Random tridiagonal matrices, β-ensembles and random Schrödinger operators

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- Generalized β-ensembles
  (GOE, GUE, Wishart...)

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Method: scaling limit of random tridiagonal matrices

Classical Random Matrix Ensembles

$x_{k,\ell}$: i.i.d. sequence of standard normals (real/complex/quaternion)

$X(n,m) = [x_{k,\ell}]_{1\le k\le n, 1\le \ell\le m}$ is an $n \times m$ random matrix

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Hermite ensemble: (GOE, GUE, GSE)	$\frac{1}{\sqrt{2}}(A + A^*)$	$A \sim X(n,n)$
Laguerre ensemble: (Wishart)	BB^*	$B \sim X(n,m)$

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Joint density of the eigenvalues:

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These are the general β-ensembles.

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tridiagonal matrix representation for the β ensembles

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- Eigenvalues of BB^T: Laguerre ensemble

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A random sample from the ensemble: $\Lambda_n = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$

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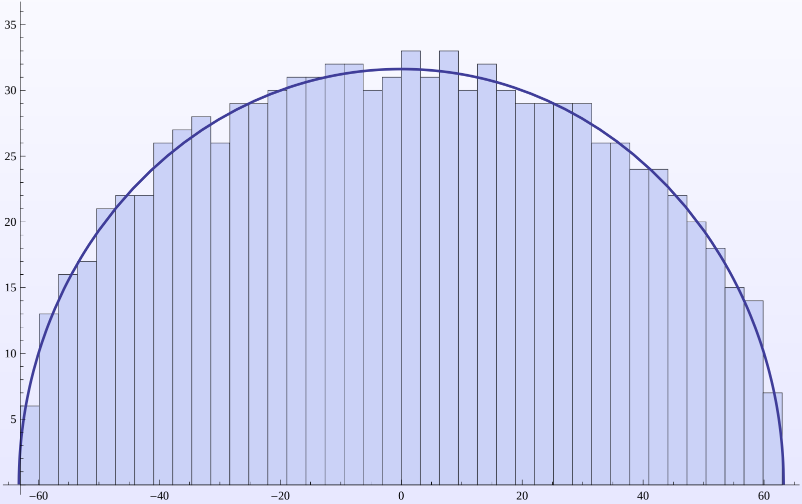
Point process limit of the spectrum:

$$b_n(\Lambda_n - a_n) \Rightarrow ?$$

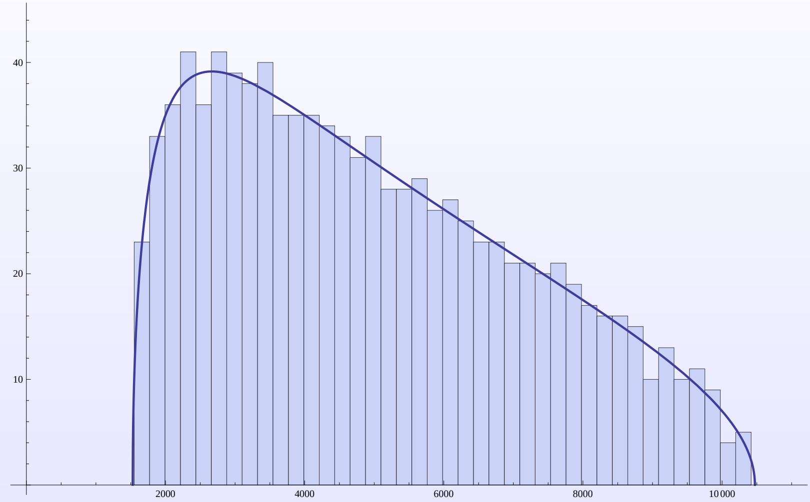
Limit of the empirical spectral measure

Histogram for a Hermite ensemble with $n = 1000$



Limit of the empirical spectral measure

Histogram for a Laguerre ensemble with $n = 1000, m = 5000$



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Thm: The properly normalized empirical spectral measure converges to a deterministic measure.

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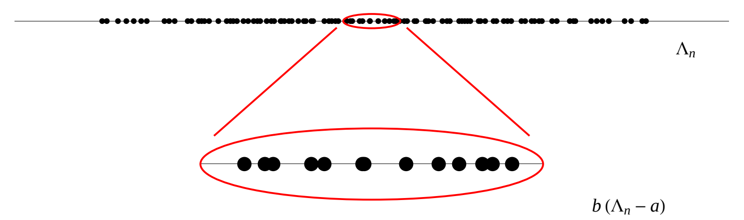
Laguerre: $n/m \to y \in (0,1]$, $\qquad a, b$ are functions of y.

Local limit

We want to 'zoom' in to see a point process in the limit.



Λ_n: the point process for n $\quad$ a_n, b_n: scaling factors

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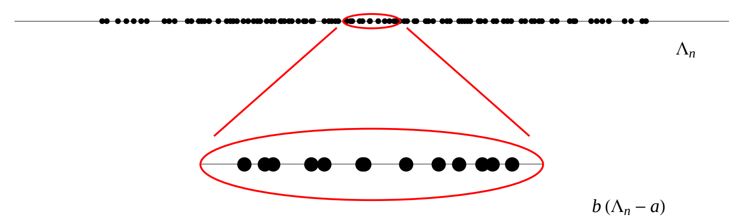

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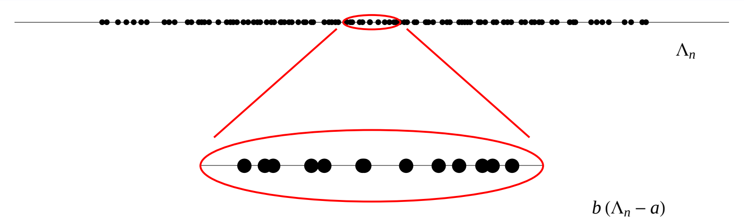

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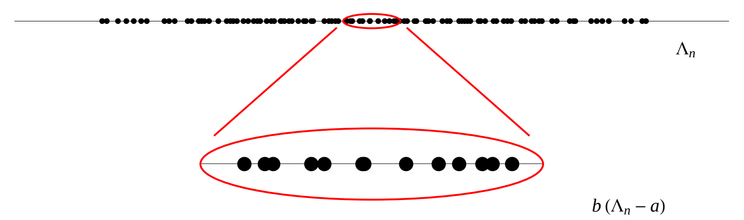

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Random Schrödinger operator

1D Schrödinger operator: $\mathcal{H} = \partial_x^2 + V$

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If we have a finite interval with Dirichlet boundary conditions$\rightsquigarrow$

$$M_n = \begin{bmatrix} V_1 & 1 & & & \\ 1 & V_2 & 1 & & \\ & \ddots & \ddots & & \\ & & 1 & V_{n-1} & 1 \\ & & & 1 & V_n \end{bmatrix}$$

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If there is no noise:

$$\text{spectrum} = \Lambda_n = \{2\cos(2\pi k/(n+1)) : k = 1 \ldots n\}$$

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Local limit (with noise): $\quad n(\Lambda_n - \mu) \Rightarrow ?$

Eigenvalue equations for tridiagonal matrices

$$M_n = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & \ddots & \ddots & & \\ & & b_{n-2} & a_{n-1} & b_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix}, \qquad M_n u = \lambda u$$

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(Discrete version of Sturm-Liouville theory)

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If we can analyze the SDE then we can also understand the limiting point process.

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Also: large deviation estimate for large gaps

The limiting process

The limit is described by the SDE

$$d\alpha_\lambda = \lambda \frac{\beta}{4} e^{-\beta/4\, t} dt + \mathrm{Re}\left[(e^{-i\alpha_\lambda} - 1) d(B_1 + iB_2)\right], \qquad \alpha_\lambda(0) = 0$$

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Killip-Stoiciu '06: similar equation for the limit of circular ensembles

1d Schrödinger operator

Thm(Kritchevski-V-Virág, '10):
Local limit of the spectrum (along an appropriate subsequence).

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- CLT for number of points in a large interval

Outline of the proof

$$f_{k-1} + f_{k+1} - \frac{\sigma\xi_k}{\sqrt{n}} f_k = \left(\mu + \frac{\lambda}{n}\right) f_k$$

$$1 \leq k \leq n, \quad \xi_k \sim N(0,1), \quad \mu \in (-2,2) \setminus \{0\}$$

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$$f_{k-1} + f_{k+1} - \frac{\sigma\xi_k}{\sqrt{n}} f_k = \left(\mu + \frac{\lambda}{n}\right) f_k$$

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T_k is close to $T = \begin{bmatrix} \mu & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow$ We should follow $Y_k = T^{-k} M_k$!

Outline of the proof

$M_k = T_k T_{k-1} \dots T_1$

Evolution of $Y_k = T^{-k} M_k$:

$$Y_k - Y_{k-1} = T^{-k} T_k T^{k-1} Y_{k-1} - Y_{k-1}$$

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&= T^{-k} \begin{bmatrix} 0 & \frac{\lambda}{n} + \frac{\sigma \xi_k}{\sqrt{n}} \\ 0 & 0 \end{bmatrix} T^k Y_{k-1}
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T can be diagonalized:

$$T = Z \begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix} Z^{-1}, \qquad Z = \begin{bmatrix} z & \bar{z} \\ 1 & 1 \end{bmatrix}$$

$z = \mu/2 + i\sqrt{1-(\mu/2)^2}$

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Explicit computation:

$$T^{-k} \left[\begin{array}{cc} 0 & \frac{\lambda}{n} + \frac{\sigma \xi_k}{\sqrt{n}} \\ 0 & 0 \end{array} \right] T^k = Z \frac{1}{i\sqrt{4 - \mu^2}} \left(\frac{\lambda}{n} + \frac{\sigma \xi_k}{\sqrt{n}} \right) \left[\begin{array}{cc} 1 & z^{-2k} \\ -z^{2k} & -1 \end{array} \right] Z^{-1}$$

Outline of the proof

SDE limit for $X_k = Z^{-1} Y_k$:

$$dX = i(\lambda dt - \sigma dB_1) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} X + \frac{i\sigma}{\sqrt{2}} \begin{bmatrix} 1 & dB_2 + idB_3 \\ -dB_2 + idB_3 & 1 \end{bmatrix} X$$

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$\Downarrow$

scaling limit of the counting function

Final comments

- V.-Virág: multidimensional version (long boxes)

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- Proof for the β-ensembles is a bit more tricky (several regions)

THANK YOU!