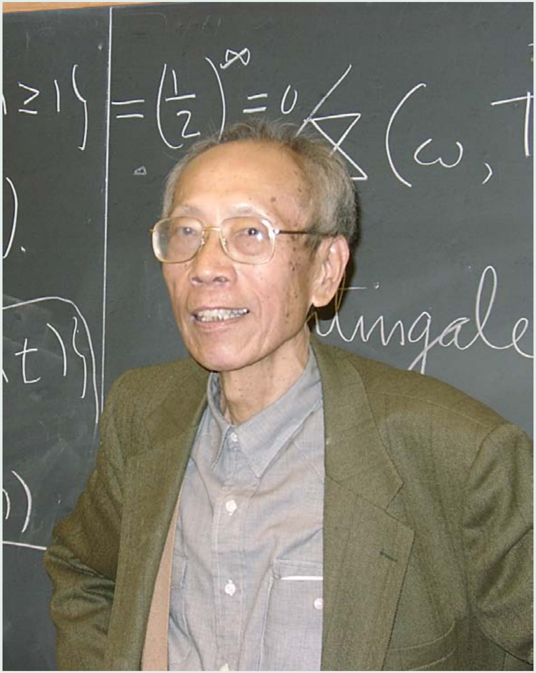
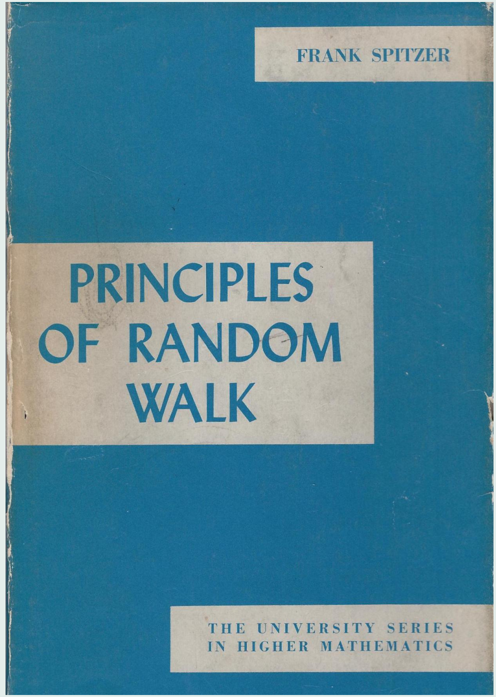
Kai Lai Chung 1917-2009

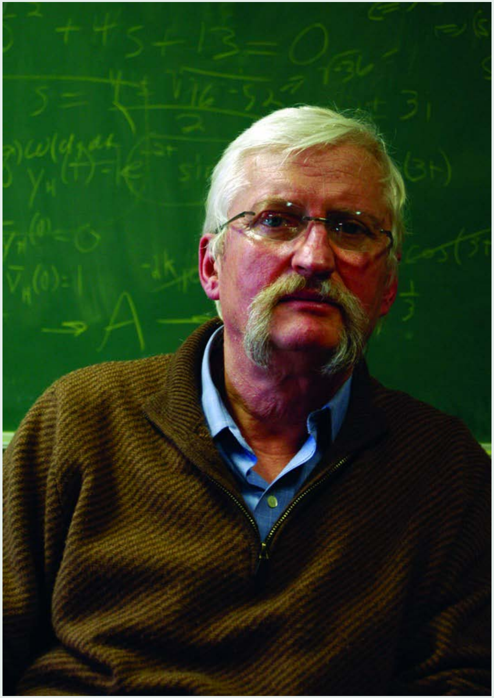


"Mathematicians are more inclined to build fire stations than to put out fires."

Courses from Chung



Peter Kotelenez 1943-2013



Also in memory of Peter Kotelenez
and the fire stations he built.

Particle representations for stochastic partial differential equations

New material joint with Dan Crisan. Earlier work with Peter Donnelly, Phil Protter, Jie Xiong, Yoonjung Lee, Peter Kotelenez

Exchangeability and de Finetti's theorem

$X_1, X_2, \ldots$ is *exchangeable* if

$$P\{X_1 \in \Gamma_1, \ldots, X_m \in \Gamma_m\} = P\{X_{s_1} \in \Gamma_1, \ldots, X_{s_m} \in \Gamma_m\}$$

$(s_1, \ldots, s_m)$ any permutation of $(1, \ldots, m)$.

Theorem 1 *(de Finetti) Let $X_1, X_2, \ldots$ be exchangeable. Then there exists a random probability measure Ξ such that for every bounded, measurable g,*

$$\lim_{n\to\infty} \frac{g(X_1) + \cdots + g(X_n)}{n} = \int g(x)\Xi(dx)$$

almost surely, and

$$E[\prod_{k=1}^m g_k(X_k)|\Xi] = \prod_{k=1}^m \int g_k d\Xi$$

Convergence of exchangeable systems

Kotelenez and Kurtz (2010)

Lemma 2 *For* $n = 1, 2, \ldots$, *let* $\{\xi_1^n, \ldots, \xi_{N_n}^n\}$ *be exchangeable (allowing* $N_n = \infty$.*) Let* Ξ^n *be the empirical measure (defined as a limit if* $N_n = \infty$*),* $\Xi^n = \frac{1}{N_n}\sum_{i=1}^{N_n} \delta_{\xi_i^n}$. *Assume*

- $N_n \to \infty$
- *For each* $m = 1, 2, \ldots$, $(\xi_1^n, \ldots, \xi_m^n) \Rightarrow (\xi_1, \ldots, \xi_m)$ *in* S^m.

Then

$\{\xi_i\}$ *is exchangeable and setting* $\xi_i^n = s_0 \in S$ *for* $i > N_n$, $\{\Xi^n, \xi_1^n, \xi_2^n \ldots\} \Rightarrow \{\Xi, \xi_1, \xi_2, \ldots\}$ *in* $\mathcal{P}(S) \times S^\infty$, *where* Ξ *is the deFinetti measure for* $\{\xi_i\}$.

If for each m, $\{\xi_1^n, \ldots, \xi_m^n\} \to \{\xi_1, \ldots, \xi_m\}$ *in probability in* S^m, *then* $\Xi^n \to \Xi$ *in probability in* $\mathcal{P}(S)$.

Lemma 3 *Let $X^n = (X_1^n, \ldots, X_{N_n}^n)$ be exchangeable families of $D_E[0,\infty)$-valued random variables such that $N_n \Rightarrow \infty$ and $X^n \Rightarrow X$ in $D_E[0,\infty)^\infty$. Define*

$$\Xi^n = \frac{1}{N_n}\sum_{i=1}^{N_n} \delta_{X_i^n} \in \mathcal{P}(D_E[0,\infty))$$

$$\Xi = \lim_{m\to\infty} \frac{1}{m}\sum_{i=}^{m} \delta_{X_i}$$

$$V^n(t) = \frac{1}{N_n}\sum_{i=1}^{N_n} \delta_{X_i^n(t)} \in \mathcal{P}(E)$$

$$V(t) = \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} \delta_{X_i(t)}$$

Then

a) For $t_1, \ldots, t_l \notin \{t : E[\Xi\{x : x(t) \neq x(t-)\}] > 0\}$

$$(\Xi_n, V^n(t_1), \ldots, V^n(t_l)) \Rightarrow (\Xi, V(t_1), \ldots, V(t_l)).$$

b) If $X^n \Rightarrow X$ in $D_{E^\infty}[0,\infty)$, then $V^n \Rightarrow V$ in $D_{\mathcal{P}(E)}[0,\infty)$. If $X^n \to X$ in probability in $D_{E^\infty}[0,\infty)$, then $V^n \to V$ in $D_{\mathcal{P}(E)}[0,\infty)$ in probability.

From particle approximation to particle representation

$$X_i^n(t) = X_i^n(0) + B_i(t) + W(t) + \frac{1}{n}\sum_{j=1}^{n}\int_0^t b(X_i^n(s) - X_j^n(s))ds$$

$$= X_i^n(0) + B_i(t) + W(t) + \int_0^t \int_{\mathbb{R}} b(X_i^n(s) - z)V^n(s, dz)ds$$

If b is bounded and continuous and $\{X_i^n(0)\} \Rightarrow \{X_i(0)\}$, then relative compactness is immediate and any limit point satisfies

$$X_i(t) = X_i(0) + B_i(t) + W(t) + \int_0^t \int_{\mathbb{R}} b(X_i(s) - z)V(s, dz)ds$$

Assuming uniqueness for the infinite system, $V^n \Rightarrow V$ where $V(t)$ is the de Finetti measure for $\{X_i(t)\}$.

Derivation of SPDE

Applying Itô's formula

$$\varphi(X_i(t)) = \varphi(X_i(0)) + \int_0^t \varphi'(X_i(s))dB_i(s) + \int_0^t \varphi'(X_i(s))dW(s) + \int_0^t L(V(s))\varphi(X_i(s))ds$$

where

$$L(v)\varphi(x) = \varphi''(x) + \int b(x-z)v(dz)\varphi'(x).$$

Averaging gives

$$\langle V(t), \varphi\rangle = \langle V(0), \varphi\rangle + \int_0^t \langle V(s), \varphi'\rangle dW(s) + \int \langle V(s), L(V(s))\varphi(\cdot)\rangle ds$$

Gaussian white noise

$W(du\times ds)$ will denote Gaussian white noise on $\mathbb{U}\times[0,\infty)$ with mean zero and variance measure $\mu(du)ds$.

For example, $W(C\times[0,t])$, $t\geq 0$, is Brownian motion with mean zero and variance $\mu(C)$.

For appropriately adapted and integrable Z,

$$M_Z(t)=\int_{\mathbb{U}\times[0,t]} Z(u,s)W(du\times ds)$$

is a square integrable martingale with quadratic variation

$$[M_Z]_t=\int_{\mathbb{U}\times[0,t]} Z(u,s)^2\mu(du)ds.$$

Coupling through the center of mass

Let $\overline{X}(t) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i(t)$ for

$$X_i(t) = X_i(0) + \int_0^t \sigma(X_i(s), \overline{X}(s))dB_i(s) + \int_0^t b(X_i(s), \overline{X}(s))ds$$
$$+ \int_{\mathbb{U}\times[0,t]} \alpha(X_i(s), \overline{X}(s), u)W(du \times ds)$$

Setting $a(x,y) = \sigma(x,y)\sigma(x,y)^T + \int \alpha(x,y,u)\alpha(x,y,u)\mu(du)$

and $L\varphi(x,y) = \frac{1}{2}\sum_{i,j} a(x,y)\partial_i\partial_j\varphi(x) + b(x,y)\cdot\nabla\varphi(x)$

$$\langle V(t),\varphi\rangle = \langle V(0),\varphi\rangle + \int_{\mathbb{U}\times[0,t]} \langle V(s), \alpha(\cdot, \overline{X}(s), u)\cdot\nabla\varphi(\cdot)\rangle W(du\times ds)$$
$$+ \int_0^t \langle V(s), L\varphi(\cdot, \overline{X}(s))\rangle ds$$

where $V(t) = \lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} \delta_{X_i(t)}$ and $\overline{X}(t) = \int zV(t,dz)$.

Stochastic Allen-Cahn equation

Consider a family of SPDEs of the form

$$\begin{aligned} dv &= \Delta v dt + F(v)dt + \text{noise}, \\ v(0,x) &= h(x), \quad x \in D, \\ v(t,x) &= g(x), \quad x \in \partial D, t > 0, \end{aligned}$$

where $F(v) = G(v)v$ and G is bounded above. For example,

$$F(v) = v - v^3 = (1 - v^2)v.$$

To be specific, in weak form the equation is

$$\begin{aligned} \langle V(t), \varphi \rangle &= \langle V(0), \varphi \rangle + \int_0^t \langle V(s), \Delta\varphi \rangle ds + \int_0^t \langle V(s), \varphi G(v(s,\cdot)) \rangle ds \\ &\quad + \int_{\mathbb{U}\times[0,t]} \int_D \varphi(x)\rho(x,u)dxW(du \times ds), \end{aligned}$$

for $\varphi \in C_c^2(D)$.

cf. Bertini, Brassesco, and Buttà (2009)

Is it a nail?

$\{X_i\}$ independent, stationary, reflecting Brownian motions in D.

$$dA_i(t) = G(v(t, X_i(t))dt + \int_{\mathbb{U}} \rho(X_i(t), u)W(du \times dt)$$

$A_i(0) = h(X_i(0)$

If X_i hits the boundary at time t, $A_i(t)$ is reset to $g(X_i(t))$.

$V(t) = \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} A_i(t)\delta_{X_i(t)}$

$\langle V(t), \varphi\rangle = \int_D \varphi(x)v(t,x)\pi(dx)$ where π is the stationary distribution for X_i (normalized Lebesgue measure on D).

Particle representation

More generally, let B be the generator of a reflecting diffusion X in D and assume that X is ergodic with stationary distribution π. Let $\{X_i, i \geq 1\}$ be independent, stationary diffusion with generator B.

Assume that the boundary of D is regular for both X_i and the time reversal of X_i. Let $\tau_i(t) = 0 \vee \sup\{s < t : X_i(s) \in \partial D$, and

$$\begin{aligned} A_i(t) &= g(X_i(\tau_i(t)))\mathbf{1}_{\{\tau_i(t)>0\}} + h(X_i(0))\mathbf{1}_{\{\tau_i(t)=0\}} \\ &\qquad + \int_{\tau_i(t)}^t G(v(s,X_i(s)),X_i(s))A_i(s)ds + \int_{\tau_i(t)}^t b(X_i(s))ds \\ &\qquad + \int_{\mathbb{U}\times(\tau_i(t),t]} \rho(X_i(s),u)W(du\times ds), \end{aligned} \tag{1}$$

where

$$\langle V(t),\varphi\rangle = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \varphi(X_i(t))A_i(t) = \int \varphi(x)v(t,x)\pi(dx).$$

Corresponding SPDE

Define $M_{\varphi,i}(t) = \varphi(X_i(t)) - \int_0^t B\varphi(X_i(s))ds$.

Then

$$
\begin{aligned}
\varphi(X_i(t))A_i(t) &= \varphi(X_i(0))A_i(0) + \int_0^t \varphi(X_i(s))dA_i(s) \\
&\quad + \int_0^t A_i(s)dM_{\varphi,i}(s) + \int_0^t B\varphi(X_i(s))A_i(s)ds \\
&= \varphi(X_i(0))A_i(0) + \int_0^t \varphi(X_i(s))G(v(s,X_i(s)),X_i(s))A_i(s)ds \\
&\quad + \int_0^t \varphi(X_i(s))b(X_i(s))ds \\
&\quad + \int_{\mathbb{U}\times[0,t]} \varphi(X_i(s))\rho(X_i(s),u)W(du\times ds) \\
&\quad + \int_0^t A_i(s)dM_{\varphi,i}(s) + \int_0^t B\varphi(X_i(s))A_i(s)ds
\end{aligned}
$$

Averaging

$$\begin{aligned}\langle V(t),\varphi\rangle &= \langle V(0),\varphi\rangle + \int_0^t \langle V(s), \varphi G(v(s,\cdot),\cdot)\rangle ds + \int_0^t \int b\varphi d\pi ds \\ &\quad + \int_{\mathbb{U}\times[0,t]}\int_D \varphi(x)\rho(x,u)\pi(dx)W(du\times ds) + \int_0^t \langle V(s), B\varphi\rangle ds\end{aligned}$$

which is the weak form of

$$\begin{aligned}v(t,x) = v(0,x) &+ \int_0^t (G(v(s,x),x)v(s,x) + b(x))ds \\ &+ \int_{\mathbb{U}\times[0,t]} \rho(x,u)W(du\times ds) + \int_0^t B^* v(x,s)ds,\end{aligned}$$

where B^* is the adjoint determined by $\int gBfd\pi = \int fB^*gd\pi$.

Approximating systems

Let ψ be an $L^1(\pi)$-valued stochastic process that is compatible with W, and assume (W,ψ) is independent of $\{X_i\}$ Define A_i^ψ to be the solution of

$$\begin{aligned} A_i^\psi(t) &= g(X_i(\tau_i(t)))\mathbf{1}_{\{\tau_i(t)>0\}} + h(X_i(0))\mathbf{1}_{\{\tau_i(t)=0\}} \\ &\quad + \int_{\tau_i(t)}^t G(\psi(s,X_i(s)),X_i(s))A_i^\psi(s)ds + \int_{\tau_i(t)}^t b(X_i(s))ds \\ &\quad + \int_{\mathbb{U}\times(\tau_i(t),t]} \rho(X_i(s),u)W(du\times ds). \end{aligned}$$

The $\{A_i^\psi\}$ will be exchangeable, so we can define $\Phi\psi(t,x)$ to be the density of the signed measure determined by

$$\langle \Phi\Psi(t),\varphi\rangle \equiv \int_D \varphi(x)\Phi\psi(t,x)\pi(dx) = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N A_i^\psi(t)\varphi(X_i(t)).$$

Apriori bounds

Assume

$$\begin{aligned} K_1 &\equiv \sup_{x,D} |b(x)| < \infty \\ K_2 &\equiv \sup_{x\in D} \int \rho(x,u)^2 \mu(du) < \infty \\ K_3 &\equiv \sup_{v\in\mathbb{R},x\in D} G(v,x) < \infty. \end{aligned}$$

Lemma 4 *Let*

$$H_i(t) = \int_{\mathbb{U}\times[0,t]} \rho(X_i(s),u)W(du\times ds) = B_i(\int_0^t \int_{\mathbb{U}} \rho(X_i(s),u)^2 \mu(du)ds).$$

Then

$$\begin{aligned} |A_i^\psi(t)| &\leq (\|g\| \vee \|h\| + K_1(t-\tau_i(t)) + \sup_{\tau_i(t)\leq r\leq t} |H_i(t) - H_i(r)|)e^{K_3(t-\tau_i(t))} \\ &\leq (\|g\| \vee \|h\| + K_1 t + \sup_{0\leq s\leq t} |H_i(t) - H_i(s)|)e^{K_3 t} \equiv \Gamma_i(t). \end{aligned}$$

Lemma 5 *Suppose that (W, ψ) is independent of $\{X_i\}$. Then $\Phi\psi$ is $\{\mathcal{F}_t^{W,\psi}\}$-adapted and for each i,*

$$E[A_i^\psi(t)|W, \psi, X_i(t)] = \Phi\psi(t, X_i(t))$$

so

$$\Phi\psi(t, X_i(t)) \leq E[\Gamma_i(t)|W, \psi, X_i(t)]$$

Remark 6 *Let $\mathcal{G}_t^{X_i} = \sigma(X_i(r) : r \geq t)$. Then the Markov property and the independence of (W, ψ) and X_i imply*

$$\Phi\psi(t, X_i(t)) = E[A_i^\psi(t)|W, \psi, X_i(t)] = E[A_i^\psi(t)|\sigma(W, \psi) \vee \mathcal{G}_t^{X_i}].$$

The properties of reverse martingales and Doob's inequality give

$$E[\sup_{0 \leq t \leq T} |\Phi\psi(t, X_i(t))|^2] \leq 4E[\sup_{0 \leq t \leq T} |A_i^\psi(t)|^2] \leq 4E[\sup_{0 \leq t \leq T} \Gamma_i(t)^2]$$

Proof. By exchangeability,

$$
\begin{aligned}
E[A_i^{\psi}(t)\varphi(X_i(t))F(W,\psi)] &= E[\int \varphi(x)\Phi\Psi(t,dx)F(W,\psi)] \\
&= E[\int \varphi(x)\Phi\psi(t,x)\pi(dx)F(W,\psi)] \\
&= E[\varphi(X_i(t))\Phi\psi(t,X_i(t))F(W,\psi)].
\end{aligned}
$$

The last equality follows by the independence of $X_i(t)$ and (W,ψ), and the lemma follows by the definition of conditional expectation. $\square$

Boundary conditions

Recall $\Phi\psi(t,x)$ is the density of the signed measure determined by

$$\langle \Phi\Psi(t), \varphi \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} A_i^{\psi}(t)\varphi(X_i(t)).$$

If $X_i(t)$ is close to ∂D, then by the regularity assumption, with high probability $t - \tau_i(t)$ is small and $A_i^{\psi}(t) \approx g(X_i(\tau_i(t)))$. Consequently, for $y \in \partial D$

$$\Phi\psi(t,y) = \lim_{\epsilon\to 0} \frac{\Phi\Psi(B_\epsilon(y))}{\pi(B_\epsilon(y))} = g(y).$$

Tightness in D of approximations

$$
\begin{aligned}
A_i^\psi(t) &= g(X_i(\tau_i(t)))\mathbf{1}_{\{\tau_i(t)>0\}} + h(X_i(0))\mathbf{1}_{\{\tau_i(t)=0\}} \\
&\quad + \int_{\tau_i(t)}^t G(\psi(s,X_i(s)),X_i(s))A_i^\psi(s)ds + \int_{\tau_i(t)}^t b(X_i(s))ds \\
&\quad + \int_{\mathbb{U}\times(\tau_i(t),t]} \rho(X_i(s),u)W(du\times ds).
\end{aligned}
$$

Let

$$
\begin{aligned}
Z_i^\psi(t) &= -g(X_i(t)) + \int_0^t G(\psi(s,X_i(s)),X_i(s))A_i^\psi(s)ds \\
&\quad + \int_0^t b(X_i(s))ds + \int_{\mathbb{U}\times[0,t]} \rho(X_i(s),u)W(du\times ds).
\end{aligned}
$$

Then

$$
A_i^\psi(t) = g(X_i(t)) + Z_i^\psi(t) - Z_i^\psi(\tau_i(t)) + (h(X_i(0)) - g(X_i(0)))\mathbf{1}_{\{\tau_i(t)=0\}}.
$$

Estimates on modulus of continuity

The Skorohod modulus of continuity of A_i^ψ can be bounded in terms of the ordinary modulus of continuity of Z_i^ψ.

Lemma 7 *Define* $\gamma_i^0 = \inf\{t : X_i(t) \in \partial D\}$. *Then with probability one,* $\gamma_i^0 > 0$, *and for* $\delta < \gamma_i^0$,

$$w'(A_i^\psi, \delta, T) \leq w(g \circ X_i, 4\delta, T) + 2w(Z_i^\psi, 4\delta, T). \tag{2}$$

The relative compactness for $\{A_i^\psi\}$ *for fixed* i *then follows from the relative compactness of* $\{Z_i^\psi\}$ *in* $C_{\mathbb{R}^d}[0,\infty)$.

Uniqueness

$L_1 \equiv \sup_{v,x\in D} \frac{|G(v,x)|}{1+|v|^2} < \infty.$

$L_2 \equiv \sup_{v_1,v_2,x\in D} \frac{|G(v_1,x)-G(v_2,x)|}{|v_1-v_2|(|v_1|+|v_2|)} < \infty.$

$$
\begin{aligned}
|A_i^{v_1}(t) - A_i^{v_2}(t)| &\le \int_{\tau_i(t)}^t |G(v_1(s,X_i(s)),X_i(s))A_i^{v_1}(s) - G(v_2(s,X_i(s)),X_i(s))A_i^{v_2}(s)|ds \\
&\le \int_{\tau_i(t)}^t L_1(1+E[\Gamma_i(s)|W,X_i(s)]^2)|A_i^{v_1}(s) - A_i^{v_2}(s)|ds \\
&\qquad + \int_{\tau_i(t)}^t 2L_2 E[\Gamma_i(s)|W,X_i(s)]\Gamma_i(s)|v_1(s,X_i(s)) - v_2(s,X_i(s))|ds \\
&\le \int_0^t L_1(1+C^2)|A_i^{v_1}(s) - A_i^{v_2}(s)|ds \\
&\qquad + \int_0^t 2L_2C^2|v_1(s,X_i(s)) - v_2(s,X_i(s))|ds \\
&\qquad + \int_0^t \mathbf{1}_{\{\Gamma_i(s)>C\}\cup\{E[\Gamma_i(s)|W,X_i(s)]>C\}}\Gamma_i(s)L_3(1+E[\Gamma_i(s)|W,X_i(s)]^2)ds
\end{aligned}
$$

References

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Peter M. Kotelenez and Thomas G. Kurtz. Macroscopic limits for stochastic partial differential equations of McKean-Vlasov type. *Probab. Theory Related Fields*, 146(1-2):189–222, 2010. ISSN 0178-8051. doi: 10.1007/s00440-008-0188-0. URL http://dx.doi.org/10.1007/s00440-008-0188-0.

Abstract

Particle representations for stochastic partial differential equations

Stochastic partial differential equations arise naturally as limits of finite systems of weighted interacting particles. For a variety of purposes, it is useful to keep the particles in the limit obtaining an infinite exchangeable system of stochastic differential equations for the particle locations and weights. The corresponding de Finetti measure then gives the solution of the SPDE. These representations frequently simplify existence, uniqueness and convergence results. The talk will focus on situations where the particle locations are given by an iid family of diffusion processes, and the weights are chosen to obtain a nonlinear driving term and to match given boundary conditions for the SPDE. (Recent results are joint work with Dan Crisan.)