The Brownian map
A continuous limit for large random planar maps

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Outline

A planar map is just a graph drawn in the plane (or on the sphere) viewed up to continuous deformation.

It should be interpreted as a discretized model of the sphere.

Goal: To show that a large planar map chosen uniformly at random in a suitable class (p-angulations) and viewed as a metric space (for the graph distance) is asymptotically close to a universal limiting object :

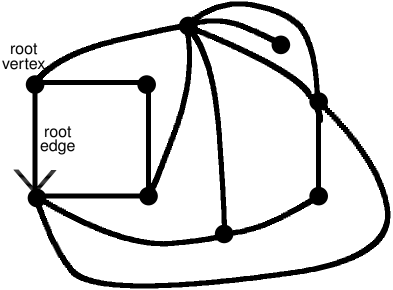
the Brownian map

Strong analogy with random paths and Brownian motion: Brownian motion is the universal continuous limit of a variety of discrete models of random paths.

1. Statement of the main result

Definition

A planar map is a proper embedding of a connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).



A rooted quadrangulation with 7 faces

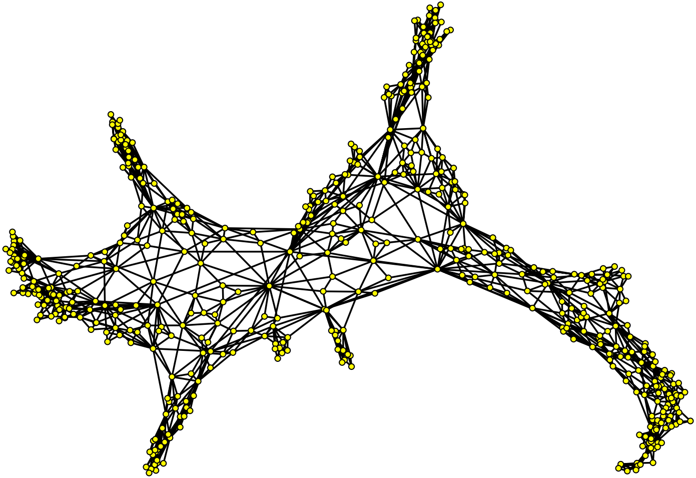
Faces = connected components of the complement of edges

p-angulation:

- each face has p adjacent edges

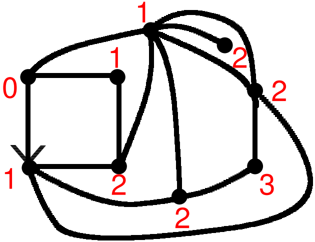
$p = 3$: triangulation
$p = 4$: quadrangulation

Rooted map: distinguished oriented edge



A large triangulation of the sphere (simulation by G. Schaeffer)
Can we get a continuous model out of this ?

Planar maps as metric spaces



In red : distances from the root vertex

M planar map

- $V(M)$ = set of vertices of M
- $d_{\rm gr}$ graph distance on $V(M)$
- $(V(M), d_{\rm gr})$ is a (finite) metric space

$\mathbb{M}_n^p = \{$rooted $p-$angulations with n faces$\}$

$\mathbb{M}_n^p$ is a finite set (*finite number of possible "shapes"*)

Choose M_n uniformly at random in $\mathbb{M}_n^p$.
View $(V(M_n), d_{\rm gr})$ as a random variable with values in

$$\mathbb{K} = \{\text{compact metric spaces, modulo isometries}\}$$

which is equipped with the Gromov-Hausdorff distance.

The Gromov-Hausdorff distance

The Hausdorff distance. K_1, K_2 compact subsets of a metric space

$$d_{\mathrm{Haus}}(K_1,K_2) = \inf\{\varepsilon > 0 : K_1 \subset U_\varepsilon(K_2) \text{ and } K_2 \subset U_\varepsilon(K_1)\}$$

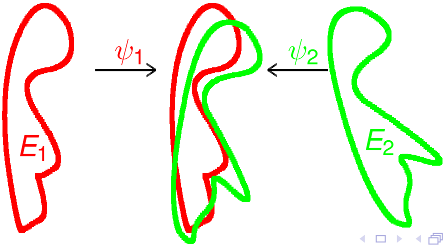
($U_\varepsilon(K_1)$ is the ε-enlargement of K_1)

Definition (Gromov-Hausdorff distance)

If (E_1,d_1) and (E_2,d_2) are two compact metric spaces,

$$d_{\mathrm{GH}}(E_1,E_2) = \inf\{d_{\mathrm{Haus}}(\psi_1(E_1),\psi_2(E_2))\}$$

the infimum is over all isometric embeddings $\psi_1 : E_1 \to E$ and $\psi_2 : E_2 \to E$ of E_1 and E_2 into the same metric space E.



Gromov-Hausdorff convergence of rescaled maps

Fact

If $\mathbb{K} = \{$*isometry classes of compact metric spaces*$\}$*, then*

$$(\mathbb{K}, d_{\mathrm{GH}}) \text{ is a separable complete metric space (Polish space)}$$

$\to$ If M_n is uniformly distributed over $\{p -$ angulations with n faces$\}$, it makes sense to study the convergence in distribution of

$$(V(M_n), n^{-a} d_{\mathrm{gr}})$$

as random variables with values in $\mathbb{K}$.

(Problem stated for triangulations by O. Schramm [ICM06])

Choice of the rescaling parameter: $a > 0$ is chosen so that $\mathrm{diam}(V(M_n)) \approx n^a$.

$\Rightarrow a = \frac{1}{4}$ [cf Chassaing-Schaeffer PTRF 2004 for quadrangulations]

The main theorem

$\mathbb{M}_n^p = \{\text{rooted } p-\text{angulations with } n \text{ faces}\}$
M_n uniform over $\mathbb{M}_n^p$, $V(M_n)$ vertex set of M_n, d_{gr} graph distance

Theorem (The scaling limit of p-angulations)

Suppose that either $p = 3$ (triangulations) or $p \geq 4$ is even. Set

$$c_3 = 6^{1/4} \quad , \quad c_p = \left(\frac{9}{p(p-2)}\right)^{1/4} \quad \text{if } p \text{ is even.}$$

Then,

$$(V(M_n), c_p \frac{1}{n^{1/4}} d_{\mathrm{gr}}) \xrightarrow[n\to\infty]{(\mathrm{d})} (\mathbf{m}_\infty, D^*)$$

in the Gromov-Hausdorff sense. The limit $(\mathbf{m}_\infty, D^*)$ is a random compact metric space that does not depend on p (universality) and is called the Brownian map (after Marckert-Mokkadem).

Remarks. Alternative approach to the case $p = 4$: Miermont (2011)
The case $p = 3$ solves Schramm's problem (2006)

Why study planar maps and their continuous limits ?

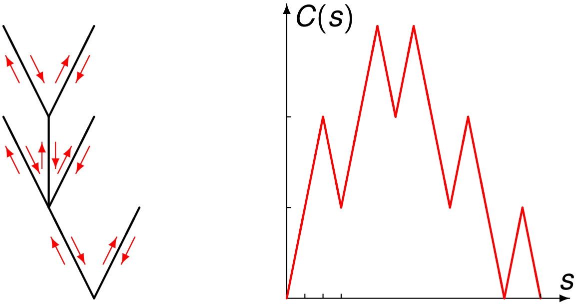
- combinatorics [Tutte '60, 4-color thm, ...]
- theoretical physics
  - enumeration of maps related to matrix integrals ['t Hooft 74, Brézin, Itzykson, Parisi, Zuber 78, etc.]
  - large random planar maps as models of random geometry (quantum gravity, cf Ambjørn, Durhuus, Jonsson 95, Duplantier-Sheffield 08, Sheffield 10)
- probability theory: models for a Brownian surface
  - analogy with Brownian motion as continuous limit of discrete paths
  - universality of the limit (conjectured by physicists)
  - asymptotic properties of large planar graphs
- algebraic and geometric motivations: cf Lando-Zvonkin 04 *Graphs on surfaces and their applications*

2. The Brownian map

The Brownian map $(\mathbf{m}_\infty, D^*)$ is constructed by identifying certain pairs of points in the Brownian continuum random tree (CRT).

Constructions of the CRT (Aldous, ...):

- As the scaling limit of many classes of discrete trees
- As the random real tree whose contour is a Brownian excursion.



A discrete tree and its contour function.

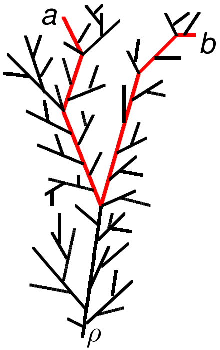
The notion of a real tree

Definition

A real tree, or $\mathbb{R}$-tree, is a (compact) metric space $\mathcal{T}$ such that:

- any two points $a, b \in \mathcal{T}$ are joined by a unique continuous and injective path (up to re-parametrization)
- this path is isometric to a line segment

$\mathcal{T}$ is a rooted real tree if there is a distinguished point ρ, called the root.



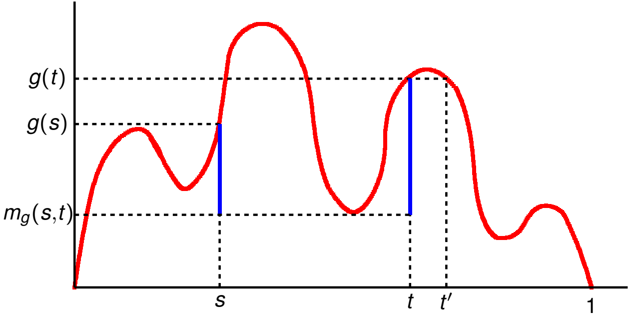
Remark. A real tree can have

- infinitely many branching points
- (uncountably) infinitely many leaves

Fact. The coding of discrete trees by contour functions (Dyck paths) can be extended to real trees.

The real tree coded by a function g

$g : [0,1] \longrightarrow [0,\infty)$
continuous,
$g(0) = g(1) = 0$



$d_g(s,t) = g(s) + g(t) - 2\min_{s\le r\le t} g(r)$ pseudo-metric on $[0,1]$

$t \sim t'$ iff $d_g(t,t') = 0$ (or equivalently $g(t) = g(t') = \min_{t\le r\le t'} g(r)$)

Proposition (Duquesne-LG)

$\mathcal{T}_g := [0,1]/\sim$ equipped with d_g is a real tree, called the tree coded by g. It is rooted at $\rho = 0$.

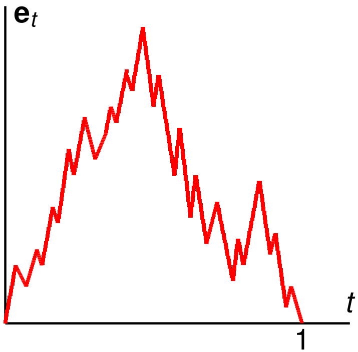
Remark. $\mathcal{T}_g$ inherits a "lexicographical order" from the coding.

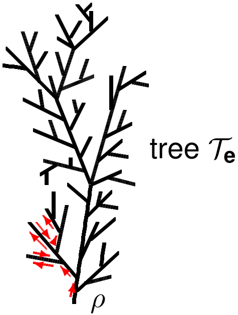
Definition of the CRT

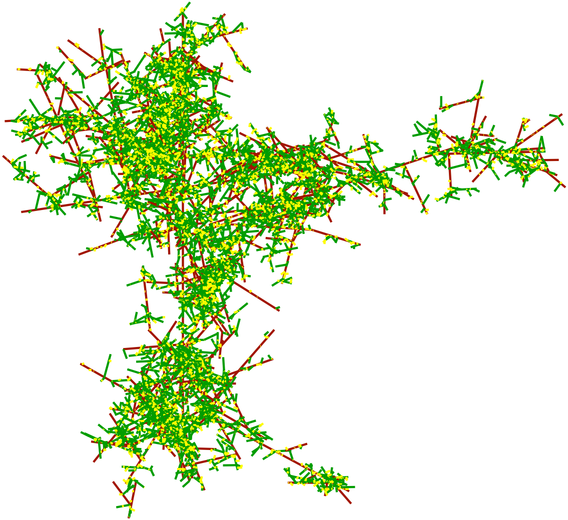
Let $\mathbf{e} = (\mathbf{e}_t)_{0 \le t \le 1}$ be a Brownian excursion with duration 1.

Definition

The CRT $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is the (random) real tree coded by the Brownian excursion $\mathbf{e}$.







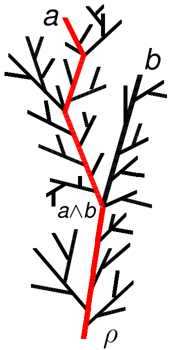
A simulation of the CRT (simulation by G. Miermont)

Assigning Brownian labels to a real tree

Let $(\mathcal{T}, d)$ be a real tree with root ρ.

$(Z_a)_{a\in\mathcal{T}}$: Brownian motion indexed by $(\mathcal{T}, d)$
= centered Gaussian process such that

- $Z_\rho = 0$
- $E[(Z_a - Z_b)^2] = d(a,b), \qquad a, b \in \mathcal{T}$



Labels evolve like Brownian motion along the branches of the tree:

- The label Z_a is the value at time $d(\rho, a)$ of a standard Brownian motion
- Similar property for Z_b, but one uses
  - the same BM between 0 and $d(\rho, a\wedge b)$
  - an independent BM between $d(\rho, a\wedge b)$ and $d(\rho, b)$

The definition of the Brownian map

$(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is the CRT, $(Z_a)_{a \in \mathcal{T}_{\mathbf{e}}}$ Brownian motion indexed by the CRT
Set, for every $a, b \in \mathcal{T}_{\mathbf{e}}$,

$$D^0(a,b) = Z_a + Z_b - 2\max\Big(\min_{c\in[a,b]} Z_c, \min_{c\in[b,a]} Z_c\Big)$$

where $[a,b]$ is the "lexicographical interval" from a to b in $\mathcal{T}_{\mathbf{e}}$ (vertices visited when going from a to b in clockwise order around the tree).
Then set

$$D^*(a,b) = \inf_{a_0=a,a_1,\ldots,a_{k-1},a_k=b} \sum_{i=1}^{k} D^0(a_{i-1}, a_i),$$

$$a \approx b \ \text{ if and only if } \ D^*(a,b) = 0 \quad (\text{equivalent to } D^0(a,b) = 0).$$

Definition

The Brownian map $\mathbf{m}_\infty$ is the quotient space $\mathbf{m}_\infty := \mathcal{T}_{\mathbf{e}} / \approx$, which is equipped with the distance induced by D^*.

Interpretation

Starting from the CRT $\mathcal{T}_{\mathbf{e}}$, with Brownian labels $Z_a, a \in \mathcal{T}_{\mathbf{e}}$,
$\rightarrow$ Identify two vertices $a, b \in \mathcal{T}_{\mathbf{e}}$ if:

- they have the same label $Z_a = Z_b$,
- one can go from a to b around the tree (in clockwise or in counterclockwise order) visiting only vertices with label greater than or equal to $Z_a = Z_b$.

Remark. Not many vertices are identified:

- A "typical" equivalence class is a singleton.
- Equivalence classes may contain at most 3 points.

Still these identifications drastically change the topology.

Two theorems about the Brownian map

Theorem (Hausdorff dimension)

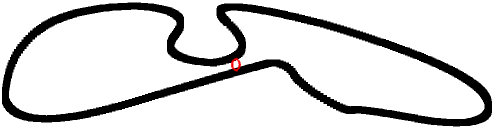
$$\dim(\mathbf{m}_\infty, D^*) = 4 \qquad \textit{a.s.}$$

(Already "known" in the physics literature.)

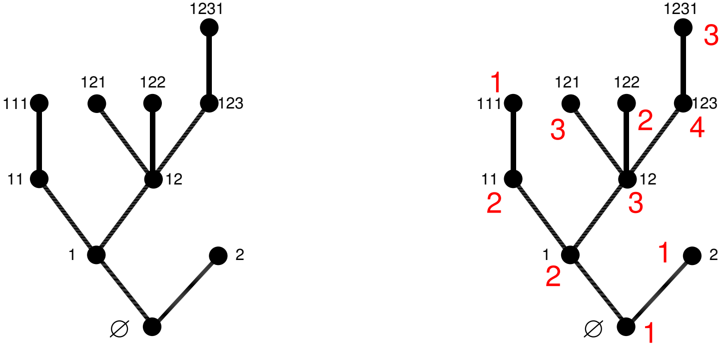
Theorem (topological type, LG-Paulin 2007)

Almost surely, $(\mathbf{m}_\infty, D^*)$ is homeomorphic to the 2-sphere $\mathbb{S}^2$.

Consequence: for a planar map M_n with n vertices, no separating cycle of size $o(n^{1/4})$ in M_n, such that both sides have diameter $\geq \varepsilon n^{1/4}$



3. The main tool: Bijections between maps and trees



A planar tree $\tau = \{\varnothing, 1, 2, 11, \ldots\}$

(rooted ordered tree)

the lexicographical order on vertices will play an important role in what follows

A well-labeled tree $(\tau, (\ell_v)_{v \in \tau})$

Properties of labels:

- $\ell_\varnothing = 1$
- $\ell_v \in \{1, 2, 3, \ldots\}, \forall v$
- $|\ell_v - \ell_{v'}| \leq 1$, if v, v' neighbors

Coding maps with trees, the case of quadrangulations

$$\mathbb{T}_n = \{\text{well-labeled trees with } n \text{ edges}\}$$
$$\mathbb{M}_n^4 = \{\text{rooted quadrangulations with } n \text{ faces}\}$$

Theorem (Cori-Vauquelin, Schaeffer)

There is a bijection $\Phi : \mathbb{T}_n \longrightarrow \mathbb{M}_n^4$ such that, if $M = \Phi(\tau, (\ell_v)_{v \in \tau})$, then

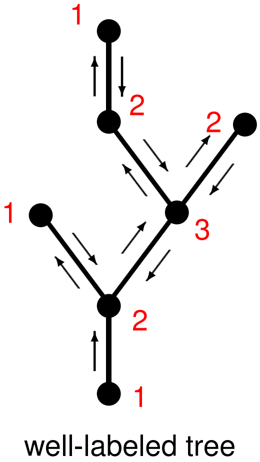
$$V(M) = \tau \cup \{\partial\} \qquad (\partial \text{ is the root vertex of } M)$$
$$d_{\mathrm{gr}}(\partial, v) = \ell_v \quad , \forall v \in \tau$$

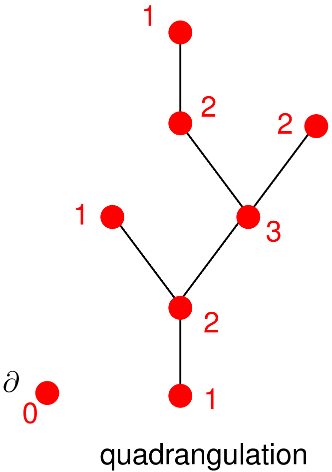
Key facts.

- Vertices of τ become vertices of M
- The label in the tree becomes the distance from the root in the map.

Coding of more general maps: Bouttier, Di Francesco, Guitter (2004)

Schaeffer's bijection between quadrangulations and well-labeled trees

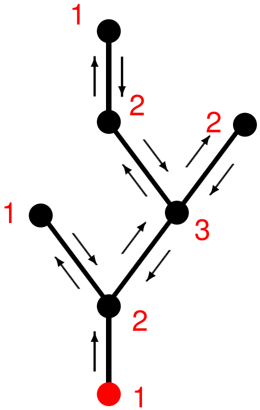




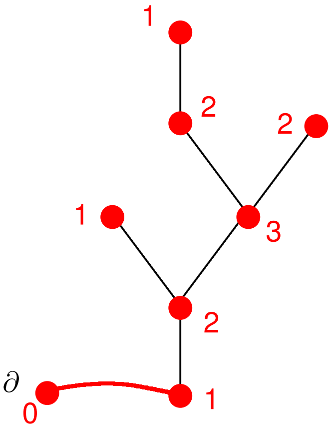
Rules.

- add extra vertex ∂ labeled 0
- follow the contour of the tree, connect each vertex to the last visited vertex with smaller label

Schaeffer's bijection between quadrangulations and well-labeled trees



well-labeled tree



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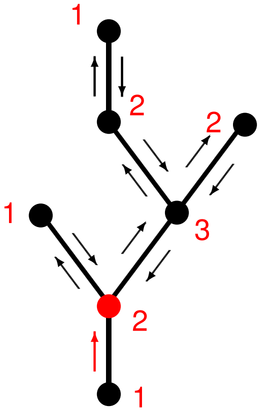

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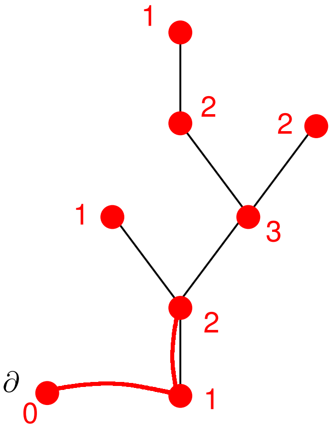

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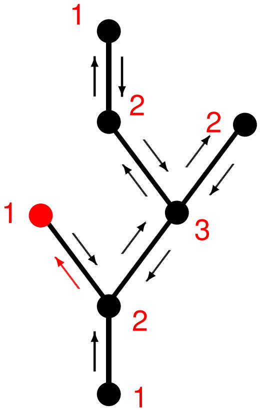

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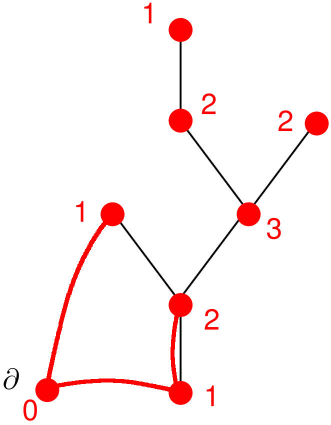

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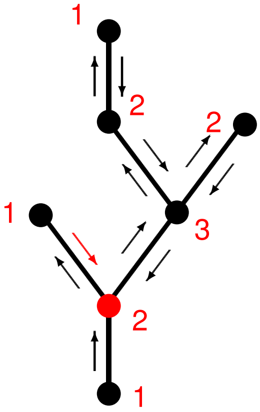

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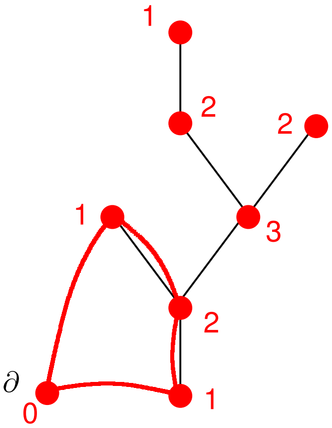

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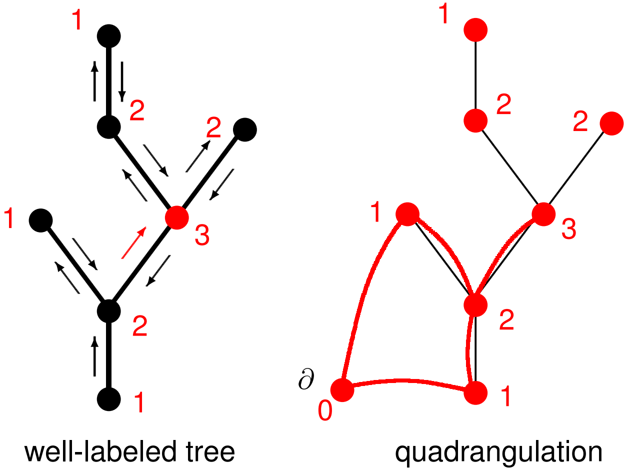

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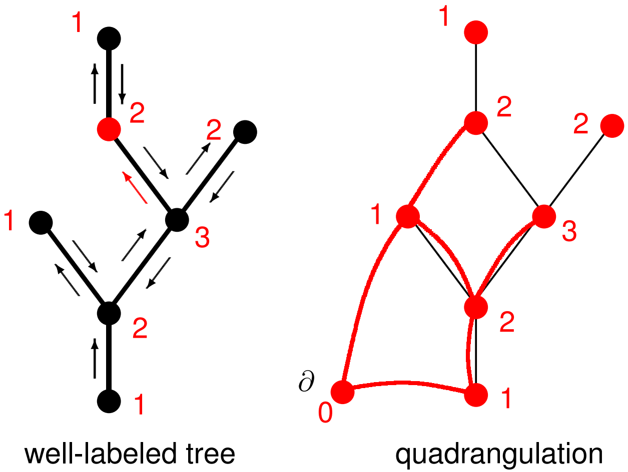

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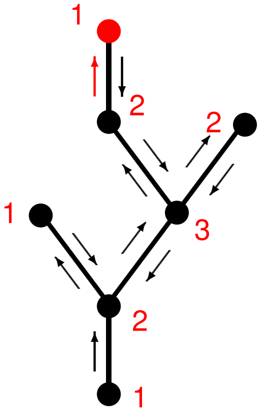
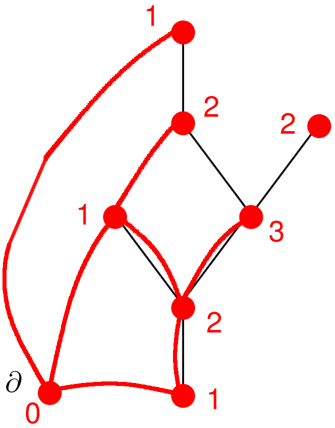


well-labeled tree



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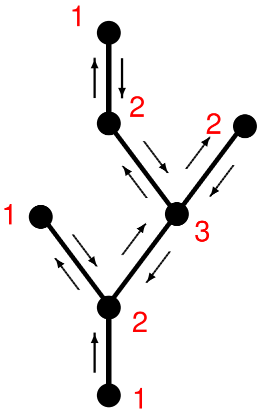

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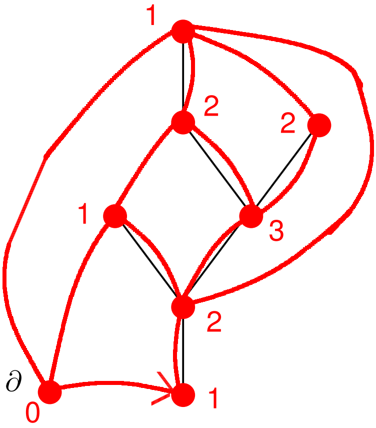


quadrangulation

Rules.

- add extra vertex ∂ labeled 0
- follow the contour of the tree, connect each vertex to the last visited vertex with smaller label

The label in the tree becomes the distance from ∂ in the graph

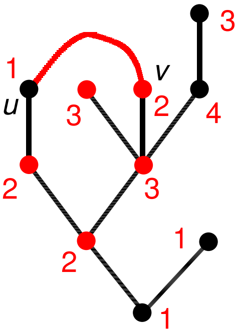
Interpretation of the equivalence relation $\approx$

In Schaeffer's bijection:
$\exists$ edge between u and v if

- $\ell_u = \ell_v - 1$
- $\ell_w \geq \ell_v$, $\forall w \in]u, v]$

Explains why in the continuous limit

$$Z_a = Z_b = \min_{c \in [a,b]} Z_c$$
$$\Rightarrow a \text{ and } b \text{ are identified}$$



Key points of the proof of the main theorem:

- Prove the converse (no other pair of points are identified)
- Obtain the formula for the limiting distance D^*

A property of distances in the Brownian map

Let ρ_* be the (unique) vertex of $\mathcal{T}_{\mathbf{e}}$ such that

$$Z_{\rho_*} = \min_{c \in \mathcal{T}_{\mathbf{e}}} Z_c$$

Then, for every $a \in \mathcal{T}_{\mathbf{e}}$,

$$D^*(\rho_*, a) = Z_a - \min Z.$$

("follows" from the analogous property in the discrete setting)

No such simple expression for $D^*(a, b)$ in terms of labels, but

$$D^*(a, b) \leq D^0(a, b) = Z_a + Z_b - 2 \max \Big(\min_{c \in [a,b]} Z_c, \min_{c \in [b,a]} Z_c \Big)$$

(also easy to interpret from the discrete setting)

D^* is the maximal metric that satisfies this inequality

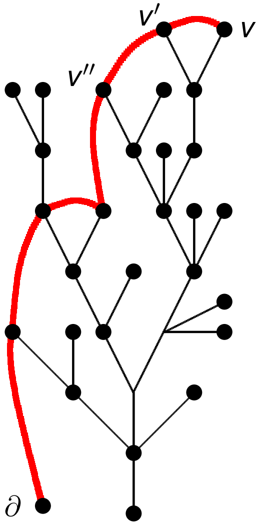
4. Geodesics in the Brownian map

Geodesics in quadrangulations

Use Schaeffer's bijection between quadrangulations and well-labeled trees.

To construct a geodesic from v to ∂:

- Look for the last visited vertex (before v) with label $\ell_v - 1$. Call it v'.
- Proceed in the same way from v' to get a vertex v''.
- And so on.
- Eventually one reaches the root ∂.



Simple geodesics in the Brownian map

Brownian map: $\mathbf{m}_\infty = \mathcal{T}_{\mathbf{e}}/\approx$

$\mathcal{T}_{\mathbf{e}}$ is re-rooted at ρ_* vertex with minimal label

$\prec$ lexicographical order on $\mathcal{T}_{\mathbf{e}}$

Recall $D^*(\rho_*, a) = \overline{Z}_a := Z_a - \min Z$.

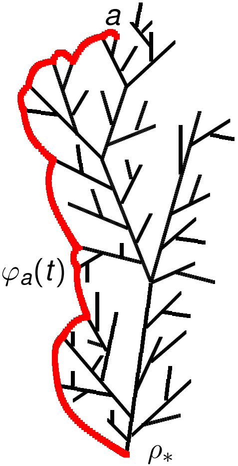
Fix $a \in \mathcal{T}_{\mathbf{e}}$ and for $t \in [0, \overline{Z}_a]$, set

$$\varphi_a(t) = \sup\{b \prec a : \overline{Z}_b = t\}$$

(same formula as in the discrete case !)

Then $(\varphi_a(t))_{0 \le t \le \overline{Z}_a}$ is a geodesic from ρ_* to a

(called a simple geodesic)



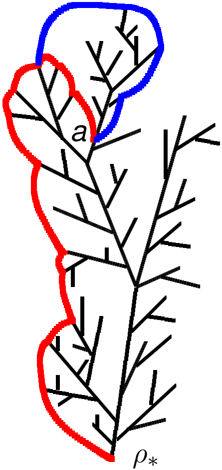
Fact

Simple geodesics visit only leaves of $\mathcal{T}_{\mathbf{e}}$ (except possibly at the endpoint)

How many simple geodesics from a given point ?

- If a is a leaf of $\mathcal{T}_{\mathbf{e}}$, there is a unique simple geodesic from ρ_* to a
- Otherwise, there are
  - 2 distinct simple geodesics if a is a simple point
  - 3 distinct simple geodesics if a is a branching point

(3 is the maximal multiplicity in $\mathcal{T}_{\mathbf{e}}$)



Proposition (key result)

All geodesics from the root are simple geodesics.

The main result about geodesics

Define the skeleton of $\mathcal{T}_{\mathbf{e}}$ by $\mathrm{Sk}(\mathcal{T}_{\mathbf{e}}) = \mathcal{T}_{\mathbf{e}} \backslash \{\text{leaves of } \mathcal{T}_{\mathbf{e}}\}$ and set

$$\mathrm{Skel} = \pi(\mathrm{Sk}(\mathcal{T}_{\mathbf{e}})) \qquad (\pi : \mathcal{T}_{\mathbf{e}} \to \mathcal{T}_{\mathbf{e}}/\approx = \mathbf{m}_\infty \text{ canonical projection})$$

Then

- the restriction of π to $\mathrm{Sk}(\mathcal{T}_{\mathbf{e}})$ is a homeomorphism onto Skel
- $\dim(\mathrm{Skel}) = 2$ (recall $\dim(\mathbf{m}_\infty) = 4$)

Theorem (Geodesics from the root)

Let $x \in \mathbf{m}_\infty$. Then,

- *if $x \notin \mathrm{Skel}$, there is a unique geodesic from ρ_* to x*
- *if $x \in \mathrm{Skel}$, the number of distinct geodesics from ρ_* to x is the multiplicity $m(x)$ of x in Skel (note: $m(x) \leq 3$).*

Remarks

- Skel is the cut-locus of $\mathbf{m}_\infty$ relative to ρ_*: cf classical Riemannian geometry [Poincaré, Myers, ...], where the cut-locus is a tree.
- same results if ρ_* replaced by a point chosen "at random" in $\mathbf{m}_\infty$.

Confluence property of geodesics

Fact: Two simple geodesics coincide near ρ_*.
(easy from the definition)

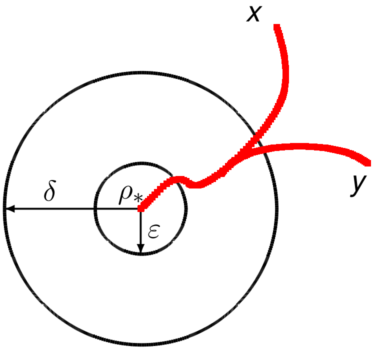
Corollary

Given $\delta > 0$, *there exists* $\varepsilon > 0$ *s.t.*

- *if* $D^*(\rho_*, x) \geq \delta$, $D^*(\rho_*, y) \geq \delta$
- *if* γ *is any geodesic from* ρ_* *to* x
- *if* γ' *is any geodesic from* ρ_* *to* y

then

$$\gamma(t) = \gamma'(t) \quad \text{for all } t \leq \varepsilon$$



"Only one way" of leaving ρ_* along a geodesic.
(also true if ρ_* is replaced by a typical point of $\mathbf{m}_\infty$)

Uniqueness of geodesics in discrete maps

M_n uniform distributed over $\mathbb{M}_n^{2p} = \{2p - \text{angulations with } n \text{ faces}\}$
$V(M_n)$ set of vertices of M_n, ∂ root vertex of M_n, $d_{\rm gr}$ graph distance

For $v \in V(M_n)$, $\mathrm{Geo}(\partial \to v) = \{\text{geodesics from } \partial \text{ to } v\}$

If γ, γ' are two discrete paths (with the same length)

$$d(\gamma, \gamma') = \max_i d_{\rm gr}(\gamma(i), \gamma'(i))$$

Corollary

Let $\delta > 0$. Then,

$$\frac{1}{n} \#\{v \in V(M_n) : \exists \gamma, \gamma' \in \mathrm{Geo}(\partial \to v),\ d(\gamma, \gamma') \geq \delta n^{1/4}\} \underset{n\to\infty}{\longrightarrow} 0$$

Macroscopic uniqueness of geodesics, also true for
"approximate geodesics"= paths with length $d_{\rm gr}(\partial, v) + o(n^{1/4})$

5. Canonical embeddings: Open problems

Recall that a planar map is defined up to (orientation-preserving) homeomorphisms of the sphere.

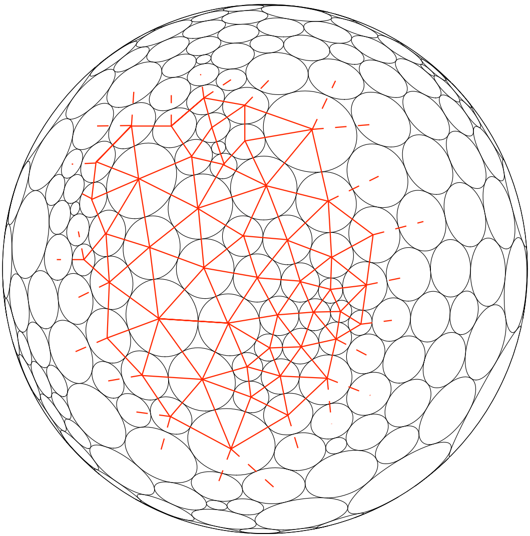
It is possible to choose a particular (canonical) embedding of the graph satisfying conformal invariance properties, and this choice is unique (at least up to the Möbius transformations, which are the conformal transformations of the sphere $\mathbb{S}^2$).

Question

Applying this canonical embedding to M_n (uniform over p-angulations with n faces), can one let n tend to infinity and get a random metric Δ on the sphere $\mathbb{S}^2$ satisfying conformal invariance properties, and such that

$$(\mathbb{S}^2, \Delta) \stackrel{\rm (d)}{=} (\mathbf{m}_\infty, D^*)$$

Canonical embeddings via circle packings 1



From a circle packing, construct a graph M :

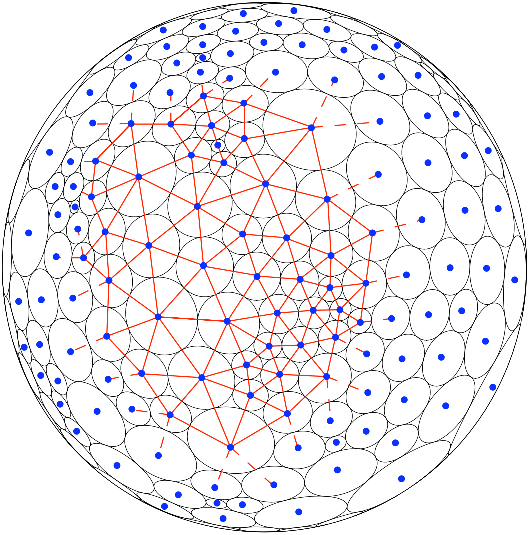
- $V(M) = \{\text{centers of circles}\}$
- edge between a and b if the corresponding circles are tangent.

A triangulation (without loops or multiple edges) can always be represented in this way.

Representation unique up to Möbius transformations.

Figure by Nicolas Curien

Canonical embeddings via circle packings 2



Apply to M_n uniform over {triangulations with n faces}. Let $n \to \infty$. Expect to get

- Random metric Δ on $\mathbb{S}^2$ (with conformal invariance properties) such that $(\mathbb{S}^2, \Delta) = (\mathbf{m}_\infty, D^*)$
- Random volume measure on $\mathbb{S}^2$

Connections with the Gaussian free field and Liouville quantum gravity ? (cf Duplantier-Sheffield).

Figure by Nicolas Curien

A few references

BENJAMINI: Random planar metrics. Proc. ICM 2010

BOUTTIER, DI FRANCESCO, GUITTER: Planar maps as labeled mobiles. *Electr. J. Combinatorics* (2004)

DUPLANTIER, SHEFFIELD: Liouville quantum gravity and KPZ. *Invent. Math.* (2011)

LE GALL: The topological structure of scaling limits of large planar maps. *Invent. Math.* (2007)

LE GALL: Geodesics in large planar maps and in the Brownian map. *Acta Math.* (2010)

LE GALL: Uniqueness and universality of the Brownian map. Preprint.

LE GALL, PAULIN: Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere. *GAFA* (2008)

MARCKERT, MOKKADEM: Limit of normalized quadrangulations: The Brownian map. *Ann. Probab.* (2006)

MIERMONT: The Brownian map is the scaling limit of uniform random plane quadrangulations. Preprint.

SCHRAMM: Conformally invariant scaling limits. Proc. ICM 2006.