Martingale transforms

The heart of the matter

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The Kai Lai Chung Lecture
UCSD
Kai Lai Chung 1917-2009
Marc Yor: 1949-2014
A word about the title, . . . , in case you noticed

Not from the 1948 Novel

But from “Singular Integrals, the heart of the matter”

Title of §2.2 in E.M. Stein’s famous (AMS Steele Prize winner) book “Singular Integrals and differentiability properties of functions”

The title borrowed from Stein with his (written) permission.
The Haar System on $[0, 1]$

\[
\psi(x) = \begin{cases} 
1, & 0 \leq x < \frac{1}{2} \\
-1, & \frac{1}{2} < x \leq 1 
\end{cases}
\]

and

\[
\psi_{jk}(x) = \psi(2^j x - k)
\]

for $j$ nonnegative and $0 \leq k \leq 2^j - 1$

\[
\int_{0}^{1} \psi_{jk}(x) \psi_{lm}(x) \, dx = 0, \quad (j, k) \neq (l, m)
\]

Any $f$ can be written as (Schauder 1928)

\[
f(x) = c_0 + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j - 1} c_{jk} \psi_{jk}(x)
\]
\[
\begin{align*}
\psi_{00} &= \psi(x) \\
\psi_{10} &= \psi(2x), \quad \psi_{11} = \psi(2x - 1) \\
\psi_{20} &= \psi(4x), \quad \psi_{21} = \psi(4x - 1), \\
\psi_{22} &= \psi(4x - 2), \quad \psi_{23} = \psi(4x - 3)
\end{align*}
\]
The Haar Martingales

\[ h_0(x) = 1, \quad h_1(x) = \psi_{00}(x), \quad h_2(x) = \psi_{10}(x), \quad h_3(x) = \psi_{11}(x), \ldots \]

\[ f_n(x) = \sum_{k=0}^{n} a_k h_k(x), \quad a_k \in \mathbb{R} \]

The sequence \( \{f_n\} \) is a martingale on the probability space \([0, 1]\).

\[ \varphi(h_1, \ldots, h_k) \] is constant whenever \( h_{k+1} \) is not zero, any function \( \varphi \). So,

\[ \int_{0}^{1} h_{k+1}(x) \varphi(h_1, \ldots, h_k) dx = C \int_{0}^{1} h_{k+1}(x) dx = 0 \]
Theorem (R.E.A.C. Paley 1932, Marcinkiewicz 1937)

For all $a_k \in \mathbb{R}$ real numbers and $\varepsilon_k \in \{1, -1\}$, $1 < p < \infty$ there is a constant $C_p$ depending only on $p$ such that

$$\left\| \sum_{k=1}^{n} \varepsilon_k a_k h_k \right\|_p \leq C_p \left\| \sum_{k=1}^{n} a_k h_k \right\|_p$$

The unconditional constant $\beta_p$ for a basis $\{e_k\}$ in $L^p$ is the least extended real number $\beta_p$ with the property that for any $n$ and any $a_k \in \mathbb{R}$ with $\left\| \sum_{k=1}^{n} a_k e_k \right\|_p = 1$, then for any choice of signs $\varepsilon \in \{1, -1\}$

$$\left\| \sum_{k=1}^{n} \varepsilon_k a_k e_k \right\|_p \leq \beta_p$$
General Martingale Transforms

\( f_n \) a martingale with difference sequence \( d_n \), \( f_n = \sum_{k=0}^{n} d_k \). \( \{v_k\} \) be predicable (measurable to \( \mathcal{F}_{k-1} \)) taking values in \([-1, 1]\) for all \( k \) (symmetric multiplier).

\[
\begin{align*}
g_n &= \sum_{k=0}^{n} v_k d_k, \\
& \text{Martingale transform of } f_n
\end{align*}
\]

**Theorem (Burkholder 1966: Paley–Marcinkiewicz holds for arbitrary martingales)**

\[
\|g_n\|_p \leq M_p \|f_n\|_p, \quad 1 < p < \infty
\]

**Theorem (Burkholder 1984–18 years later)**

\[
\|g_n\|_p \leq (p^* - 1) \|f_n\|_p,
\]

\[
p^* - 1 = \begin{cases} 
  p - 1, & 2 \leq p < \infty \\
  \frac{1}{p-1}, & 1 < p \leq 2
\end{cases}
\]

and this constant \((p^* - 1)\) cannot be improved!

(The unconditional constant for the Haar system is \((p^* - 1)\))
Three steps in the 1966 paper, all rather simple (even for the time):

**Step 1: Boundedness on $L^2$**

Straight from martingale from orthogonality of martingale difference sequence:

$$
\|g_n\|_2^2 = \mathbb{E} \sum_{k=0}^{n} |v_k d_k|^2 \leq \mathbb{E} \sum_{k=0}^{n} |d_k|^2 = \|f_n\|_2^2
$$

**Step 2: Weak-type bound on $L^1$**

$$
\mathbb{P}\{|g_n| > \lambda\} \leq \frac{C}{\lambda} \mathbb{E}|f_n|, \text{ for all } \lambda > 0,
$$
The martingale square function

\[ S_n(f) = \left( \sum_{k=0}^{n} |d_k|^2 \right)^{1/2} \]

Note: With our assumption that \(|v_k| \leq 1,\]

\[ S_n(g) = \left( \sum_{k=0}^{n} |v_k d_k|^2 \right)^{1/2} \leq \left( \sum_{k=0}^{n} |d_k|^2 \right)^{1/2} = S_n(f) \]

If we knew the following “Square Function” inequality:

\[ (*) \quad a_p \| f^* \|_p \leq \| S(f) \|_p \leq b_p \| f^* \|_p, \quad 1 < p < \infty \quad \text{(Note: trivial case } p = 2) \]

with \(a_p\) and \(b_p\) depending only on \(p\), Burkholder’s 1966 inequality would follow.

Burkholder proved this inequality as a consequence of the boundedness of martingale transforms.
Many questions arose from $a_p \|f^*\|_p \leq \|S(f)\|_p \leq b_p \|f^*\|_p$, $1 < p < \infty$

**Theorem**

- **Inequality does not hold for** $0 < p < 1$ **but there is an LlogL inequality** (Burkholder)

- **Burgess Davis (1970)** Inequality holds for $p = 1$.

- **Burkholder-Gundy (1970)** Inequality holds for all $0 < p < \infty$ provided the martingales are “regular”, in particular for all martingales $X_t$, $t \geq 0$, indexed by continuous time for which the function $t \to X_t$, is continuous a.s. (continuous trajectories!)

$B_t = (B^1_t, B^2_t, \ldots, B^n_t)$ Brownian motion on $\mathbb{R}^n$

$$X_t = \int_0^t H_s \cdot dB_s$$

Max Function $X^*_t = \sup_{0 \leq s \leq t} |X_t|$. “Square Function” $\langle X \rangle_t^{1/2} = \left( \int_0^t |H_s|^2 ds \right)^{1/2}$
“Burkholder-Gundy good-\( \lambda \) principle” (widely used even now in norm comparison problems).

**Lemma (Burkholder-Gundy 1970 (on any measure space) )**

Suppose \( f \) and \( g \) non-negative satisfy: For \( \varepsilon > 0 \) and \( \lambda > 0 \),

\[
\mu\{f > 2\lambda, \ g \leq \varepsilon \lambda\} \leq C\varepsilon^2 \mu\{f > \lambda\}.
\]

Then

\[
\|f\|_p \leq C_p \|g\|_p, \quad 0 < p < \infty
\]

Where \( C_p \) is a constant depending on \( C \) and \( p \).

**More: There are “\( \Phi \)”–Inequalities.**

\( \Phi(0) = 0 \), increasing, \( \Phi(2x) \leq C\Phi(x) \Rightarrow \mathbb{E}\Phi(f) \leq C\mathbb{E}\Phi(g) \).
\[
\frac{1}{2p} \| f \|_p^p = \left\| \frac{f}{2} \right\|_p^p = p \int_0^\infty \lambda^{p-1} \mu \{ f > 2\lambda \} d\lambda
\]

\[
\leq p \int_0^\infty \lambda^{p-1} \mu \{ f > 2\lambda, g \leq \varepsilon \lambda \} d\lambda + p \int_0^\infty \lambda^{p-1} \mu \{ g > \varepsilon \lambda \} d\lambda
\]

\[
\leq C \varepsilon^2 p \int_0^\infty \lambda^{p-1} \mu \{ f > \lambda \} d\lambda + p \int_0^\infty \lambda^{p-1} \mu \{ g > \varepsilon \lambda \} d\lambda
\]

\[
\leq C \varepsilon^2 \| f \|_p^p + \frac{1}{\varepsilon p} \| g \|_p^p
\]

Theorem (Burkholder-Gundy, Acta Math (1970))

The pairs \((X_t^*, \langle X \rangle_t^{1/2}), (\langle X \rangle_t^{1/2}, X_t^*)\) satisfy the good-\(\lambda\) principle.

A major achievement of these inequalities in analysis was the Burkholder, Gundy and Silverstein solution in 1971 of a 1930 problem of Hardy and Littlewood: The Hardy spaces \(H^p\) are characterized by the integrability of the maximal function of its real part. The birth of huge activity in analysis which lasted many years.
The sharper the good-$\lambda$, the better its applications.

**Sharp good-$\lambda$: R.B. 1987 but known to Burkholder earlier**

For all $\varepsilon > 0$, $\lambda > 0$ and $1 < \delta$,

$$\mathbb{P}\{X^*_t > \delta \lambda, \langle X \rangle^{1/2}_t \leq \varepsilon \lambda\} \leq 2 \exp \left( -\frac{\left(\delta - 1\right)^2}{2\varepsilon^2} \right) \mathbb{P}\{X^*_t > \lambda\}$$

$$\mathbb{P}\{\langle X \rangle^{1/2}_t > \delta \lambda, X^*_t \leq \varepsilon \lambda\} \leq C \exp \left( -\frac{\pi^2}{8} \frac{\left(\delta^2 - 1\right)}{\varepsilon^2} \right) \mathbb{P}\{\langle X \rangle^{1/2}_t > \lambda\}$$

and these are best possible.
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\]

\[
\mathbb{P}\{\langle X \rangle_t^{1/2} > \delta \lambda, X_t^* \leq \varepsilon \lambda\} \leq C \exp \left(-\frac{\pi^2}{8} \frac{\left(\delta^2 - 1\right)}{\varepsilon^2}\right) \mathbb{P}\{\langle X \rangle_t^{1/2} > \lambda\}
\]

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**Corollary:** There are positive constants $C_1$, $C_2$, $C_3$, $C_4$ such that

(Classic) \[ C_1 \leq \limsup_{t \to \infty} \frac{X_t^*}{\sqrt{\langle X \rangle_t \log \log(\langle X \rangle_t)}} \leq C_2, \text{ a.s. on } \{\langle X \rangle = \infty\} \]

(Chung) \[ C_3 \leq \liminf_{t \to \infty} \left(\frac{\log \log(\langle X \rangle_t)}{\langle X \rangle_t}\right)^{1/2} X_t^* \leq C_4, \text{ a.s. on } \{\langle X \rangle = \infty\} \]
$u$ harmonic functions in upper-half space of $\mathbb{R}^d$

\[ \mathbb{R}^{d+1} = \{(x, y) : x \in \mathbb{R}^n, y > 0\} \]

\[ \Gamma_\alpha^1(x) = \{(\bar{x}, y) \in \mathbb{R}^{d+1}_+ : |x - \bar{x}| < \alpha y, 0 < y < 1\}, \]

\[ N_\alpha^1 u(x) = \sup_{(\bar{x}, y) \in \Gamma_\alpha^1(x)} |u(\bar{x}, y)|, \text{ non-tangential maximal function} \]

\[ A_\alpha^1 u(x) = \left( \int_{\Gamma_\alpha^1(x)} y^{1-n} |\nabla u(\bar{x}, y)|^2 d\bar{x} dy \right)^{1/2}, \text{ Lusin square (area) function} \]

**Theorem** (Privalov (1916), Marcinkiewicz–Zygmund (1938), Spencer (1943), Calderón (1950, 1951), Stein (1961): Except for sets of Lebesgue measure zero)

\[ \{x \in \mathbb{R}^d : A_\alpha^1 u(x) < \infty\} = \{x \in \mathbb{R}^d : N_\alpha^1 u(x) < \infty\} = \{x \in \mathbb{R}^d : \lim_{(y, t) \to (x, 0) \atop (y, t) \in \Gamma_\alpha^1(x)} u(y, t) \text{ exists and is finite}\}. \]
Burkholder-Gundy 1972

- The pairs \((A, N)\) and \((N, A)\) have the good-\(\lambda\) principle.
- The good-\(\lambda\) inequalities imply Privalov et al . . .

Question (Richard Gundy (1970’s))

*Does the LIL hold for harmonic functions, \(X^*\) replace by \(N\), \(\langle X \rangle^{1/2}\) replaced by \(A\)?
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- The pairs \((A, N)\) and \((N, A)\) have the good-\(\lambda\) principle.
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1. Kolmogorov-type LIL holds
2. Chung-type LIL holds
3. Kensten-type LIL holds

\(\limsup\)’s, \(\liminf\)’s between constants a.e. on the complement of Privalov et al set.

Need sharp (Gaussian) good-\(\lambda\)’s (like the martingale above). More: need them on Lipschitz domains. Proved via an “invariance principle” for harmonic functions.

Burkholder, Ann of Prob. 1981. For a Banach space $B$ TFAE

1. $B$ is $\zeta$-convex: There exist a biconvex function $\zeta: B \times B \to \mathbb{R}$ such that $\zeta(0, 0) > 0$ and $\zeta(x, y) \leq |x + y|$, if $|x| = 1 = |y|$.

2. $B \in UMD$: Martingale transforms of $B$ valued martingales are bounded on $L^p$, $1 < p < \infty$.


A geometric characterization of such Banach spaces had been pondered by many

- Maurey, Pisier mid 70’s introduced the acronym UMD (unconditionality of martingale difference sequences) and raised the question of characterizing the Banach spaces with this property. Aldous, Lindenstrauss, Pelczyński, . . .

- Hilbert transform. S. Bochner and A.E. Taylor (1938) raised the question and many investigated it: Schwartz, Benedek, Calderón, Panzone, Stein, Vagi, . . .

Bañuelos (Purdue)
Gilles Pisier (2012) in “Selected works of D.L. Burkholder” writes:

“One of the main sobering features of zeta-convex is that it is not easy to find the \( \zeta \)-function directly. Actually, at first only the Hilbert space case was available (\( \zeta(x, y) = 1 + \langle x, y \rangle \)) and even the \( L^p \)-case with \( p \neq 2 \) was elusive”. But later on, in an analytic tour the force, Burkholder managed to identify the \( \zeta \)-function for \( B = L^p \) as a solution to non-linear PDE.”

For the $B = \mathbb{R}$ case: Let $V(x, y) = |y|^p - (p^* - 1)^p |x|^p$. There is a $U$ such that

(i) $V(x, y) \leq U(x, y)$,

(ii) $\mathbb{E} U(f_n, g_n) \leq \mathbb{E} U(f_{n-1}, g_{n-1}) \leq \ldots \leq \mathbb{E} U(f_0, g_0) \leq 0$.

He show the existence of such a $U$ by solving the non-linear PDE

$$(p - 1)[yF_y - xF_x]F_{yy} - [(p - 1)F_y - xF_{xy}]^2 + x^2 F_{xx} F_{yy} = 0$$

with suitable boundary conditions in certain domains of $\mathbb{R}^2$. The solutions to such equation leads to a system of several nonlinear differential inequalities with boundary conditions. From this system, a function $u(x, y, t)$ is constructed in the domain

$$\Omega = \left\{(x, y, t) \in \mathbb{R}^3 : \left|\frac{x - y}{2}\right|^p < t\right\}$$

with certain convexity properties for which

$$u(0, 0, 1) \|g_n\|^p \leq \|f_n\|^p$$

for $1 < p \leq 2$. Burkholder then shows $u(0, 0, 1) = (p - 1)^p$. 
1986 Burkholder wrote the function down, removing all difficulties. (Not exactly!)

\[ U(x, y) = \alpha_p(|y| - (p^* - 1)|x|)(|x| + |y|)^{p-1}, \]
\[ \alpha_p = p(1 - 1/p^*)^{p-1}. \]

\(x\) and \(y\) can be in a Hilbert space.

**Theorem (Burkholder 1986)**

\[ \{e_k\}, \{d_k\} \text{ Hilbert space } \mathcal{H}\text{-valued martingale difference sequences with} \]
\[ \|e_k(\omega)\|_{\mathcal{H}} \leq \|d_k(\omega)\|_{\mathcal{H}}, \quad \forall \omega \in \Omega, k \geq 0. \]
\[ g_n = \sum_{k=0}^{n} e_k, \quad f_n = \sum_{k=0}^{n} d_k \]
\[ \|g\|_p \leq (p^* - 1)\|f\|_p, \]

**Corollary**

\[ \left\| \sum_{k=0}^{\infty} e^{i\theta_k} a_k h_k \right\|_p \leq (p^* - 1)\left\| \sum_{k=0}^{\infty} a_k h_k \right\|_p \quad (\text{Conjectured by Pełczynski}) \]

With this function one can use Itô calculus to get sharp martingale inequalities that have applications to Fourier analysis (R. B. & G. Wang 1995.)
Definition

$m : \mathbb{R}^d \rightarrow \mathbb{C}$ in $L^\infty$ produces the **Fourier multiplier** operator $\mathcal{M}_m$

$$\widehat{\mathcal{M}_m f}(\xi) = m(\xi) \hat{f}(\xi), \text{ with } \mathcal{M}_m : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$$

Question

*When do such operators have extensions to $L^p$, $1 < p < \infty$?*

Theorem (Hörmander 1960: If $m$ is smooth ”enough” with)

$$\sup_{x \in \mathbb{R}^d} \left\{ |x|^{|\alpha|} \left| \frac{\partial^\alpha m(x)}{\partial x^\alpha} \right| \right\} = C < \infty$$

Then

$$\|\mathcal{M}_m f\|_p \leq C_p \|f\|_p, \quad 1 < p < \infty$$

with $C_p$ depending on $C$, $d$ and $p$.

Theorem (C. Fefferman 1971–“The Multiplier Problem for the Ball”)

*If $m = \chi_B$ where $B$ is the unit ball in $\mathbb{R}^d$, $d > 1$, then $\mathcal{M}_m$ is an $L^p$–multiplier if and only if $p = 2$.***
Lévy measure \( \nu \geq 0 \) on \( \mathbb{R}^d \). So, \( \nu(\{0\}) = 0 \) and

\[
\int_{\mathbb{R}^d} \min(|z|^2, 1) d\nu(z) < \infty
\]

Let \( \mu \geq 0 \) be a finite Borel measure on the unit sphere \( S \subset \mathbb{R}^d \), and

\[
\varphi : \mathbb{R}^d \to \mathbb{C}, \quad \psi : S \to \mathbb{C}, \quad \varphi, \psi \in L^\infty(\mathbb{C})
\]

Consider the "Lévy multiplier"

\[
m(\xi) = \frac{\int_{\mathbb{R}^d} \left(1 - \cos \xi \cdot z\right) \varphi(z) d\nu(z) + \int_S |\xi \cdot \theta|^2 \psi(\theta) d\mu(\theta)}{\int_{\mathbb{R}^d} \left(1 - \cos \xi \cdot z\right) d\nu(z) + \int_S |\xi \cdot \theta|^2 d\mu(\theta)},
\]

Note that \( \|m\|_\infty \leq \max\{\|\varphi\|_\infty, \|\psi\|_\infty\} \).

\[
m(\xi) = \frac{\int_{\mathbb{R}^d} \left(1 - \cos \xi \cdot z\right) \varphi(z) d\nu(z) + A\xi \cdot \xi}{\int_{\mathbb{R}^d} \left(1 - \cos \xi \cdot z\right) d\nu(z) + B\xi \cdot \xi},
\]
\[ A = \left[ \int_S \varphi(\theta) \theta_i \theta_j \, d\mu(\theta) \right]_{i,j=1 \ldots d} \] and \[ B = \left[ \int_S \theta_i \theta_j \, d\mu(\theta) \right]_{i,j=1 \ldots d} \]

with both \( A \) and \( B \) symmetric and \( B \) non-negative definite.

**The Lévy–Khintchine Formula**

\( \{X_t\} \) a Lévy process in \( \mathbb{R}^n \). The Lévy-Khintchine formula: \( \mathbb{E} \left[ e^{i\xi \cdot X_t} \right] = e^{t \rho(\xi)} \)

\[ \rho(\xi) = i b \cdot \xi - B \xi \cdot \xi + \int_{\mathbb{R}^n} \left[ e^{i \xi \cdot y} - 1 - i (\xi \cdot y) \mathbb{1}_{\{|y|<1\}}(y) \right] \nu(dy) \]

\[ = \Re \rho(\xi) + i \Im \rho(\xi) \]

\[ \Re \rho(\xi) = -B \xi \cdot \xi + \int_{\mathbb{R}^n} \left[ \cos (\xi \cdot y) - 1 \right] \nu(dy), \]

\[ \Im \rho(\xi) = b \cdot \xi + \int_{\mathbb{R}^n} \left[ \sin (\xi \cdot y) - (\xi \cdot y) \mathbb{1}_{\{|y|<1\}}(y) \right] \nu(dy). \]
Martingale transforms

**Theorem**

\[
\|\phi\|_\infty \quad \& \quad \|\psi\|_\infty \leq 1 \Rightarrow \|M_m f\|_p \leq (p^* - 1)\|f\|_p, \quad 1 < p < \infty,
\]

\[
p^* - 1 = \begin{cases} 
\frac{1}{p-1}, & 1 < p \leq 2, \\
\phantom{\frac{1}{p-1}}, & \\
\phantom{1 < p \leq 2,}, & \\
\phantom{2 < p < \infty}, & \end{cases}
\]

*The constant is best possible.*


2. These multiplier include many classical multiplier.
Theorem (R.B. & A. Osękowski–2011)

Suppose \( \varphi, \psi \) take values in \( [b, B] \) for some \(-\infty < b < B < \infty\). Then \( \mathcal{M}_m \) with Lévy multiplier

\[
m(\xi) = \frac{\int_{\mathbb{R}^d} (1 - \cos \xi \cdot z) \varphi(z) \, d\nu(z) + \int_S |\xi \cdot \theta|^2 \psi(\theta) \, d\mu(\theta)}{\int_{\mathbb{R}^d} (1 - \cos \xi \cdot z) \, d\nu(z) + \int_S |\xi \cdot \theta|^2 \, d\mu(\theta)}
\]

\[\Rightarrow \|\mathcal{M}_m f\|_p \leq C_{p,b,B} \|f\|_p, \quad 1 < p < \infty.\]

and the inequality is sharp.

Here \( C_{p,b,B} \) is the best constant in the martingale transform inequality

\[\|\sum_{k=0}^{m} v_k d_k\|_p \leq C_{p,b,B} \|\sum_{k} d_k\|_p, \quad v_k \in [b, B]\]

Of particular interest are the one-sided, non-symmetric, multipliers where \( b = 0 \).

Conjecture of T. Iwaniec 1982:

\[ \| \partial f \|_p \leq (p^* - 1) \| \bar{\partial} f \|_p, \quad f \in C_0^\infty(\mathbb{C}) \]

With \( V(z, w) = |w|^p - (p^* - 1)^p |z|^p \) same as

\[ (*) \quad \int_{\mathbb{C}} V(\bar{\partial} f, \partial f) \, dm(z) \leq 0, \quad f \in C_0^\infty(\mathbb{C}). \]


\( (*) \) holds with \( 4\bar{\partial} f \).


For all \( f \in C_0^\infty(\mathbb{C}) \)

\[ \int_{\mathbb{C}} U(\bar{\partial} f, \partial f) \, dm(z) \leq 0. \] (1)
C. Morrey (1952): “Quasi–convexity and lower semicontinuity of multiple integrals.”

F : \( \mathbb{R}^{d \times d} \to \mathbb{R} \),

\[
I(f) = \int_{\Omega} F \left( \frac{\partial f_i}{\partial x_j}(x) \right) \, dx, \quad f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n, \quad f \in W^{1,p}(\Omega, \mathbb{R}^n).
\]

- \( I \) is (weakly) lower semicontinuous \( \iff \) \( F \) quasi-convex
- The Euler equations \( I'(f) = 0 \) are elliptic \( \iff \) \( F \) is rank–one convex

**Quasi-convexity:** \( F : \mathbb{R}^{d \times d} \to \mathbb{R} \) for each \( A \in \mathbb{R}^{d \times d} \), each bounded \( D \subset \mathbb{R}^d \), each compactly supported Lipschitz function \( f : D \to \mathbb{R}^n \),

\[
F(A) \leq \frac{1}{|D|} \int_D F(A + \frac{\partial f_i}{\partial x_j})
\]

**Rank-one convexity:** \( F : \mathbb{R}^{d \times d} \to \mathbb{R} \), \( A, \ B \in \mathbb{R}^{d \times d} \), rank \( B = 1 \),

\[
h(t) = F(A + tB) \quad \text{is convex}
\]
\( d = 1, \) quasi-convex or rank–one convex \( \iff \) convex.

\[\text{If } d \geq 2, \text{ convexity } \implies \text{ quasi–convexity } \implies \text{ rank–one convexity}.\]

**Conjecture (Morrey 1952:)**

*Rank–one convexity does not imply quasi–convexity.*

**Šverak 1992:** Morrey is correct for \( d \geq 3. \) Case \( d = 2, \) open.

**Enter the Burkholder function:** For all \( \forall \ z, \ w, \ h, \ k \in \mathbb{C}, \ |k| \leq |h|, \)

\[ h(t) = -U(z + th, \ w + tk) \text{ is convex} \]

Define \( \Gamma : \mathbb{R}^{2 \times 2} \to \mathbb{C} \times \mathbb{C} \) by

\[ \Gamma \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = (z, w), \]

\[ z = (a + d) + i(c - b), \quad w = (a - d) + i(c + b) \]

\( F_U = -U \circ \Gamma, \) is rank–one convex–(R.B–Lindeman 1997).
Martingale transforms

\[ f : \mathbb{C} \to \mathbb{C}, \quad \left( \frac{\partial f_i}{\partial x_j} \right) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}, \quad f = u + iv \in C_0^\infty(\mathbb{C}) \]

\[ F_U \left( \frac{\partial f_i}{\partial x_j} \right) = -U(\overline{\partial f}, \partial f). \]

Quasiconvexity of \( F_U \) at \( 0 \in \mathbb{R}^{2 \times 2} \) \( \iff \)

\[ 0 = F_U(0) \leq \int_{\text{supp } f} F_U \left( \frac{\partial f_i}{\partial x_j} \right) = -\int_{\text{supp } f} U(\overline{\partial f}, \partial f) \]


Is \( F_U \) quasiconvex?

1. **If true:** Iwaniec’s conjecture follows
2. **If false:** Morrey’s conjecture follows.

If we have seen a little further it is not because we have such good sight but because we have been standing on the shoulders of giants.

John of Salisbury (12th Century English Theologian)