Malliavin calculus and central limit theorems

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Outline

1. A brief introduction to Malliavin calculus
2. Stochastic integral representations
3. Central and noncentral limit theorems
4. Central limit theorem for the self-intersection local time of the fractional Brownian motion

Multiple stochastic integrals

- H is a separable Hilbert space.
- $\mathcal{H}_1 = \{X(h), h \in H\}$ is a Gaussian family of random variables in $(\Omega, \mathcal{F}, P)$ with zero mean and covariance

$$E(X(h)X(g)) = \langle h, g \rangle_H.$$

- For $q \geq 2$ we define the qth *Wiener chaos* as

$$\mathcal{H}_q = \overline{\text{Span}}\{h_q(X(g)), g \in H, \|g\|_H = 1\},$$

  where $h_q(x)$ is the qth Hermite polynomial.
- *Multiple stochastic integral* of order q:

$$I_q : \left(H^{\hat{\otimes} q}, \sqrt{q!}\|\cdot\|_{H^{\otimes q}}\right) \to \mathcal{H}_q$$

  is a linear isometry defined by $I_q(g^{\otimes q}) = h_q(X(g))$.

Wiener chaos expansion

Assume $\mathcal{F}$ is generated by $\mathcal{H}_1$. We have the orthogonal decomposition

$$L^2(\Omega) = \bigoplus_{q=0}^{\infty} \mathcal{H}_q,$$

where $\mathcal{H}_0 = \mathbb{R}$. Any $F \in L^2(\Omega)$ can be written as

$$F = E(F) + \sum_{q=1}^{\infty} I_q(f_q),$$

where $f_q \in H^{\hat{\otimes} q}$ are determined by F.

Example: Let $B = \{B_t, t \in [0,T]\}$ be a Brownian motion. Then, $H = L^2([0,T])$ and $X(h) = \int_0^T h_t dB_t$. For any $q \geq 2$, $H^{\hat{\otimes} q} = L^2_{sym}([0,T]^q)$ and I_q is the iterated Itô stochastic integral:

$$I_q(h) = q! \int_0^T \cdots \int_0^{t_2} h(t_1, \dots, t_q) dB_{t_1} \dots dB_{t_q}.$$

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Malliavin Calculus

- $\mathcal{S}$ is the space of random variables of the form

$$F = f(X(h_1), ..., X(h_n)),$$

where $h_i \in H$ and $f \in C_b^\infty(\mathbb{R}^n)$.

- If $F \in \mathcal{S}$ we define its *derivative* by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X(h_1), ..., X(h_n))h_i.$$

DF is a random variable with values in H.

- For any $p \geq 1$, $\mathbb{D}^{1,p} \subset L^p(\Omega; H)$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|DF\|_{1,p} = \left(E(|F|^p) + E(\|DF\|_H^p)\right)^{1/p}.$$

- The adjoint of D is the *divergence* operator δ defined by the duality relationship

$$E(\langle DF, u \rangle_H) = E(F\delta(u))$$

for any $F \in \mathbb{D}^{1,q}$ and $u \in \mathrm{Dom}\,\delta \subset L^p(\Omega; H)$, with $\frac{1}{p} + \frac{1}{q} = 1$.

- In the Brownian motion case, an adapted process $u \in L^2(\Omega \times [0,T]) \sim L^2(\Omega; H)$ belongs to $\mathrm{Dom}\,\delta$ and $\delta(u)$ coincides with the Itô's stochastic integral:

$$\boxed{\delta(u) = \int_0^T u_t dB_t}$$

- If u not adapted $\delta(u)$ coincides with an anticipating stochastic integral introduced by Skorohod in 1975. A stochastic calculus can be developed for the Skorohod integral (N.-Pardoux 1988).

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Meyer inequalities

Theorem (Meyer 1983)

For any $p > 1$ and $u \in \mathbb{D}^{1,p}(H)$ satisfying $E(u) = 0$,

$$\boxed{E(|\delta(u)|^p) \leq c_p \left(E(\|Du\|^p_{H \otimes H})\right)}.$$

1. A proof based on the boundedness in L^p of the Riesz transform was given by Pisier.

The Ornstein-Uhlenbeck semigroup

- $\{T_t, t \geq 0\}$ is a one-parameter semigroup of contraction operators on $L^2(\Omega)$, defined by

$$T_t(F) = \sum_{q=0}^{\infty} e^{-qt} I_q(f_q),$$

where $F = \sum_{q=0}^{\infty} I_q(f_q)$.

- *Mehler's formula*: For any $t \geq 0$ and $F \in L^2(\Omega)$ we have

$$T_t(F) = \tilde{E}(F(e^{-t}X + \sqrt{1-e^{-2t}}\tilde{X})),$$

where $\tilde{X}$ is an independent copy of X and $\tilde{E}$ denotes the mathematical expectation with respect to $\tilde{X}$.

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The generator of the Ornstein-Uhlenbeck semigroup

- The infinitesimal generator of the semigroup T_t in $L^2(\Omega)$ is given by

$$LF = \lim_{t \downarrow 0} \frac{T_t F - F}{t} = \sum_{q=1}^{\infty} -q I_q(f_q),$$

  if $F = \sum_{q=0}^{\infty} I_q(f_q)$.
- The domain of L is

$$\mathrm{Dom}\, L = \{F \in L^2(\Omega), \sum_{q=1}^{\infty} q^2 q! \|f_q\|_2^2 < \infty\} = \mathbb{D}^{2,2}.$$

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Stochastic integral representations

Question: Given $F \in L^2(\Omega)$, with $E[F] = 0$, find $u \in \mathrm{Dom}\,\delta$ such that

$$\boxed{F = \delta(u)}.$$

(I) Brownian motion case: If we require u to be *adapted*, then it is unique and given by the *Clark-Ocone* formula:

$$u_t = E(D_t F | \mathcal{F}_t),$$

where $\mathcal{F}_t$ is the filtration generated by the Brownian motion. That is,

$$F = E[F] + \int_0^T E(D_t F | \mathcal{F}_t) dB_t.$$

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CLT for local time increments

- Let

$$L_t^x = \int_0^t \delta_x(B_s)ds$$

be the Brownian local time. Clark-Ocone formula has been used to give a simple proof of the following central limit theorem (Hu-N. '09):

Theorem

$$h^{-\frac{3}{2}} \left(\int_{\mathbb{R}} (L_t^{x+h} - L_t^x)^2 dx - 4th \right) \Rightarrow 8\sqrt{\frac{\alpha_t}{3}}\eta,$$

as $h \to 0$, where $\alpha_t = \int_{\mathbb{R}} (L_t^x)^2 dx$ and η is a $N(0,1)$ random variable independent of B.

(II) Second integral representation:

$$\boxed{u = -DL^{-1}F}.$$

Proof: The proof is based on the following property:

Proposition

$F \in \mathbb{D}^{1,2}$ and $DF \in \mathrm{Dom}\delta$ if and only if $F \in \mathrm{Dom}L$ and in this case

$$\delta(DF) = -LF.$$

Therefore, taking into account that $E[F] = 0$, we get

$$F = LL^{-1}F = -\delta(DL^{-1}F).$$

- In the Brownian motion case, u is unique among all processes with a chaos expansion $u_t = \sum_{q=0}^{\infty} I_q(f_{q,t})$, such that $f_{q,t}(t_1, \ldots, t_q)$ is symmetric in all $q+1$ variables $t, t_1, \ldots, t_q$.

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Convergence to a mixture of normal laws

- Let $F = \delta(u)$. Then

$$\boxed{\left.\begin{array}{l} \langle DF, u\rangle_H \sim S^2 \\ \langle u, h\rangle_H \sim 0,\ \forall h \in H \end{array}\right\} \quad \Longrightarrow \quad \mathcal{L}(F) \sim \mathcal{L}(S\eta)},$$

where η is a $N(0,1)$ random variable independent of X.

Theorem (Nourdin-N. '10)

Let $F_n = \delta(u_n)$, where $u_n \in \mathbb{D}^{2,2}(H)$. Suppose that $\sup_n E(|F_n|) < \infty$ and

(i) $\langle u_n, DF_n\rangle_H \xrightarrow{L^1} S^2$, *as* $n \to \infty$.

(ii) *For all* $h \in H$, $\langle u_n, h\rangle_H \xrightarrow{L^1} 0$ *as* $n \to \infty$.

Then, F_n *converges stably to* ηS, *where* η *is a* $N(0,1)$ *random variable independent of* X *(that is,* $(F_n, X) \xrightarrow{\mathcal{L}} (S\eta, X)$*).*

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Sketch of the proof:

- Suppose that $(F_n, X) \stackrel{\mathcal{L}}{\to} (F_\infty, X)$. We need to show that

$$E\left(e^{i\lambda F_\infty}|X\right) = e^{-\frac{\lambda^2}{2}S^2}. \tag{1}$$

- Set $Y \in \mathcal{S}$ and define

$$\phi_n(\lambda) = E(e^{i\lambda F_n} Y).$$

- We compute the limit of $\phi_n'(\lambda)$ in two ways:
  1. Using weak convergence:

$$\phi_n'(\lambda) = iE(e^{i\lambda F_n} F_n Y) \to iE(e^{i\lambda F_\infty} F_\infty Y).$$

  2. Using Malliavin calculus and our assumptions:

$$\begin{aligned}\phi_n'(\lambda) &= iE(e^{i\lambda F_n} F_n Y) = iE(e^{i\lambda F_n}\delta(u_n)Y) \\ &= iE\left(\left\langle D\left(e^{i\lambda F_n}Y\right), u_n\right\rangle_H\right) \\ &= -\lambda E\left(e^{i\lambda F_n}\langle u_n, DF_n\rangle_H Y\right) + iE\left(e^{i\lambda F_n}\langle u_n, DY\rangle_H\right) \\ &\to -\lambda E(e^{i\lambda F_\infty} S^2 Y).\end{aligned}$$

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- As a consequence,

$$iE(e^{i\lambda F_\infty} F_\infty Y) = -\lambda E(e^{i\lambda F_\infty} S^2 Y).$$

- This leads to a linear differential equation satisfied by the conditional characteristic function of F_∞:

$$\frac{\partial}{\partial \lambda} E(e^{i\lambda F_\infty} | X) = -S^2 \lambda E(e^{i\lambda F_\infty} | X),$$

  and we obtain

$$E(e^{i\lambda F_\infty} | X) = e^{-\frac{\lambda^2}{2} S^2}.$$

Remarks:

- It turns out that

$$\langle u_n, DF_n \rangle_H = \|u_n\|_H^2 + \langle u_n, \delta(Du_n) \rangle_H.$$

Therefore, a sufficient condition for (i) is:

(i') $\|u_n\|_H^2 \xrightarrow{L^1} S^2$ and $\langle u_n, \delta(Du_n) \rangle_H \xrightarrow{L^1} 0$.

- Comparison with the *Asymptotic Knight Theorem* for Brownian martingales (Revuz-Yor):

If $\{u_n, n \geq 1\}$ are square-integrable adapted processes, then, $F_n = \delta(u_n) = \int_0^T u_n(s) dB_s$ and the stable convergence of F_n to $N(0, S^2)$ is implied by the following conditions:

(A) $\int_0^t u_n(s) ds \xrightarrow{P} 0$, uniformly in t.

(B) $\int_0^T u_n(s)^2 ds \to S^2$ in $L^1(\Omega)$.

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Applications

- *Weighted Hermite variations of the fractional Brownian motion:* Assume $\frac{1}{2q} < H < 1 - \frac{1}{2q}$. Then,

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(B^H_{\frac{k-1}{n}}) h_q(n^H(B^H_{\frac{k}{n}} - B^H_{\frac{k-1}{n}})) \xrightarrow{\text{Stably}} \sigma_{H,q} \int_0^1 f(B^H_s) dW_s,$$

  where W is a Brownian motion independent of B^H (Nourdin, Réveillac, Tudor, N.).

- *Itô's formulas in law:* Noncentral limit theorem for symmetric integrals with respect to the fractional Brownian motion for critical values of the Hurst parameter (Burdzy, Swanson, Nourdin, Réveillac, Harnett, N., Binotto).

Rate of convergence:

Theorem (Nourdin-N.-Peccati '16)

Let $F = \delta(u)$, where $u \in \mathbb{D}^{2,2}(H)$. Let $S \geq 0$ be such that $S^2 \in \mathbb{D}^{1,2}$ and let η be a $N(0,1)$ random variable independent of X. Then for any $\varphi \in C_b^3$

$$\begin{aligned} |E[\varphi(F)] - E[\varphi(S\eta)]| &\leq \frac{1}{2}\|\varphi''\|_\infty E\left[|\langle u, DF\rangle_H - S^2|\right] \\ &+\frac{1}{3}\|\varphi'''\|_\infty E\left[|\langle u, DS^2\rangle_H|\right]. \end{aligned}$$

Proof: Use interpolation method:

$$E[\varphi(F)] - E[\varphi(S\eta)] = \int_0^1 g'(t)dt,$$

where $g(t) = E[\varphi(\sqrt{t}F + \sqrt{1-t}S\eta)]$.

Stein's method for normal approximation

- *Stein's lemma:*

$$Z \sim N(0,\sigma^2) \quad \Leftrightarrow \quad E(f(Z)Z) = \sigma^2 E(f'(Z)) \quad \forall f \in C_b^1(\mathbb{R}).$$

- Let $Z \sim N(0,\sigma^2)$, and fix h such that $E(|h(Z)|) < \infty$. Stein's equation

$$\boxed{\sigma^2 f'(x) - xf(x) = h(x) - E(h(Z))}$$

has a unique solution f_h satisfying $\lim_{x\to\pm\infty} e^{-x^2/2\sigma^2} f_h(x) = 0$.

- If $\|h\|_\infty \le 1$, then $\|f_h\|_\infty \le \frac{1}{\sigma}\sqrt{\pi/2}$ and $\|f_h'\|_\infty \le \frac{2}{\sigma^2}$. So, for any random variable F, taking $h = \mathbf{1}_B$,

$$\begin{aligned} d_{TV}(F,Z) &= \sup_{B\in\mathcal{B}(\mathbb{R})} |P(F\in B) - P(Z\in B)| \\ &\le \sup_{f\in\mathcal{C}_{TV}} |E[\sigma^2 f'(F) - Ff(F)]|, \end{aligned}$$

where $\mathcal{C}_{TV}$ is the class of functions with $\|f\|_\infty \le \frac{1}{\sigma}\sqrt{\pi/2}$ and $\|f'\|_\infty \le \frac{2}{\sigma^2}$.

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Case $S^2 = \sigma^2$

This leads to the following result:

Theorem

Suppose that $F \in \mathbb{D}^{1,2}$ satisfies $F = \delta(u)$, where $u \in \mathrm{Dom}\,\delta$. Let Z be a $N(0, \sigma^2)$ random variable. Then,

$$\boxed{d_{TV}(F, Z) \leq \frac{2}{\sigma^2} E[|\sigma^2 - \langle DF, u \rangle_H|].}$$

Proof:

$$\begin{aligned} E[\sigma^2 f'(F) - Ff(F)] &= E[\sigma^2 f'(F) - \delta(u) f(F)] \\ &= E[\sigma^2 f'(F) - \langle u, D[f(F)] \rangle_H] \\ &= E[f'(F)(\sigma^2 - \langle u, DF \rangle_H)]. \end{aligned}$$

- In particular, taking $u = -DL^{-1}F$, we get

$$d_{TV}(F, Z) \leq \frac{2}{\sigma^2} E[|\sigma^2 - \langle DF, -DL^{-1}F \rangle_H|].$$

Case $S^2 = \sigma^2$

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Normal approximation on a fixed Wiener chaos

Proposition

Suppose $F \in \mathcal{H}_q$ *for some* $q \geq 2$ *and* $E(F^2) = \sigma^2$. *Then,*

$$\boxed{d_{TV}(F, Z) \leq \frac{2}{q\sigma^2}\sqrt{\mathrm{Var}\left(\|DF\|_H^2\right)}}.$$

Proof: Using $L^{-1}F = -\frac{1}{q}F$ and $E[\|DF\|_H^2] = q\sigma^2$, we obtain

$$\begin{aligned} E[|\sigma^2 - \langle DF, -DL^{-1}F\rangle_H|] &= E\left[\left|\sigma^2 - \frac{1}{q}\|DF\|_H^2\right|\right] \\ &\leq \frac{1}{q}\sqrt{\mathrm{Var}\left(\|DF\|_H^2\right)}. \end{aligned}$$

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- Using the product formula for multiple stochastic integrals, one can show that $\mathrm{Var}\left(\|DF\|_H^2\right)$ is equivalent up to a constant to $E[F^4] - 3\sigma^4$. This leads to the Fourth Moment Theorem (Nualart-Peccati-'05, Nourdin-Peccati'08):

Theorem

Fix $q \geq 2$. Let $F_n \in \mathcal{H}_q$, $n \geq 1$ be such that

$$\lim_{n\to\infty} E(F_n^2) = \sigma^2.$$

The following conditions are equivalent:

(i) $d_{TV}(F_n, Z) \to 0$, *as* $n \to \infty$ *where* $Z \sim N(0, \sigma^2)$.

(ii) $E(F_n^4) \to 3\sigma^4$, *as* $n \to \infty$.

(iii) $\|DF_n\|_H^2 \to q\sigma^2$ *in* $L^2(\Omega)$, *as* $n \to \infty$.

Applications and extensions

- Book: Nourdin and G. Peccati '12: *Normal Approximations with Malliavin Calculus : From Stein's Method to Universality*.
- Webpage: https ://sites.google.com/site/malliavinstein/home.
- Applications: Exact Berry Esséen asymptotics, quantitative Breuer-Major theorems, ...
- Generalizations: Functionals of the Poisson processes, convergence to nongaussian distributions (Gamma, second chaos, invariant measures of diffusions,...).

Self-intersection local time of the fBm

- The d-dimensional fractional Brownian motion ($d \geq 2$) with Hurst parameter $H \in (0,1)$ is a zero mean Gaussian process $\{B_t^H, t \geq 0\}$ with covariance
$$E[B_t^{H,i} B_s^{H,j}] = \delta_{ij} \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$
- Its *self-intersection local time* on $[0, T]$ is formally defined by
$$L_T = \int_0^T \int_0^t \delta_0(B_t^H - B_s^H) ds dt.$$

Notice that
$$\begin{aligned} E[L_T] &= \int_0^T \int_0^t E\left[\delta_0(B_t^H - B_s^H)\right] ds dt \\ &= (2\pi)^{-\frac{d}{2}} \int_0^T \int_0^t |t-s|^{-Hd} ds dt < \infty \Leftrightarrow Hd < 1. \end{aligned}$$

Varadhan's renormalization

Let $p_\epsilon(x) = (2\pi\epsilon)^{-d/2} e^{-|x|^2/2\epsilon}$, and set

$$L_{T,\epsilon} = \int_0^T \int_0^t p_\epsilon(B_t^H - B_s^H) ds dt.$$

The following results were proved in [Hu-Nualart '05]:

(i) If $Hd < 1$, then $L_{T,\epsilon} \stackrel{L^2}{\to} L_T$, as $\epsilon \downarrow 0$.

(ii) If $\frac{1}{d} \leq H < \frac{3}{2d}$, then $L_{T,\epsilon} - E[L_{T,\epsilon}] \stackrel{L^2}{\to} \widetilde{L}_T$, as $\epsilon \downarrow 0$.

(iii) If $\frac{3}{2d} < H < \frac{3}{4}$, then we have the convergence in law:

$$\boxed{\epsilon^{\frac{d}{2} - \frac{3}{4H}} \left[L_{T,\epsilon} - E[L_{T,\epsilon}] \right] \stackrel{\mathcal{L}}{\to} N(0, T\sigma^2_{H,d})} \tag{2}$$

as $\epsilon \downarrow 0$. (*Example*: $H = \frac{1}{2}$ and $d \geq 3$)

- The proof of (2) is based on the chaos expansion

$$\Phi_{T,\epsilon} := \epsilon^{\frac{d}{2}-\frac{3}{4H}} \left[L_{T,\epsilon} - E[L_{T,\epsilon}] \right] = \sum_{m=2}^{\infty} J_m(L_{T,\epsilon})$$

and the application of the Fourth Moment Theorem to each projection $J_m(L_{T,\epsilon})$, $m \geq 2$, as $\epsilon \downarrow 0$.

- A first ingredient in the proof is the convergence of the variance:

$$E\left[\Phi_{T,\epsilon}^2\right] \stackrel{\epsilon\downarrow 0}{\to} I_T,$$

where

$$I_T = (2\pi)^{-d} \int_{\substack{0<s<t<T \\ 0<s'<t'<T}} \left[(\lambda\rho - \mu^2)^{-\frac{d}{2}} - (\lambda\rho)^{-\frac{d}{2}} \right] dsdtds'dt',$$

with the notation $\lambda = |t-s|^{2H}$, $\rho = |t'-s'|^{2H}$ and $\mu = E[(B_t^{H,1} - B_s^{H,1})(B_{t'}^{H,1} - B_{s'}^{H,1})]$.

Functional CLT

Theorem (Jaramillo-N. '17)

If $\frac{3}{2d} < H < \frac{3}{4}$*, then*

$$\{\epsilon^{\frac{d}{2}-\frac{3}{4H}}\left[L_{T,\epsilon} - E[L_{T,\epsilon}]\right], T \geq 0\} \xrightarrow{\mathcal{L}} \{\sigma_{H,d} W_T, T \geq 0\},$$

where W *is a standard Brownian motion.*

- The proof of the convergence of the finite dimensional distributions can be done by the same method as the CLT for T fixed.
- The main difficulty is to show the tightness property of the laws. For this we need an estimate of the form

$$E\left[|\Phi_{T,\epsilon} - \Phi_{S,\epsilon}|^p\right] \leq C_{p,d,H}|T - S|^{p/2}, \tag{3}$$

for some $p > 2$.

- We know that this is true for $p = 2$, and in this case, we need to estimate a double integral over essentially three types of regions:
  1. $[s', t'] \subset [s, t]$
  2. $[s', t'] \cap [s, t] = \emptyset$
  3. $s < s' < t < t'$ (the intervals overlap)
- However, for $p = 4$ we have to deal with 4 intervals $[s_i, t_i]$, $i = 1, 2, 3, 4$, the number of different regions is very large and each integral is too complicated.
- Question: How to show the estimate (2)?

Proving tightness using Malliavin calculus

- Fix $S \le T$ and define $Z_\epsilon = \Phi_{T,\epsilon} - \Phi_{S,\epsilon}$.
- We can write
$$Z_\epsilon = -\delta D L^{-1} Z_\epsilon.$$
Using that $E[DL^{-1}Z_\epsilon] = 0$ and Meyer inequalities, yields
$$\|Z_\epsilon\|_p \le c_p \|D^2 L^{-1} Z_\epsilon\|_{L^p(\Omega;(\mathfrak{H}^d)^{\otimes 2})},$$
where $\mathfrak{H}$ is the Hilbert space associated with the covariance of the fBm.
- We know that
$$Z_\epsilon = \epsilon^{\frac{d}{2} - \frac{3}{4H}} \int_{\substack{s<t \\ S<t<T}} \left(p_\epsilon(B_t^H - B_s^H) - E[p_\epsilon(B_t^H - B_s^H)] \right) dsdt.$$
- Also $L^{-1}(F - E[F]) = -\int_0^\infty T_\theta F d\theta$, where $\{T_\theta, \theta \ge 0\}$ is the Ornstein-Uhlenbeck semigroup.

- As a consequence,

$$
\begin{aligned}
D^2 L^{-1} Z_\epsilon &= \int_0^\infty D^2 T_\theta Z_\epsilon d\theta \\
&= \int_0^\infty \int_{\substack{s<t \\ S<t<T}} D^2 T_\theta [p_\epsilon(B_t^H - B_s^H)] dsdtd\theta \\
&= \int_0^\infty \int_{\substack{s<t \\ S<t<T}} D^2 \tilde{E}[p_\epsilon(e^{-\theta}(B_t^H - B_s^H) + \sqrt{1-e^{-2\theta}}(\tilde{B}_t^H - \tilde{B}_s^H))] dsdtd\theta \\
&= \int_0^\infty \int_{\substack{s<t \\ S<t<T}} D^2 p_{\epsilon+(1-e^{-2\theta})(t-s)^{2H}}(e^{-\theta}(B_t^H - B_s^H)) dsdtd\theta \\
&= \int_0^\infty \int_{\substack{s<t \\ S<t<T}} e^{-2\theta} p''_{\epsilon+(1-e^{-2\theta})(t-s)^{2H}}(e^{-\theta}(B_t^H - B_s^H)) \mathbf{1}_{[s,t]}^{\otimes 2} dsdtd\theta.
\end{aligned}
$$

- Finally, using Minkowski inequality,

$$\begin{aligned} \|Z_\epsilon\|_p^2 &\le c_p^2 \|D^2 L^{-1} Z_\epsilon\|^2_{L^p(\Omega;(\mathfrak{H}^d)^{\otimes 2})} = c_p^2 \| \| DL^{-1} Z_\epsilon\|^2_{(\mathfrak{H}^d)^{\otimes 2})} \|_{p/2} \\ &\le c_p^2 \int_{\mathbb{R}_+^2} \int_{\substack{s<t \\ S<t<T}} \int_{\substack{s'<t' \\ S<t'<T}} e^{-2\theta-2\beta} \mu^2 \\ &\quad \times \|p''_{\epsilon+(1-e^{-2\theta})}(e^{-\theta}(B^H_{t'} - B^H_{s'}))\|_p \\ &\quad \times \|p''_{\epsilon+(1-e^{-2\beta})}(e^{-\beta}(B^H_{t} - B^H_{s}))\|_p dsdtd'sdt'd\theta d\beta. \end{aligned}$$

- This leads to

$$\|Z_\epsilon\|_p^2 \le C_{p,d,H} \mathcal{I} |T-S|^p,$$

where

$$\mathcal{I} = \int_{\substack{0<s<t<T \\ 0<s'<t'<T}} \frac{\mu^2}{\lambda\rho} ((1+\lambda)(1+\rho)-\mu^2)^{-\frac{d}{p}} dsdtds'dt' < \infty,$$

provided $2 < p < \frac{4Hd}{3}$.

Remarks:

- Critical case $H = \frac{3d}{2}$: A logarithmic factor is needed for the central limit theorem to hold true, but the functional version is open.
- Case $H > \frac{3}{4}$: In this case,

$$\epsilon^{-\frac{d}{2}-\frac{3}{2H}+1}(L_{T,\epsilon} - E[L_{T,\epsilon}]) \xrightarrow{L^2} c_{d,H} \sum_{j=1}^{d} X_T^j,$$

  where X_T^j is a Rosenblatt-type random variable, defined as the double stochastic integral of $\delta_{\{s=t\}}$ with respect to $B^{H,j}$ on $[0,T]^2$.
- The case $H = \frac{3}{4}$ is open.

Thanks for your attention !