Quenched and annealed heat kernel bounds for the Uniform Spanning Tree

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Kai Lai Chung 1982

Random walk on random subgraphs of $\mathbb{Z}^d$

Let $G(\omega) = (\mathbb{Z}^d, E(\omega))$ be a family of random subgraphs of $\mathbb{Z}^d$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Write $d(x,y) = d_\omega(x,y)$ for the graph distance on $G(\omega) = (\mathbb{Z}^d, E(\omega))$.

Let X be the lazy SRW on $G(\omega)$: this moves to a (uniformly chosen) neighbouring point with probability $\frac{1}{2}$, and stays where it is with probability $\frac{1}{2}$.

Write P^x_ω for the law of X started at $x \in \mathbb{Z}^d$. Let $\mu_x(\omega)$ be the degree of x. *Quenched (discrete time) heat kernel* on G:

$$p^\omega_n(x,y) = p^\omega_n(y,x) = \frac{P^x_\omega(X_n = y)}{\mu_y}, \quad n \in \mathbb{Z}_+, \quad x,y \in \mathbb{Z}^d.$$

The *annealed or averaged heat kernel* is

$$\mathbb{E}\, p^\omega_n(x,y).$$

Example 1: Percolation on $\mathbb{Z}^d$

This was introduced by Broadbent and Hammersley (1957).

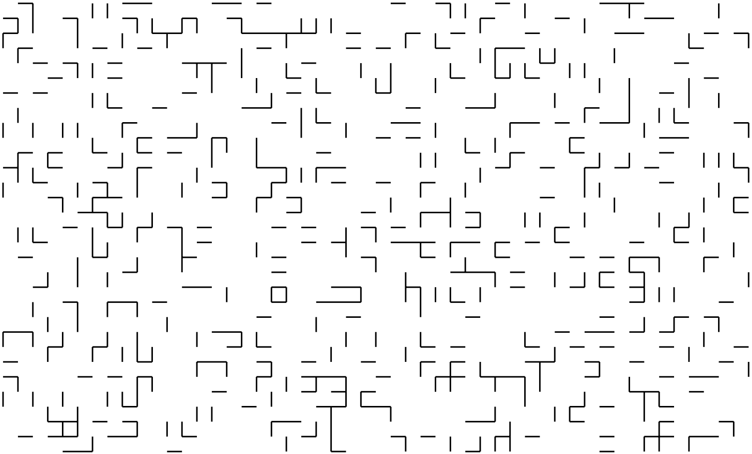
Fix $p \in [0, 1]$. For each edge $e = \{x, y\}$ keep the edge with probability p, delete it with probability $1 - p$, independently of all the others. Let $\mathcal{O}(\omega)$ be the set of edges which are kept, which are called open edges. The connected components of the graph $(\mathbb{Z}^d, \mathcal{O})$ are called (open) clusters.

There exists $p_c = p_c(d) \in (0, 1)$ such that, with probability 1:

- if $p < p_c$ all clusters are finite (subcritical regime),
- if $p > p_c$ then there exists a unique infinite cluster, C_∞ (supercritical regime),

If $p = p_c$ (critical regime) it is conjectured that all clusters are finite, but only proved in some cases ($d = 2$, $d \geq 11$).

Percolation



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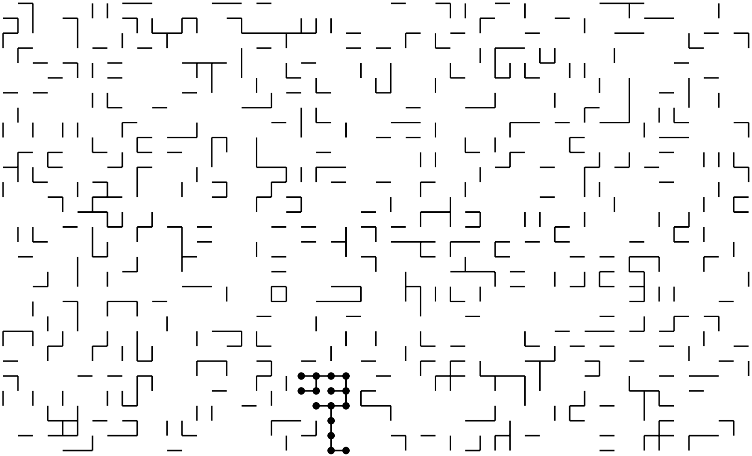

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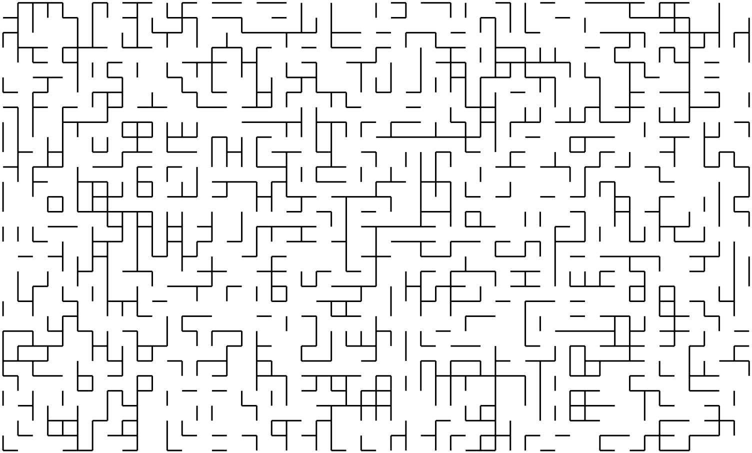

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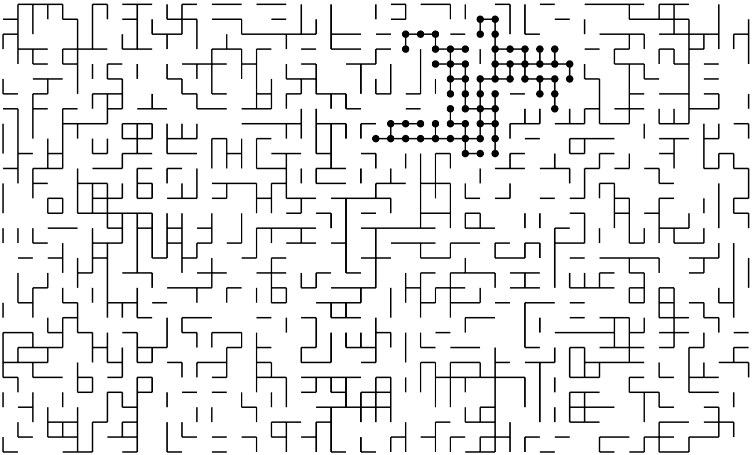

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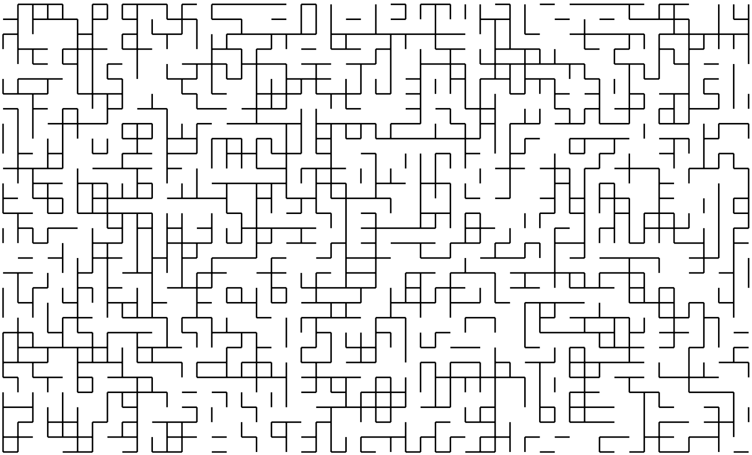

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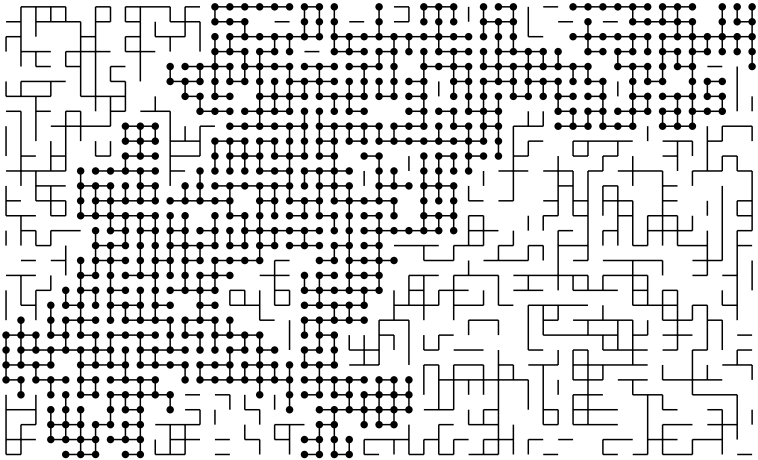

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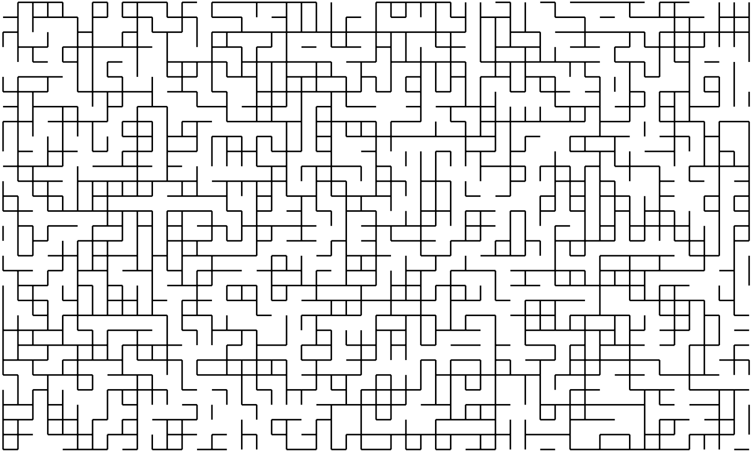

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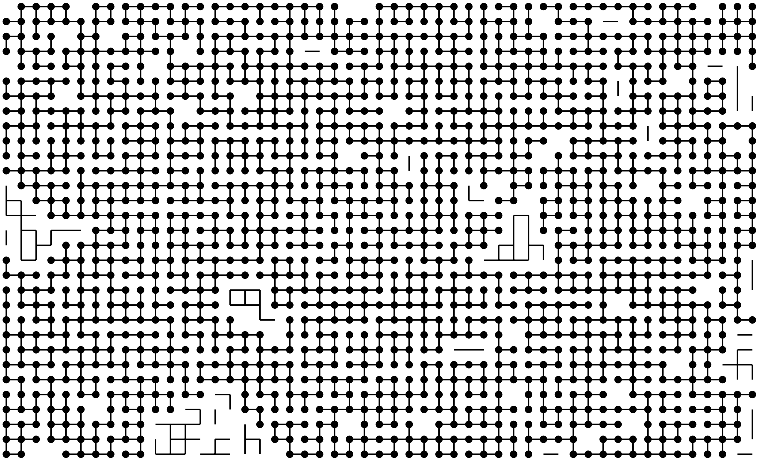

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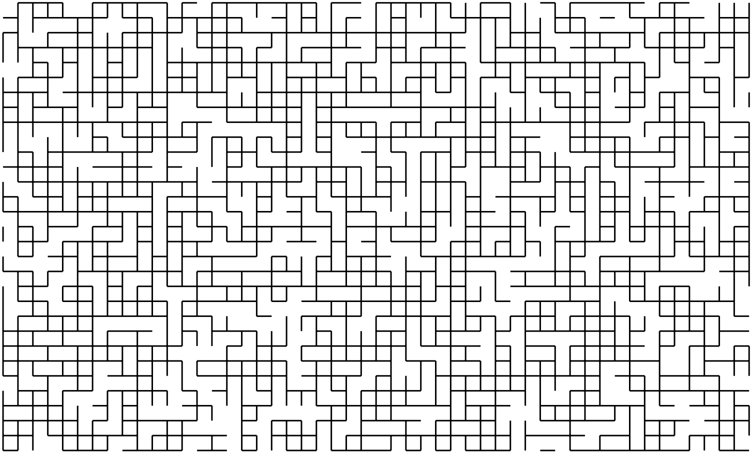

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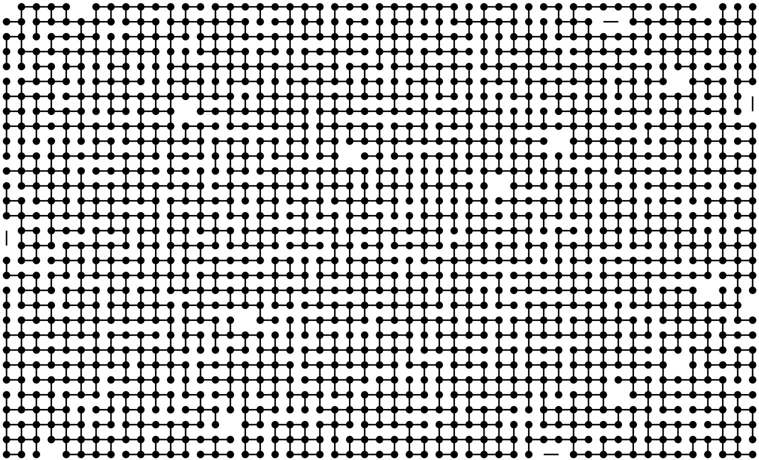


Random walk an percolation on $\mathbb{Z}^d$

Subcritical case, i.e. $p \in [0, p_c)$. For a *fixed p* little interesting to say: the random walk is trapped in a 'small' finite cluster and rapidly reaches equilibrium.

Critical case (i.e. $p = p_c$) – we know exponents in high dimensions (Kozma & Nachmias, 2009), but little known in detail about the heat kernel.

Supercritical case i.e. $p \in (p_c, 1]$. In this case there exists a unique infinite cluster $\mathcal{C}_\infty$. This looks roughly like a d-dimensional net: given a cube Λ side k, with probability about $1 - e^{-ck}$ the cluster $\mathcal{C}_\infty$ has many connected crossings between all the faces of Λ. We expect the SRW on $\mathcal{C}_\infty$ to behave in a similar fashion to SRW on $\mathbb{Z}^d$.

What is the effect on the r.w. of the small irregularities in $\mathcal{C}_\infty$?

Supercritical percolation – quenched bounds

Theorem A. (MB, 2004). Let $d \geq 2$, $p > p_c$. There exist (non-random) constants $c_i = c_i(d)$, $\delta = \delta(d)$, and r.v. $T_x,\ x \in \mathbb{Z}^d$ with

$$\mathbb{P}_p(T_x \geq n) \leq e^{-n^\delta}$$

such that for $x, y \in \mathcal{C}_\infty(\omega)$, $n \geq T_x(\omega) \vee d_\omega(x,y)$,

$$p_n^\omega(x,y) \gtreqless c_1 n^{-d/2} \exp(-c_2 d_\omega(x,y)^2/n).$$

Remarks. 1. The r.v. T_x handles possible irregularities in $\mathcal{C}_\infty$ close to x.
2. Antal, Pisztora (1996): $c|x-y| \leq d_\omega(x,y) \leq c'|x-y|$ with probability greater than $1 - e^{-c|x-y|}$.

Supercritical percolation – averaged bounds

Why no log type oscillations in the quenched bounds?

In a box of side n, the largest irregularities in $\mathcal{C}_\infty$ are of size $(\log n)^c$, and heat homogenizes over these on a time scale of at most $(\log n)^{2c} \ll n^2$.

Theorem B. (MB, 2004). There exist constants c_i such that for $x, y \in \mathbb{Z}^d$, $n \geq |x-y|$,

$$\mathbb{E}_p\big(p_n^\omega(x,y)\big|x,y \in \mathcal{C}_\infty\big) \leq c_1 n^{-d/2} \exp(-c_2|x-y|^2/n),$$
$$\mathbb{E}_p\big(p_n^\omega(x,y)\big|x,y \in \mathcal{C}_\infty\big) \geq c_3 n^{-d/2} \exp(-c_4|x-y|^2/n).$$

Which part of the graph affects $p_n(x,y)$?

To calculate $p_n(x,y)$ completely we need to know about the structure of G in the region

$$\{z \in V : d(x,z) + d(z,y) \leq n\}.$$

However, good bounds can be obtained with less information:
- If $d(x,y) \leq n^{1/2}$ we need to know about $B(x, cn^{1/2})$,
- if $n \geq R = d(x,y) \geq n^{1/2}$ then we need to know about the 'sausage' width

$$r = \frac{n}{R}$$

which connects x and y.

Example 2: Uniform spanning tree (UST)

On a finite graph the UST is a spanning tree (i.e. a connected subgraph which is a tree and contains all the vertices) chosen uniformly at random.

Pemantle (1991) defined UST on $\mathbb{Z}^d$ as limit of UST on cubes $[-N,N]^d$. (One gets a forest if $d \geq 5$).

Haggstrom (1995): UST is a limit as $q \to 0$ of the FK(p,q) random cluster model.

Wilson (1996): algorithm for construction of UST from loop erased random walk (LERW).

Wilson's algorithm (1996)

Write $\mathrm{LEW}(x, A)$ for the loop-erased RW from x to $A \subset \mathbb{Z}^2$; this is obtained by chronological erasure of the loops in a SRW started at x and run until it hits A.

Wilson's algorithm:

(0) Choose (z_k) so that $\mathbb{Z}^2 = \{z_0, z_1, \dots\}$.

(1) Let $\mathcal{T}_0 = \{z_0\}$.

(2) For $k \geq 1$ let $\mathcal{T}_k = \mathcal{T}_{k-1} \cup \mathrm{LEW}(z_k, \mathcal{T}_{k-1})$.

(3) $\mathcal{U} = \cup_k \mathcal{T}_k$ is the UST in $\mathbb{Z}^2$, and the law does not depend on the particular sequence (z_k).

This implies that the geodesic path between x and y has the same law as a LEW from x to y.

UST in two dimensions

Key estimate (Lawler (2014)). Let $X^{(n)}$ be SRW on $\mathbb{Z}^2$ run until it first hits $\partial B(0,n)$ and L be the loop erasure of $X^{(n)}$. Then $E^0|L| \asymp n^{5/4}$.

Let $\mathcal{U}$ be the UST in $\mathbb{Z}^2$. Write $d_{\mathcal{U}}$ for the shortest path metric in $\mathcal{U}$, and $B_{\mathcal{U}}(x,r)$ for balls in $(\mathcal{U}, d_{\mathcal{U}})$. Write $B_E(x,r)$ for balls in the Euclidean metric.

Set $\kappa = 5/4$. We should expect that very roughly

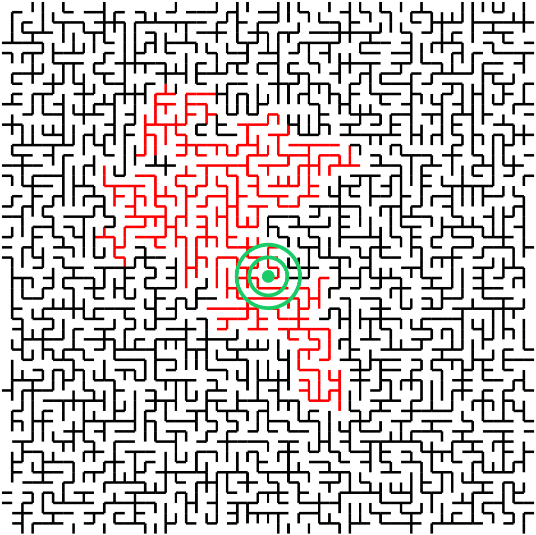
$$B_E(x,r) \approx B_{\mathcal{U}}(x, r^{\kappa}).$$

So for the UST in $\mathbb{Z}^2$ one expects (and finds) that

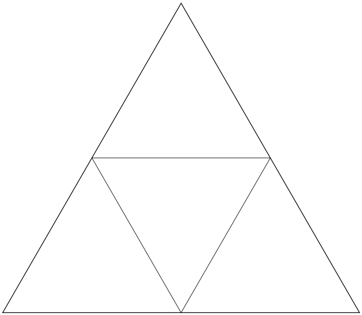
$$|B_{\mathcal{U}}(x,R)| \approx |B_E(x, R^{1/\kappa})| \asymp R^{2/\kappa},$$

Since the UST has 'fractal' properties, look at SRW on some simpler fractals.

Intrinsic ball (radius 43) in UST in 50 × 50 box

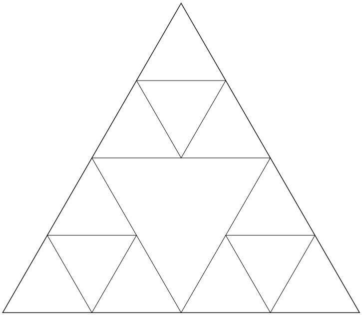


Example of an exact fractal graph: Sierpinski gasket



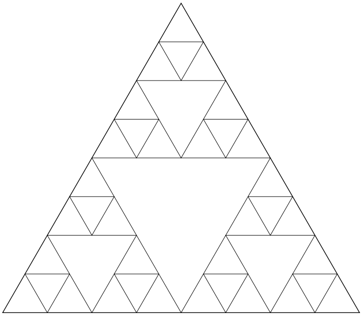
Mean number of steps to cross triangle is 5.

Example of an exact fractal graph: Sierpinski gasket



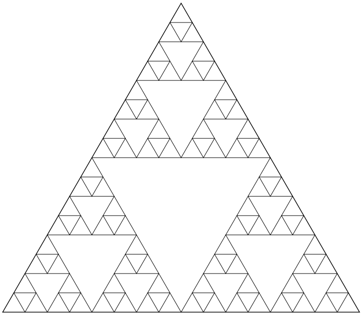
Mean number of steps to cross triangle is 5^2.

Example of an exact fractal graph: Sierpinski gasket



Mean number of steps to cross triangle is 5^3.

Example of an exact fractal graph: Sierpinski gasket



Mean number of steps to cross triangle is 5^4.

General picture for exact symmetric fractal graphs

The behaviour of the RW is described by two indices, d_f and d_w.

The *fractal dimension* d_f is given by the geometry of the set:

$$c_1 r^{d_f} \le |B(x,r)| \le c_2 r^{d_f}, \qquad (\text{or } |B(x,r)| \asymp r^{d_f}.)$$

The *walk dimension* d_w gives the space/time scaling of the RW; in time n it moves distance roughly n^{1/d_w}, and one finds that

$$E^x d(x, X_n)^2 \asymp n^{2/d_w} \qquad (\text{anomalous diffusion if } d_w \neq 2)\ .$$

In time n the SRW X moves about $R = n^{1/d_w}$. Since $|B(x,R)| \asymp R^{d_f} = n^{d_f/d_w}$, if the RW "mixes well" then

$$P^x(X_n = x) \asymp n^{-d_f/d_w}.$$

Note that $d_f(\mathbb{Z}^d) = d$ and $d_w(\mathbb{Z}^d) = 2$.

Theorem C. (MB-Perkins, Kumagai, Hambly-Kumagai, MB-Bass, Jones ...) For various classes of exact fractal graphs one finds that

$$p_n(x,y) \gtrless c_1 n^{-\frac{d_f}{d_w}} \exp\left(-c_2 \left(\frac{d(x,y)^{d_w}}{n} \right)^{\frac{1}{d_w - 1}} \right).$$

Remarks. 1. These are often called sub-Gaussian estimates.
2. Taking $d_f = d$, $d_w = 2$ gives the usual Gaussian bounds for $\mathbb{Z}^d$.
3. If these estimates hold on a graph then $2 \le d_w \le 1 + d_f$.
4. The SRW is recurrent if and only if $d_f \le d_w$.
5. The proofs are much simpler if $d_f < d_w$; sometimes called the strongly recurrent case. The bounds above follow if we can prove a 'volume' and an 'electrical resistance' estimate.

Volume: $|B(x,r)| \asymp r^{d_f}$ for all x, r.

Resistance: $R_{\text{eff}}(x,y) \asymp r^{d_w - d_f}$ for all x, y.

Quenched heat kernel on UST

Set
$$d_f = \frac{2}{\kappa}, \quad d_w = \frac{2+\kappa}{\kappa} \quad \Phi(T,R) = \left(\frac{R^{d_w}}{T}\right)^{1/(d_w-1)}.$$

Theorem D. (MB, Masson 2012) There exist r.v T_x with $\mathbb{P}(T_x > n) \le \exp(-c(\log n)^2)$ such that writing

$$A = A(x,y,n) = (\log n d_\omega(x,y))^\alpha$$

one has for $n \ge T_x \vee d_\omega(x,y)$,

$$p_n^\omega(x,y) \le n^{-d_f/d_w} A \exp(-A^{-1}\Phi(n, d_\omega(x,y)),$$
$$p_n^\omega(x,y) \ge n^{-d_f/d_w} A^{-1} \exp(-A\Phi(n, d_\omega(x,y)).$$

(Bounds of the same type as for exact fractal graphs, but, so far anyway, with log type errors.)

What about annealed bounds?

Recall

$$\Phi(n,R) = \left(\frac{R^{d_w}}{n}\right)^{1/(d_w-1)}.$$

Since $d(0,x)$ is roughly $|x|^\kappa$ we conjectured that averaging would remove the log type errors, and we would have

$$T^{d_f/d_w}\mathbb{E}\, p_T(0,x) \gtrless \exp\left(-c_2\Phi(T,|x|^\kappa)\right).$$

(Lower bound is easy from MB-Masson.)

We tried several times to prove the upper bound...

Theorem 1. (MB, Croydon, Kumagai 2020+) There exist $0 < \beta_2 \le \beta_1 < 1$ such that for $x \in \mathbb{Z}^d$, $T \ge |x|^\kappa$,

$$c_2 \exp(-c_1 \Phi(T, |x|^\kappa)^{\beta_1}) \le T^{d_f/d_w} \mathbb{E}\, p_T(0,x) \le c_1 \exp(-c_2 \Phi(T, |x|^\kappa)^{\beta_2}).$$

Remark. Our value of β_2 is poor, but we have

$$\beta_1 = \frac{d_w - 1}{\kappa d_w - 1},$$

and we conjecture that this is the right exponent, i.e. that the upper bound also holds with this value of β_2.

Main ideas for proofs

Our proofs use Wilson's algorithm to construct exceptional events for the UST, which then force exceptional behaviour of the heat kernel.

To show that the averaged bound $\mathbb{E}p_n(0,x)$ is larger than our (incorrect) conjectured value, one looks for an event F such that on F the graph distance $d_\omega(0,x)$ is much smaller than the 'usual' value of $|x|^\kappa$.

Short paths in the UST

By Wilson's algorithm $d(0,x)$ and $|\mathrm{LEW}(0,\{x\})|$ have the same law.

Theorem 2. (MB, Croydon, Kumagai 2019+) For $\lambda \geq 1$

$$\exp(-c_1\lambda^4) \leq \mathbb{P}\left(|\mathrm{LEW}(0,\{x\})| \leq \frac{|x|^\kappa}{\lambda}\right) \leq \exp(-c_2\lambda^4).$$

Upper bound. A bound with exponent $\lambda^{4/5-\varepsilon}$ was obtained by MB and Masson (2011). Small changes give the much better estimate above.

The 4 here is actually $1/(\kappa-1)$; recall that $\kappa = 5/4$.

Sketch for lower bound.

Choose $m \in \mathbb{N}$, let $N \geq 1$ and for simplicity take $x = (mN, 0)$. Tile $\mathbb{Z}^2$ with boxes side m and centres in $m\mathbb{Z}^2$.

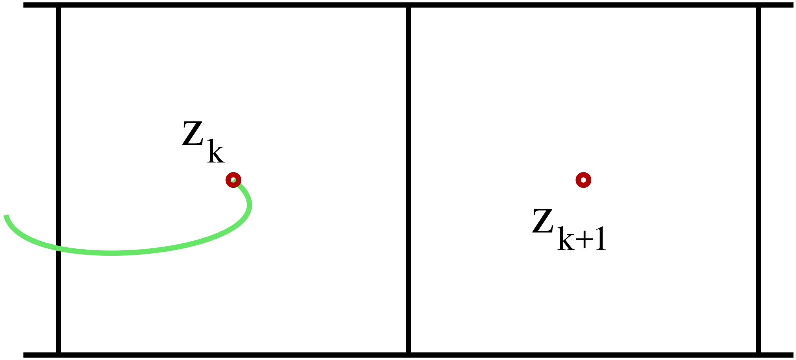
Let $z_0 = 0$, $z_j = (jm, 0)$ for $1 \leq j \leq N$ and write Q_j for the box side m centre z_j.

Run WA with the initial part of the sequence being $\{z_0, z_1, \ldots, z_N\}$. Recall that at stage $k \geq 1$ we take S^{z_k} to be a SRW started at z_k and killed on its first hit on T_{k-1}, and set

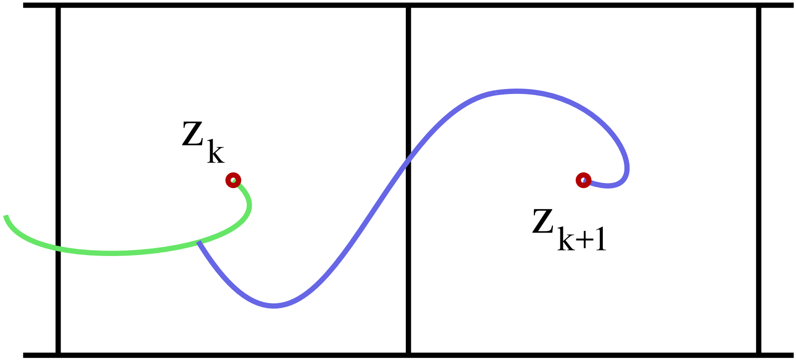
$$\mathrm{LEW}(z_k, T_{k-1}) = LE(S^{z_k}), \quad T_k = T_{k-1} \cup \mathrm{LEW}(z_k, T_{k-1}).$$

We declare stage k a *success* if S^{z_k} hits T_{k-1} before it leaves $Q_k \cup Q_{k-1}$, and

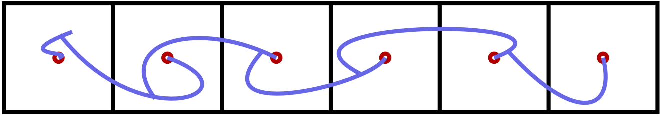
$$|\mathrm{LEW}(z_k, T_{k-1})| \leq m^{\kappa}.$$



z_k
z_{k+1}


z_k
z_{k+1}

Sketch for lower bound II



For $k \geq 1$ the probability of success, given that the previous stages have all been successful, is at least $p = e^{-a_1} > 0$. (Independent of m.) Let F be the event that all N stages are successful, so that

$$\mathbb{P}(F) \geq e^{-a_1 N}.$$

On F we have $d(0,x) \leq Nm^\kappa$, while $|x| = Nm$. So

$$\frac{d(0,x)}{|x|^\kappa} \leq \frac{Nm^\kappa}{N^\kappa m^\kappa} = \frac{1}{N^{\kappa-1}} = \frac{1}{N^{1/4}}.$$

Set $\lambda = N^{1/4}$ to obtain the lower bound.

Averaged heat kernel lower bounds

Let $x \in \mathbb{Z}^2$, $R = |x|^\kappa$ and let $R^{d_w} \gg T \gg R$. Recall that

$$\Phi(T,R) = \left(\frac{R^{d_w}}{T}\right)^{1/(d_w-1)}.$$

We expect that for most ω

$$p_T^\omega(0,x) \approx T^{-d_f/d_w} \exp(-c\Phi(T, d_\omega(0,x))).$$

Let $\lambda \gg 1$ and $F = F_\lambda = \{\omega : d_\omega(0,x) \leq \lambda^{-1}|x|^\kappa\}$ so

$$\mathbb{P}(F) \geq \exp(-c\lambda^{1/(\kappa-1)}).$$

Then

$$\mathbb{E}p_T(0,x) \geq \mathbb{E}(1_F p_T(0,x)).$$

We can hope that on F since $d_\omega(0,x) \le R/\lambda$, we will have

$$T^{d_f/d_w} p_T^\omega(0,x) \ge c\exp(-c\Phi(T,\lambda^{-1}R)). \qquad (*)$$

If so then

$$T^{d_f/d_w}\mathbb{E}(1_F p_T(0,x)) \ge \exp(-c\lambda^{\frac{1}{(\kappa-1)}})\exp(-c\Phi(T,\lambda^{-1}R)).$$

(Minus) the term in the exponential is

$$\lambda^{1/(\kappa-1)} + \Big(\frac{R^{d_w}}{\lambda^{d_w}T}\Big)^{1/(d_w-1)},$$

and optimizing over λ one obtains

$$\Big(\frac{R^{d_w}}{T}\Big)^{1/(\kappa d_w-1)} = \Phi(T,R)^{(d_w-1)/(\kappa d_w-1)}.$$

Remarks

1. We have the 'usual' heat kernel lower bound

$$p_T^\omega(0,x) \geq c' T^{-d_f/d_w} \exp(-c\Phi(T, d_\omega(0,x)))$$

on typical environments, but we have conditioned ω to be in an atypical set $F \subset \Omega$ which has very small probability.
So we need 'separation of scales': we want $\mathcal{U}$ conditioned on F to be well behaved over Euclidean distances of order m, and the conditioning only to have an effect on scales of order km with $k \gg 1$.

This is proved by first requiring an unusual event F in the early stages of WA, but then making sure that the later stages behave as expected.

2. The lower bound on the probability of having $d(0,x)$ small uses boxes of side m_1, and the chaining argument to obtain the heat kernel lower bound uses boxes of side m_2. Fortunately $m_1 \asymp m_2$.

3. What about supercritical percolation?

Recall for the UST one gets for x with $R = |x|^\kappa$,

$$\mathbb{E}\, p_T(0,x) \geq cT^{-d_f/d_w} \exp\Big(- \Big(\frac{R^{d_w}}{T}\Big)^{1/(\kappa d_w - 1)} \Big).$$

For supercritical percolation one has

$$\mathbb{P}(d_\omega(0,x) > c|x|) \leq e^{-c'|x|}$$

so the index $\kappa = 1$.

Since $d_w = 2$, $d_f = d$ the formula above does give Gaussian lower bounds.