. Bochner and A.E. Taylor (1938) raised the question and many investigated it: Schwartz, Benedek, Calderón, Panzone, Stein, Vagi, . . .

Gilles Pisier (2012) in "Selected works of D.L. Burkholder" writes:

"One of the main sobering features of zeta-convex is that it is not easy to find the ζ-function directly. Actually, at first only the Hilbert space case was available ($\zeta(x, y) = 1 + \langle x, y\rangle$) and even the L^p-case with $p \neq 2$ was elusive". But later on, in an analytic tour the force, Burkholder managed to identify the ζ-function for $B = L^p$ as a solution to non-linear PDE."

Theorem (Burkholder (1984) Ann. of Prob. "Special Invited Paper")

For the $B = \mathbb{R}$ case: Let $V(x,y) = |y|^p - (p^* - 1)^p |x|^p$. There is a U such that

$$(i) \quad V(x,y) \leqslant U(x,y),$$

$$(ii) \quad \mathbb{E}U(f_n, g_n) \leqslant \mathbb{E}U(f_{n-1}, g_{n-1}) \leqslant \ldots \leqslant \mathbb{E}U(f_0, g_0) \leqslant 0.$$

He show the existence of such a U by solving the non-linear PDE

$$(p-1)[yF_y - xF_x]F_{yy} - [(p-1)F_y - xF_{xy}]^2 + x^2 F_{xx} F_{yy} = 0$$

with suitable boundary conditions in certain domains of $\mathbb{R}^2$. The solutions to such equation leads to a system of several nonlinear differential inequalities with boundary conditions. From this system, a function $u(x,y,t)$ is constructed in the domain

$$\Omega = \left\{ (x,y,t) \in \mathbb{R}^3 : \left| \frac{x-y}{2} \right|^p < t \right\}$$

with certain convexity properties for which

$$u(0,0,1) \|g_n\|_p^p \leqslant \|f_n\|_p^p$$

for $1 < p \leqslant 2$. Burkholder then shows $u(0,0,1) = (p-1)^p$.

1986 Burkholder wrote the function down, removing all difficulties. (Not exactly!)

$$U(x,y) = \alpha_p(|y| - (p^* - 1)|x|)(|x| + |y|)^{p-1},$$
$$\alpha_p = p(1 - 1/p^*)^{p-1}.$$

x and y can be in a Hilbert space.

Theorem (Burkholder 1986)

$\{e_k\}$, $\{d_k\}$ Hilbert space $\mathcal{H}$-valued martingale difference sequences with $\|e_k(\omega)\|_{\mathcal{H}} \leqslant \|d_k(\omega)\|_{\mathcal{H}}, \quad \forall \omega \in \Omega, k \geqslant 0$. $g_n = \sum_{k=0}^n e_k$, $f_n = \sum_{k=0}^n d_k$

$$\|g\|_p \leqslant (p^* - 1)\|f\|_p,$$

Corollary

$$\Big\| \sum_{k=0}^{\infty} e^{i\theta_k} a_k h_k \Big\|_p \leqslant (p^* - 1) \Big\| \sum_{k=0}^{\infty} a_k h_k \Big\|_p$$ **(Conjectured by Pełczyński)**

With this function one can use Itô calculus to get sharp martingale inequalities that have applications to Fourier analysis (R. B. & G. Wang 1995.)

Definition

$m : \mathbb{R}^d \to \mathbb{C}$ in L^∞ produces the **Fourier multiplier** operator $\mathcal{M}_m$

$$\widehat{\mathcal{M}_m f}(\xi) = m(\xi)\, \widehat{f}(\xi), \text{ with } \mathcal{M}_m : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

Question

When do such operators have extensions to L^p, $1 < p < \infty$?

Theorem (Hörmander 1960: If m is smooth "enough" with)

$$\sup_{x \in \mathbb{R}^d} \left\{ |x|^{|\alpha|} \left| \frac{\partial^\alpha m(x)}{\partial x^\alpha} \right| \right\} = C < \infty$$

Then

$$\|\mathcal{M}_m f\|_p \leqslant C_p \|f\|_p, \quad 1 < p < \infty$$

with C_p depending on C, d and p.

Theorem (C. Fefferman 1971–"The Multiplier Problem for the Ball")

If $m = \chi_B$ where B is the unit ball in $\mathbb{R}^d$, $d > 1$, then $\mathcal{M}_m$ is an L^p–multiplier if and only if $p = 2$.

Lévy measure $\nu \geqslant 0$ on $\mathbb{R}^d$. So, $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} \min(|z|^2, 1) d\nu(z) < \infty$$

Let $\mu \geqslant 0$ be a finite Borel measure on the unit sphere $\mathbb{S} \subset \mathbb{R}^d$, and

$$\varphi : \mathbb{R}^d \to \mathbb{C}, \quad \psi : \mathbb{S} \to \mathbb{C}, \quad \varphi, \psi \in L^\infty(\mathbb{C})$$

Consider the **"Lévy multiplier"**

$$m(\xi) = \frac{\int_{\mathbb{R}^d} \left(1 - \cos \xi \cdot z\right) \varphi(z)\, d\nu(z) + \int_{\mathbb{S}} |\xi \cdot \theta|^2 \psi(\theta)\, d\mu(\theta)}{\int_{\mathbb{R}^d} \left(1 - \cos \xi \cdot z\right) d\nu(z) + \int_{\mathbb{S}} |\xi \cdot \theta|^2 d\mu(\theta)},$$

Note that $\|m\|_\infty \leqslant \max\{\|\phi\|_\infty, \|\psi\|_\infty\}$.

$$m(\xi) = \frac{\int_{\mathbb{R}^d} \left(1 - \cos \xi \cdot z\right) \varphi(z)\, d\nu(z) + \mathbb{A}\xi \cdot \xi}{\int_{\mathbb{R}^d} \left(1 - \cos \xi \cdot z\right) d\nu(z) + \mathbb{B}\xi \cdot \xi},$$

$$\mathbb{A} = \left[\int_{\mathbb{S}} \varphi(\theta)\, \theta_i \theta_j \, d\mu(\theta)\right]_{i,j=1\ldots d} \quad \text{and} \quad \mathbb{B} = \left[\int_{\mathbb{S}} \theta_i \theta_j \, d\mu(\theta)\right]_{i,j=1\ldots d}$$

with both $\mathbb{A}$ and $\mathbb{B}$ symmetric and $\mathbb{B}$ non–negative definite.

The Lévy–Khintchine Formula

$\{X_t\}$ a Lévy process in $\mathbb{R}^n$. The Lévy-Khintchine formula: $\mathbb{E}\left[\, e^{i\xi\cdot X_t} \,\right] = e^{t\rho(\xi)}$

$$\begin{aligned} \rho(\xi) &= ib\cdot\xi - \mathbb{B}\xi\cdot\xi + \int_{\mathbb{R}^n} \left[\, e^{i\,\xi\cdot y} - 1 - i(\xi\cdot y)\,\mathbb{I}_{\{|y|<1\}}(y) \,\right] \nu(dy) \\ &= \Re\rho(\xi) + i\Im\rho(\xi) \end{aligned}$$

$$\Re\rho(\xi) = -\mathbb{B}\xi\cdot\xi + \int_{\mathbb{R}^n} [\cos(\xi\cdot y) - 1]\, \nu(dy),$$

$$\Im\rho(\xi) = b\cdot\xi + \int_{\mathbb{R}^n} \left[\sin(\xi\cdot y) - (\xi\cdot y)\,\mathbb{I}_{\{|y|<1\}}(y)\right] \nu(dy).$$

Theorem

$$\|\phi\|_\infty \;\&\; \|\psi\|_\infty \leqslant 1 \Rightarrow \|\mathcal{M}_m f\|_p \leqslant (p^* - 1)\|f\|_p, \quad 1 < p < \infty,$$

$$p^* - 1 = \begin{cases} \frac{1}{p-1}, & 1 < p \leqslant 2, \\ \\ p - 1, & 2 \leqslant p < \infty. \end{cases}$$

The constant is best possible.

1. Proved in a series of papers: R.B & P. Hernández-Méndez (2003), R.B & K. Bogdan (2007), R.B., A. Bielaszewski & K. Bogdan (2010)
2. These multiplier include many classical multiplier.

Theorem (R.B. & A. Osękowski–2011)

Suppose φ, ψ take values in $[b, B]$ for some $-\infty < b < B < \infty$. Then $\mathcal{M}_m$ with Lévy multipler

$$m(\xi) = \frac{\int_{\mathbb{R}^d} (1 - \cos \xi \cdot z)\, \varphi(z)\, d\nu(z) + \int_{\mathbb{S}} |\xi \cdot \theta|^2 \psi(\theta)\, d\mu(\theta)}{\int_{\mathbb{R}^d} (1 - \cos \xi \cdot z)\, d\nu(z) + \int_{\mathbb{S}} |\xi \cdot \theta|^2 d\mu(\theta)}$$

$$\Rightarrow \ \|\mathcal{M}_m f\|_p \leqslant C_{p,b,B} \|f\|_p, \qquad 1 < p < \infty.$$

and the inequality is sharp.

Here $C_{p,b,B}$ is the best constant in the martingale transform inequality

$$\|\sum_{k=0}^{m} v_k d_k\|_p \leqslant C_{p,b,B} \|\sum_{k}^{n} d_k\|_p, \quad v_k \in [b, B]$$

Of particular interest are the one-sided, non-symmetric, multipliers where $b = 0$.

There are versions of the above sharp bounds on Manifolds and sub–elliptic Laplacians (R.B. & F. Baoudoin, 2013) Lie groups (R.B. & Applebaum 2013), and for the Ornstein-Uhlenbeck operator (R. B. & Osękowski, 2013-14).

C. Morrey (1952), T. Iwaniec (1982), D. Burkholder (1986)

Conjecture of T. Iwaneic 1982:

$$\|\partial f\|_p \leqslant (p^* - 1)\|\overline{\partial} f\|_p, \ , \ f \in C_0^\infty(\mathbb{C})$$

With $V(z, w) = |w|^p - (p^* - 1)^p |z|^p$ same as

$$(*) \quad \int_{\mathbb{C}} V(\overline{\partial} f, \partial f) dm(z) \leqslant 0, \quad f \in C_0^\infty(\mathbb{C}).$$

R.B. & G. Wang–1995.

(*) holds with $4\overline{\partial} f$.

Conjecture (R.B. G. Wang (1995))

For all $f \in C_0^\infty(\mathbb{C})$

$$\int_{\mathbb{C}} U(\overline{\partial} f, \partial f) dm(z) \leqslant 0. \qquad (1)$$

C. Morrey (1952): "Quasi–convexity and lower semicontinuity of multiple integrals." $F : \mathbb{R}^{d\times d} \to \mathbb{R}$,

$$I(f) = \int_\Omega F\left(\frac{\partial f_i}{\partial x_j}(x)\right) dx, \qquad f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n, \quad f \in W^{1,p}(\Omega, \mathbb{R}^n).$$

- I is (weakly) lower semicontinuous $\iff$ F quasi-convex
- The Euler equations $I'(f) = 0$ are elliptic $\iff$ F is rank–one convex

- **Quasi-convexity:** $F\colon \mathbb{R}^{d\times d} \to \mathbb{R}$ for each $A \in \mathbb{R}^{d\times d}$, each bounded $D \subset \mathbb{R}^d$, each compactly supported Lipschitz function $f\colon D \to \mathbb{R}^n$,

$$F(A) \leqslant \frac{1}{|D|}\int_D F(A + \frac{\partial f_i}{\partial x_j})$$

- **Rank-one convexity:** $F\colon \mathbb{R}^{d\times d} \to \mathbb{R}$, $A,\ B \in \mathbb{R}^{d\times d}$, rank $B = 1$,

$$h(t) = F(A + tB) \qquad \text{is convex}$$

- $d = 1$, quasi-convex or rank–one convex $\iff$ convex.
- If $d \geqslant 2$, convexity $\implies$ quasi–convexity $\implies$ rank–one convexity.

Conjecture (Morrey 1952:)

Rank–one convexity does not imply quasi–convexity.

Šverak 1992: Morrey is correct for $d \geqslant 3$. Case $d = 2$, open.

Enter the Burkholder function: For all $\forall\ z,\ w,\ h,\ k \in \mathbb{C}$, $|k| \leqslant |h|$,

$$h(t) = -U(z + th,\ w + tk) \quad \text{is convex}$$

$$\text{Define} \quad \Gamma : \mathbb{R}^{2\times 2} \to \mathbb{C} \times \mathbb{C} \quad \text{by} \quad \Gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (z, w),$$

$$z = (a + d) + i(c - b), \qquad w = (a - d) + i(c + b)$$

$F_U = -U \circ \Gamma$, is rank–one convex–(**R.B–Lindeman 1997).**

$$f:\mathbb{C}\to\mathbb{C}, \qquad \left(\frac{\partial f_i}{\partial x_j}\right)=\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}, \quad f=u+iv\in C_0^\infty(\mathbb{C})$$

$$F_U\left(\frac{\partial f_i}{\partial x_j}\right)=-U\left(\overline{\partial} f,\ \partial f\right).$$

Quasiconvexity of F_U at $0\in\mathbb{R}^{2\times 2}$ $\iff$

$$0=F_U(0)\leqslant\int_{\operatorname{supp} f}F_U\left(\frac{\partial f_i}{\partial x_j}\right)=-\int_{\operatorname{supp} f}U\left(\overline{\partial} f,\ \partial f\right)$$

Question (The "Win-Win Question"–R.B. Wang 1995, R.B. Lindeman 1997)

Is F_U quasiconvex?

1. **If true:** *Iwaniec's conjecture follows*
2. **If false:** *Morrey's conjecture follows.*

1. Astala, Iwaniec, Prause, Saksman, "Burkholder integrals, Morrey's problem and quasiconformal mappings," J. Amer. Math. Soc. 25 (2012).
2. Astala, Iwaniec, Prause, Saksman, Bilipschitz and quasiconformal rotation, stretching and multifractal spectra, 2014 preprint.

If we have seen a little further it is not because we have such good sight but because we have been standing on the shoulders of giants.

John of Salisbury (12th Century English Theologian)

THANK YOU!