Spatial Epidemics: Critical Behavior in Dimensions 1, 2, and 3

Steve Lalley[1] and Xinghua Zheng[2]

[1]University of Chicago

[2]University of Hong Kong

March 12, 2010

Reed-Frost (SIR) Model

- Population Size $N < \infty$
- Individuals susceptible (S), infected (I), or recovered (R).
- Recovered individuals immune from further infection.
- Infecteds recover in time 1.
- Infecteds infect susceptibles with probability p.

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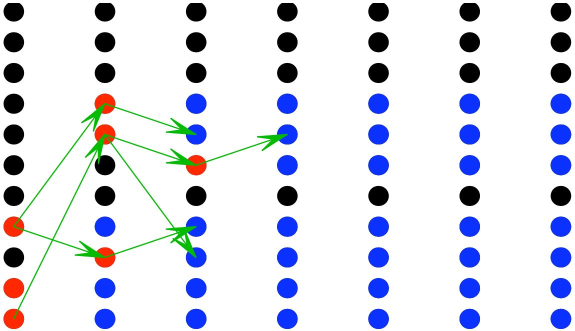
Critical Case: $p = 1/N$

SIR Model:Example



Reed-Frost and Random Graphs

Reed-Frost model is equivalent to the Erdös-Renyi random graph model:

$$
\begin{aligned}
\text{Individuals} &\longleftrightarrow \text{Vertices} \\
\text{Infections} &\longleftrightarrow \text{Edges} \\
\text{Epidemic} &\longleftrightarrow \text{Connected Components}
\end{aligned}
$$

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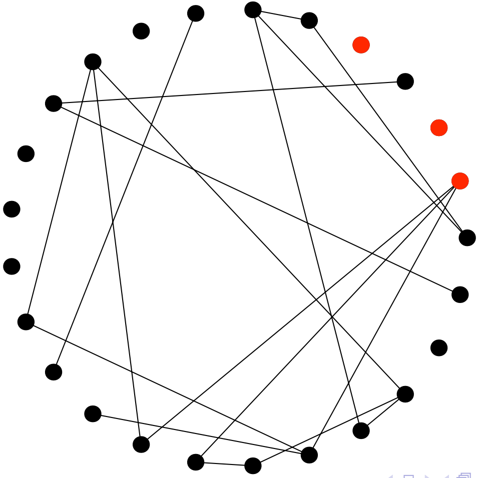

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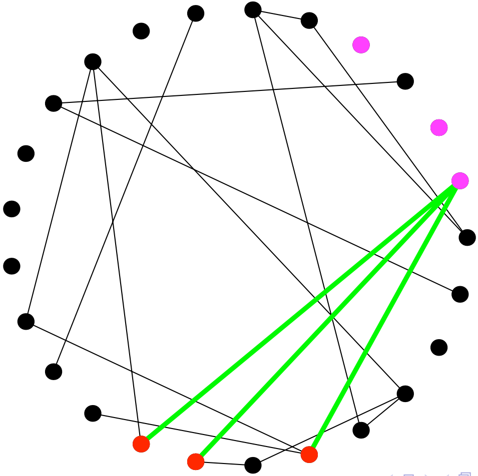

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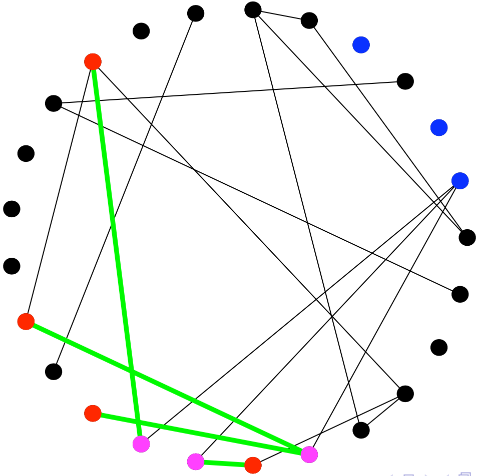

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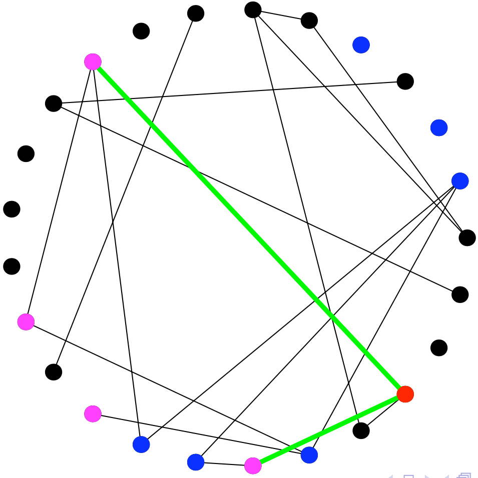

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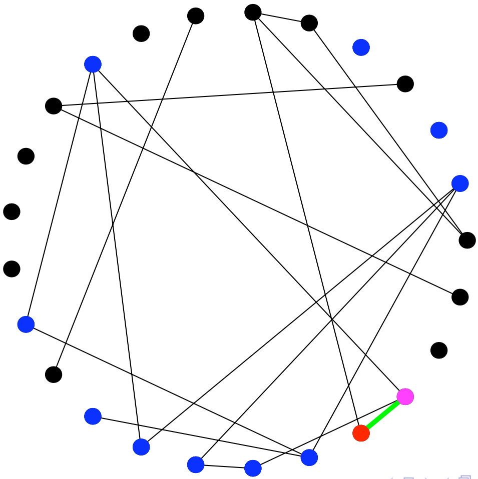

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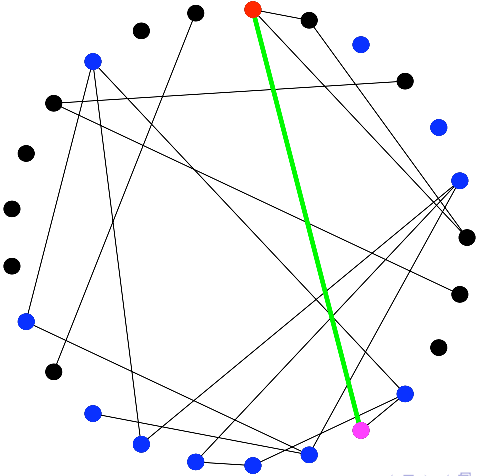

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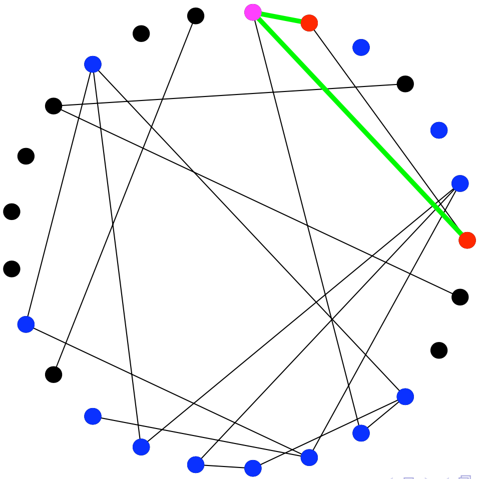

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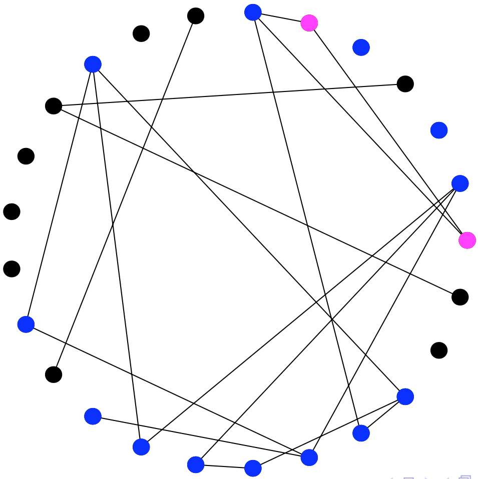
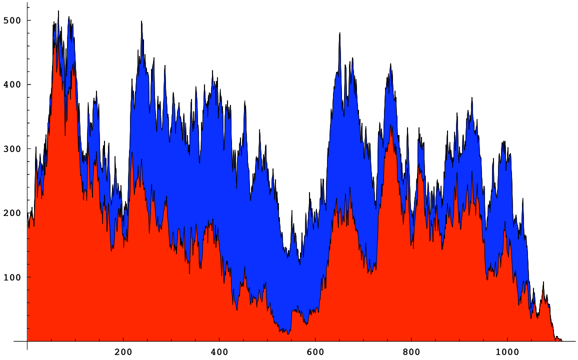


Branching Envelope of an Epidemic

- Each epidemic has a branching envelope (GW process)
- Offspring distribution: Binomial-(N, p)
- Epidemic is dominated by its branching envelope
- When $I_t \ll S_t$, infected set grows $\approx$ branching envelope

Example: SIS Epidemic and its Branching Envelope



$$N = 80000$$
$$I_0 = 200$$
$$p = 1/80000$$

Critical Behavior: Reed-Frost (SIR) Epidemics

- # Infected in Generation t:= I_t^N
- # Recovered in Generation t: $= R_t^N$
- Initial Condition: $I_0^N \sim bN^\alpha$

Theorem: (Dolgoarshinnykh &L.) As population size $N \to \infty$,

$$\begin{pmatrix} N^{-\alpha} I_t^N \\ N^{-2\alpha} R_t^N \end{pmatrix} \xrightarrow{\mathcal{D}} \begin{pmatrix} I(t) \\ R(t) \end{pmatrix}$$

The limit process satisfies $I(0) = b$ and

$$\begin{aligned} dR(t) &= I(t)\,dt \\ dI(t) &= +\sqrt{I(t)}\,dB_t && \text{if } \alpha < 1/3 \\ dI(t) &= +\sqrt{I(t)}\,dB_t - I(t)R(t)\,dt && \text{if } \alpha = 1/3 \end{aligned}$$

Critical Behavior: Reed-Frost (SIR) Epidemics

- Population size: $N \to \infty$
- # Infected in Generation t:= I_t^N
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- Initial Condition: $I_0^N \sim bN^\alpha$

Corollary: If $\alpha = 1/3$ then

$$R_\infty^N / N^{2/3} \Longrightarrow \tau(b)$$

where $\tau(b) =$ first passage time of $B(t) + t^2/2$ to b.

(Martin-Lof; Aldous)

Critical Behavior: Heuristics

- Critical Epidemic with $I_0 = m$ should last $\approx m$ generations.
- Number R_t recovered should be $\approx m^2$.
- Offspring in branching envelope :: attempted infections.
- Misfires: Infections of immunes not allowed.
- Critical Threshold: $\#$ misfires/generations $\approx O(1)$

Critical SIR Epidemic:

$$E(\#\text{misfires in generation } t+1) \approx I_t R_t / N$$

so there will be observable deviation from branching envelope when

$$I_t \approx N^{1/3} \quad \text{and} \quad R_t \approx N^{2/3}$$

Spatial SIR Epidemic:

- Villages V_x at Sites $x \in \mathbb{Z}^d$
- Village Size:=N
- Nearest Neighbor Disease Propagation
- SIR Rules Locally:
  - Infected individuals infect susceptibles at same or neighboring site with probability p_N
  - Infecteds recover in time 1.
  - Recovered individuals immune from further infection.

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Critical Case: Infection probability $p_N = 1/((2d+1)N)$.

Percolation Representation

Spatial SIR epidemic is equivalent to critical bond percolation on the graph $G_N := K_N \times \mathbb{Z}^d$ with nearest neighbor connections:

- Vertex set $[N] \times \mathbb{Z}^d$
- Edges connect vertices (i, x) and (j, y) if $\text{dist}(x, y) \leq 1$

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Problem: At critical point $p = 1/(2d + 1)N$,

- How does connectivity probability decay?
- How does size of largest connected cluster scale with N?
- Joint distribution of largest, 2nd largest, $\cdots$?

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Conjecture: Let $R_N =$ maximum x such that a vertex (j, x) is connected to a vertex $(i, 0)$. Then $R_N/N^{4/(6-d)}$ converges in distribution as $N \to \infty$ in dimensions $d = 1, 2, 3$.

Branching Envelope

The branching envelope of a spatial SIR epidemic is a branching random walk: In each generation,

- A particle at x puts offspring at x or neighbors $x + e$.
- #Offspring are independent Binomial$-(N, p_N)$ or Poisson-Np_N
- Critical BRW : $p = p_N = 1/((2d + 1)N)$.

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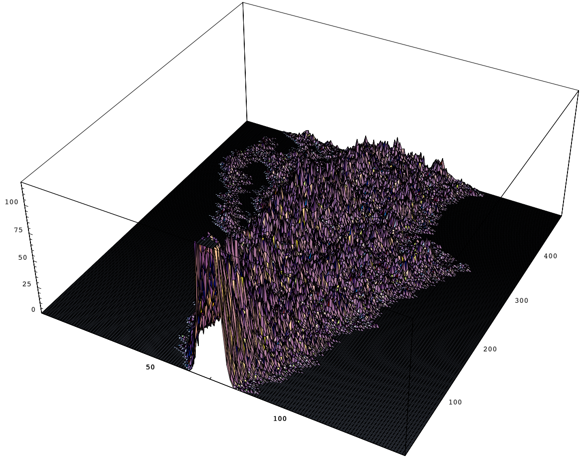
Associated Measure-Valued Processes

$X_t^M = X_t^{M,N}$: measure that puts mass $1/M$ at $x/\sqrt{M}$ for each particle at site x at time t.

Critical Spatial SIS Epidemic: Simulation



Watanabe's Theorem

Let X_t^M be the measure-valued process associated to a critical nearest neighbor branching random walk. If

$$X_0^M \Longrightarrow X_0$$

then

$$X_{Mt}^M \Longrightarrow X_t$$

where X_t is the Dawson-Watanabe process (superBM). The DW process is a measure-valued diffusion.

Note 1: The total mass $\|X_t\|$ is a Feller diffusion.
Note 2: Watanabe is the spatial analogue of Feller.
Note 3: In $1D$, X_t has a continuous density $X(t,x)$.

Scaling Limits: SIR Spatial Epidemics in $d = 1$

Recall: $X_t^{M,N}$ is the measure that puts mass $1/M$ at $x/\sqrt{M}$ for each infected individual at site x at time t.

Theorem: Assume that the epidemic is critical and that $M = N^\alpha$. If $X_0^{M,N} \Rightarrow X_0$ then

$$X_{Mt}^{M,N} \Longrightarrow X_t$$

where

- If $\alpha < 2/5$ then X_t is the Dawson-Watanabe process.
- If $\alpha = 2/5$ then X_t is the Dawson-Watanabe process with killing rate

$$\int_0^t X(s,x)\,ds$$

Scaling Limits: SIR Spatial Epidemics in $d = 2, 3$

Recall: $X_t^{M,N}$ is the measure that puts mass $1/M$ at $x/\sqrt{M}$ for each infected individual at site x at time t.

Theorem: Assume that the epidemic is critical and that $M = N^\alpha$. If $X_0^{M,N} \Rightarrow X_0$ and X_0 satisfies a smoothness condition then

$$X_{Mt}^{M,N} \Longrightarrow X_t$$

where

- If $\alpha < 2/(6-d)$ then X_t is the Dawson-Watanabe process.
- If $\alpha = 2/(6-d)$ then X_t is the Dawson-Watanabe process with killing rate $L(t,x) =$ Sugitani local time density.

Sugitani's Local Time

Theorem: Assume that $d = 2$ or 3 and that the initial configuration $X_0 = \mu$ of the super-BM X_t satisfies

Smoothness Condition:

$$\int_0^t \int_{x \in \mathbb{R}^d} \phi_t(x - y)\, d\mu(y)$$

is jointly continuous in t, x, where $\phi_t(x)$ is the heat kernel (Gaussian density). Then for each $t \geq 0$ the occupation measure

$$L_t := \int_0^t X_s\, ds$$

is absolutely continuous with jointly continuous density $L(t, x)$.

Critical Scaling: Heuristics (SIR Epidemics, $d = 1$)

- # Infected Per Generation: $\approx M$
- Duration: $\approx M$ generations.
- # Infected Per Site: $\approx \sqrt{M}$
- # Recovered Per Site: $\approx M\sqrt{M}$
- # Misfires Per Site: $\approx M^2/N$
- # Misfires Per Generation: $\approx M^{5/2}/N$

So if $M \approx N^{2/5}$ then # Misfires Per Generation ≈ 1.

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But how do we know that the infected individuals in generation n don't "clump"?

Critical Scaling: Heuristics (SIR Epidemics, $d = 3$)

- # Infected Per Generation ≈ Duration ≈ M
- # Sites Reachable ≈ $M^{3/2}$.
- # Infected Per Infected Site: ≈ $O(1)$
- # Recovered Per Site: ≈ $M^2/M^{3/2} = \sqrt{M}$
- # Misfires Per Generation: ≈ $M \times \sqrt{M}/N$

So if $M \approx N^{2/3}$ then # Misfires Per Generation ≈ 1.

Proof Strategy I

Lemma: Assume that L_n and L are likelihood ratios under P_n and P, and define Q_n and Q by

$$dQ_n = L_n \, dP_n,$$
$$dQ = L \, dP.$$

Assume that X_n and X are random variables whose distributions under P_n and P satisfy

$$(X_n, L_n) \Longrightarrow (X, L).$$

Then the Q_n-distribution of X_n converges weakly to the $Q-$distribution of X.

Proof Strategy II

Theorem: (Dawson) The law Q of the Dawson-Watanabe process with location-dependent killing rate $\theta(x, t)$ is mutually a.c. relative to the law P of the Dawson-Watanabe process with no killing (superBM), and the likelihood ratio is

$$dQ/dP = \exp\left\{-\int \theta(t,x)\,dM(t,x) - \frac{1}{2}\int \langle X_t, \theta(t,\cdot)^2\rangle\,dt\right\}$$

where M is the orthogonal martingale measure attached to the superBM X_t.

Proof Strategy III

P^M = Law of Mth branching random walk.
$Q^{M,N}$ = Law of corresponding spatial epidemic.

$$\frac{dQ^{M,N}}{dP^M} = \prod_{\text{times} t} \prod_{\text{sites} x} \left(1 + R^{M,N}(t,x)\right)$$

where $R^{M,N}(t,x)$ is a function of the number of misfires at site x at time t. So the problem is to show that under P^M, as $M \to \infty$,

$$\sum_t \sum_x R^{M,N}(t,x)$$

converges to the exponent in Dawson's likelihood ratio.

Local Behavior for Branching Random Walk: $d = 1$

$Y_n^k(\cdot) =$ branching random walk on $\mathbb{Z}$ with Poisson-1 offspring distribution and initial state Y_0^k scaling as in Watanabe's theorem.

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Theorem: If $Y_0^k([\sqrt{k}x]) \to Y_0(x)$ where $Y_0(x)$ is continuous with compact support then

$$\frac{Y_{kt}^k([\sqrt{k}x])}{\sqrt{k}} \Longrightarrow X(t,x)$$

where $X(t,x)$ is the Dawson-Watanabe density process.

Local Time for Branching Random Walk: $d = 2, 3$

$Y_n^k(\cdot) =$ branching random walk on $\mathbb{Z}^d$ with Poisson-1 offspring distribution and initial state Y_0^k scaling as in Watanabe's theorem.

$U_n^k(\cdot) = \sum_{i=0}^t Y_i^k(\cdot)$

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$U_n^k(\cdot) = \sum_{i=0}^t Y_i^k(\cdot)$

Theorem: If $Y_0^k([\sqrt{k}x]) \to Y_0(x)$ where Y_0 satisfies hypotheses of Sugitani then in $d = 2, 3$,

$$\frac{U_{kt}^k([\sqrt{k}x])}{k^{2-d/2}} \Longrightarrow L(t, x)$$

where $L(t, x)$ is Sugitani local time.

Occupation Statistics in $d = 2$

For branching random walk with Poisson-1 offspring distribution and nearest neighbor steps initiated by a single particle at $(0, 0)$, let

$$G_n = \{\text{process survives to generation } n\}$$
$$G_n^0 = \{\exists \text{ particle at origin in generation } n\}$$
$$\Omega_n = \#\{\text{occupied sites in generation } n\}$$

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Theorem: Conditional on G_n, the number Ω_n of occupied sites is $O_p(n/\log n)$, that is, the conditional distributions of $\Omega_n \log n/n$ are tight.

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Theorem: Conditional on G_n^0, the trajectory of a particle randomly chosen from those particles at the origin in generation n is not a pinned random walk.

Spatial Extent of Super-BM in $d = 1$

- X_t = Dawson-Watanabe process,
- $\mathcal{R} := \bigcup_{t \geq 0} \text{support}(X_t)$
- $u_D(x) := -\log P(\mathcal{R} \subset D \mid X_0 = \delta_x)$

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Theorem (Dynkin): For any finite interval D, $u_D(x)$ is the maximal nonnegative solution in D of the differential equation

$$u'' = u^2$$

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Solution: Weierstrass $\mathcal{P}-$ Function

$$u_D(x) = \mathcal{P}_L(x/\sqrt{6}) = \frac{1}{6x^2} + \sum_{\omega \in L^*} \left\{ \frac{1}{6(x-\omega)^2} - \frac{1}{6\omega^2} \right\}$$

where the period lattice L is generated by $Ce^{\pi i/3}$ for $C > 0$ depending on $D = [0, a]$ as follows:

$$C = \sqrt{6}a$$

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General Initial Conditions: For any finite Borel measure μ with support $\subset D$,

$$-\log P(\mathcal{R} \subset D \mid X_0 = \mu) = \int u_{D(x)} \, \mu(dx)$$
$$= \int \mathcal{P}_L(x/\sqrt{6}) \, \mu(dx)$$