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Harmonic deformation of Delaunay triangulations

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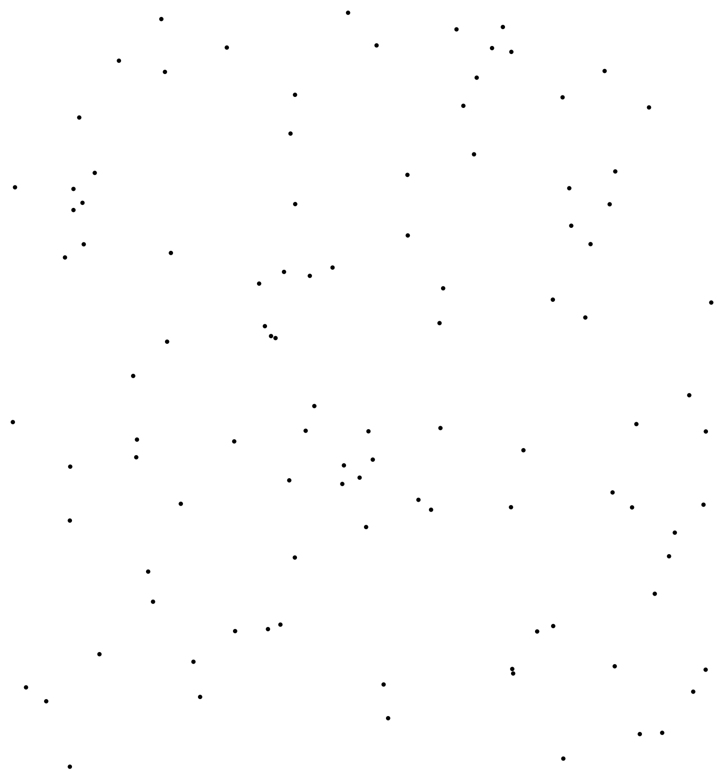
Poisson process in $\mathbb{R}^d$. S is the Palm version of a homogeneous point process on $\mathbb{R}^d$ (with a point at the origin).

$\mathbb{P}$, $\mathbb{E}$ probability and expectation on $\mathcal{N}$ induced by S.

points $s \in S$ and *sites* $x \in \mathbb{R}^d$

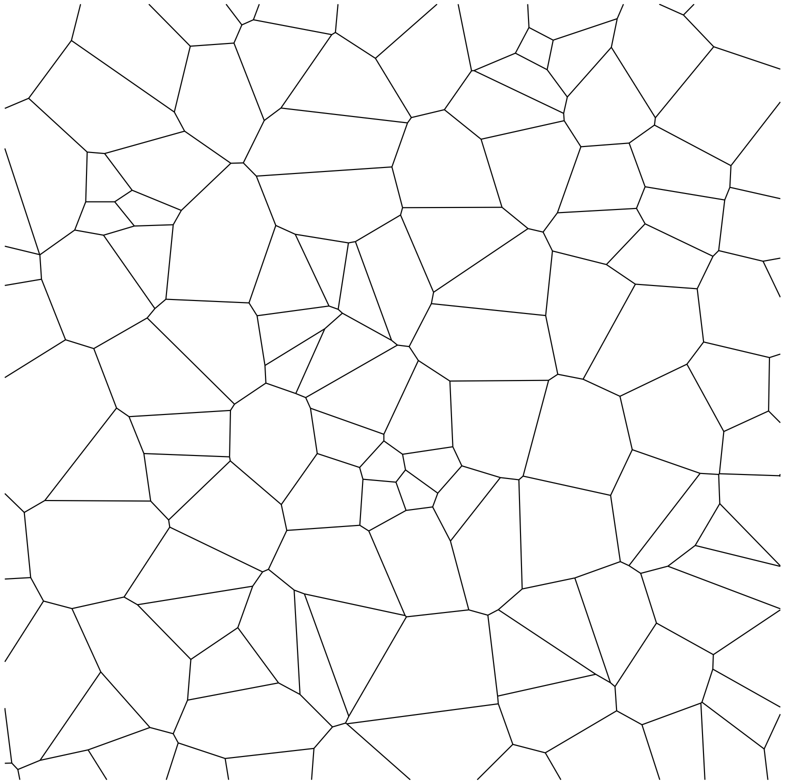


Voronoi tessellation. Voronoi cell

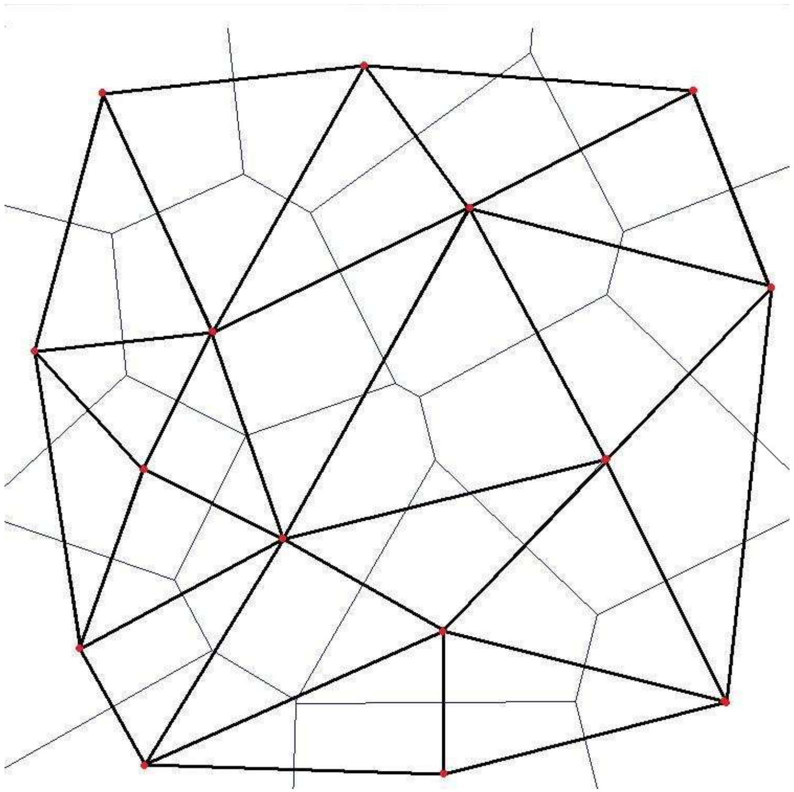
$$\mathrm{Vor}(s) = \{x \in \mathbb{R}^d : |x - s| \leq |x - s'|, \text{ for all } s' \in S \setminus \{s\}\}$$

Voronoi neighbors share a $(d-1)$-dimensional boundary.

$a(s, s') := \mathbf{1}\{s \text{ and } s' \text{ are neighbors}\}$



Delaunay triangulation is the *graph* $\mathcal{G} = (S, \mathcal{E})$ with
$\mathcal{E} := \{(s, s') : s \text{ and } s' \text{ are neighbors}\}$.



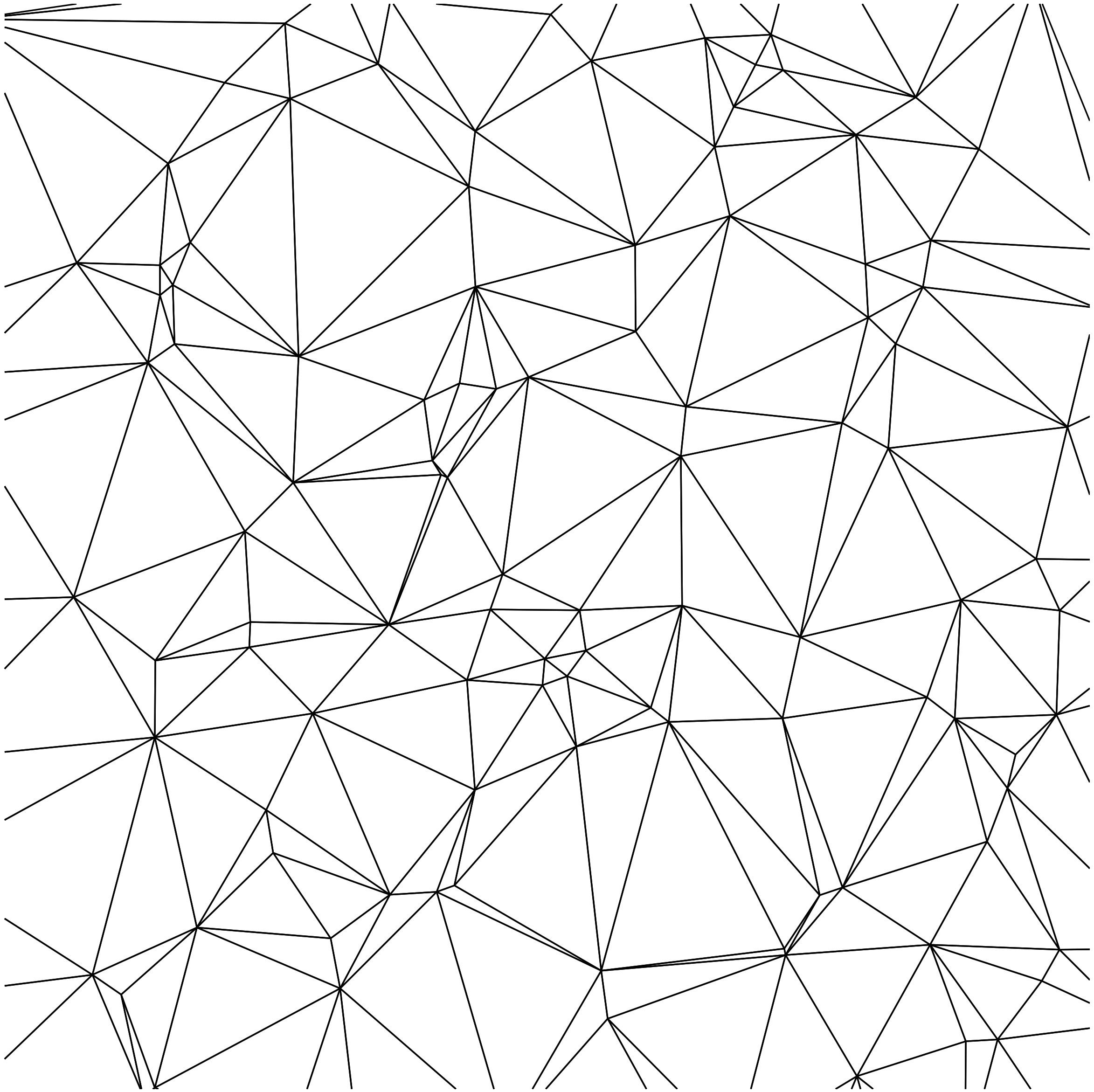


Figure 1: Delaunay triangulation of the above Poisson process.

Harmonic sub-linear deformation

Goal: move the points of S such that keeping the Delaunay neighborhood, the resulting graph is harmonic.

More precisely, find $H : S \to \mathbb{R}^d$ such that

(1) $H(S)$ is harmonic:

$$H(s) = \frac{1}{a(s)} \sum_{s' \in S} a(s, s')\, H(s'), \qquad \text{for all } s \in S.$$

where $a(s) = \sum_{s' \in S} a(s, s')$ is the number of neighbors of s.

(2) $H(S)$ is a sublinear deformation of S:

$$\lim_{K \to \infty} \frac{|H(\mathrm{Cen}(Ku)) - \mathrm{Cen}(Ku)|}{|K|} = 0, \qquad u \text{ unit vector.}$$

Corrector. $H(s) - s$ is called *corrector*.

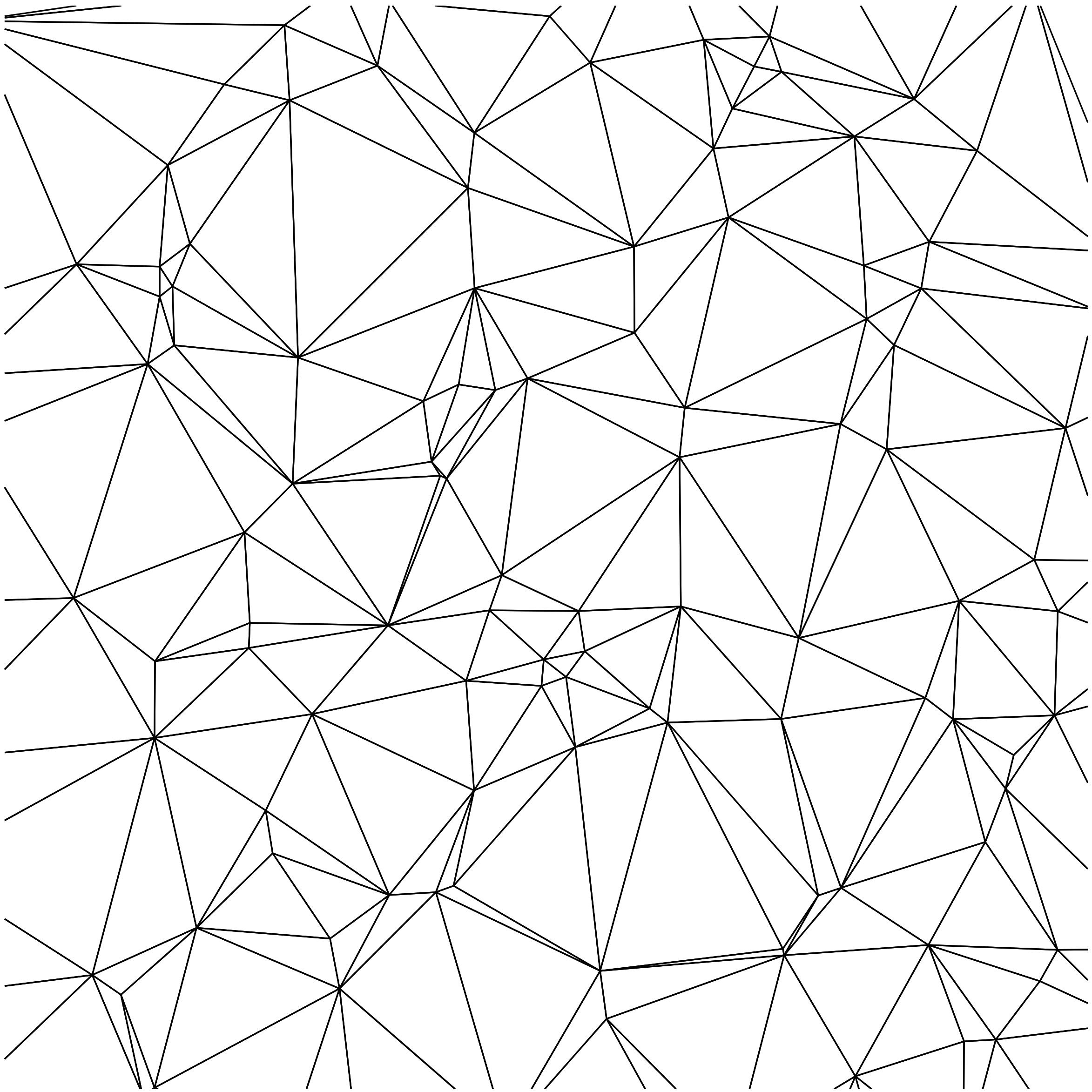


Figure 2: Delaunay triangulation of the above Poisson process.

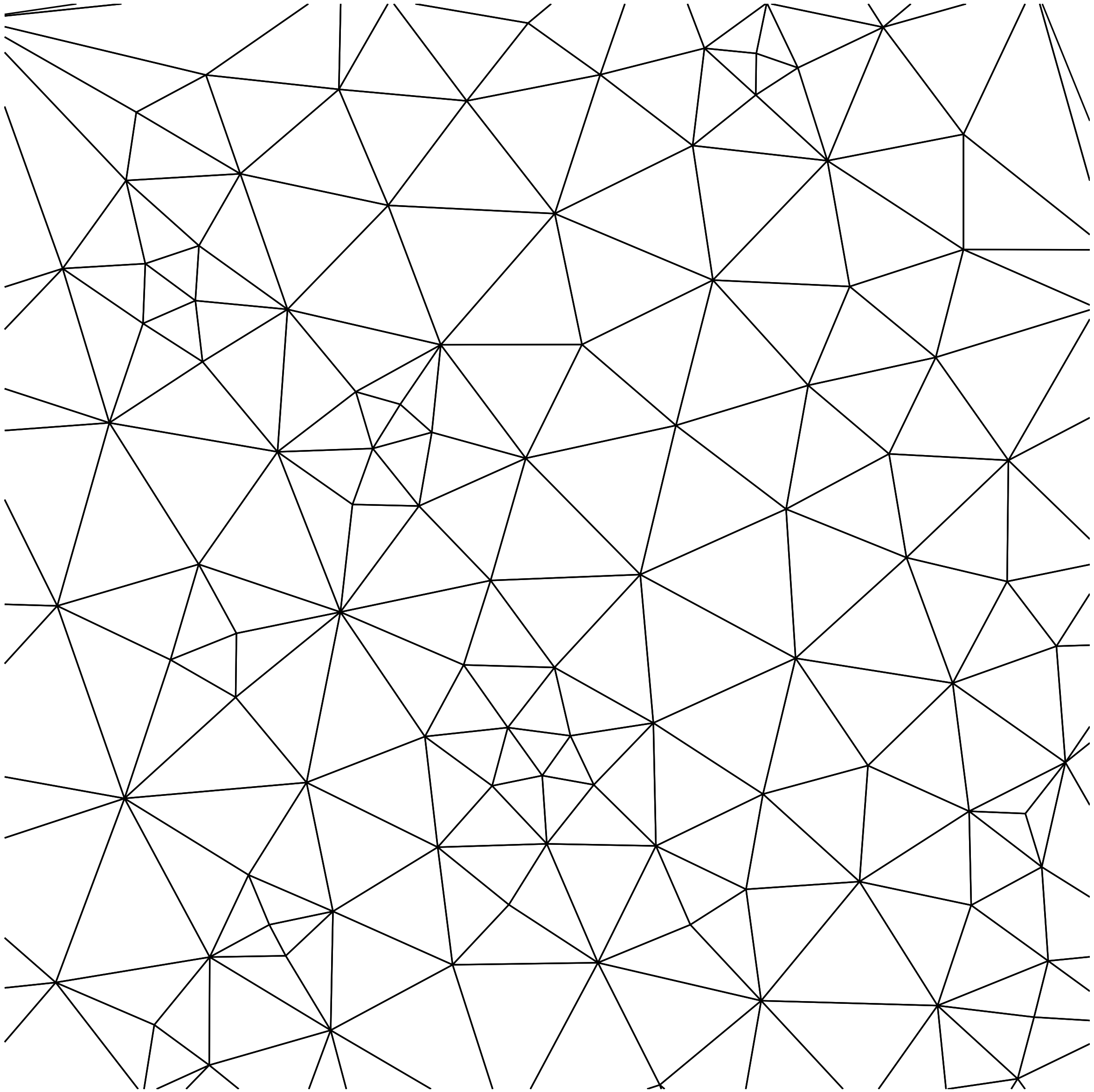


Figure 3: Harmonic deformation of above Delaunay triangulation.

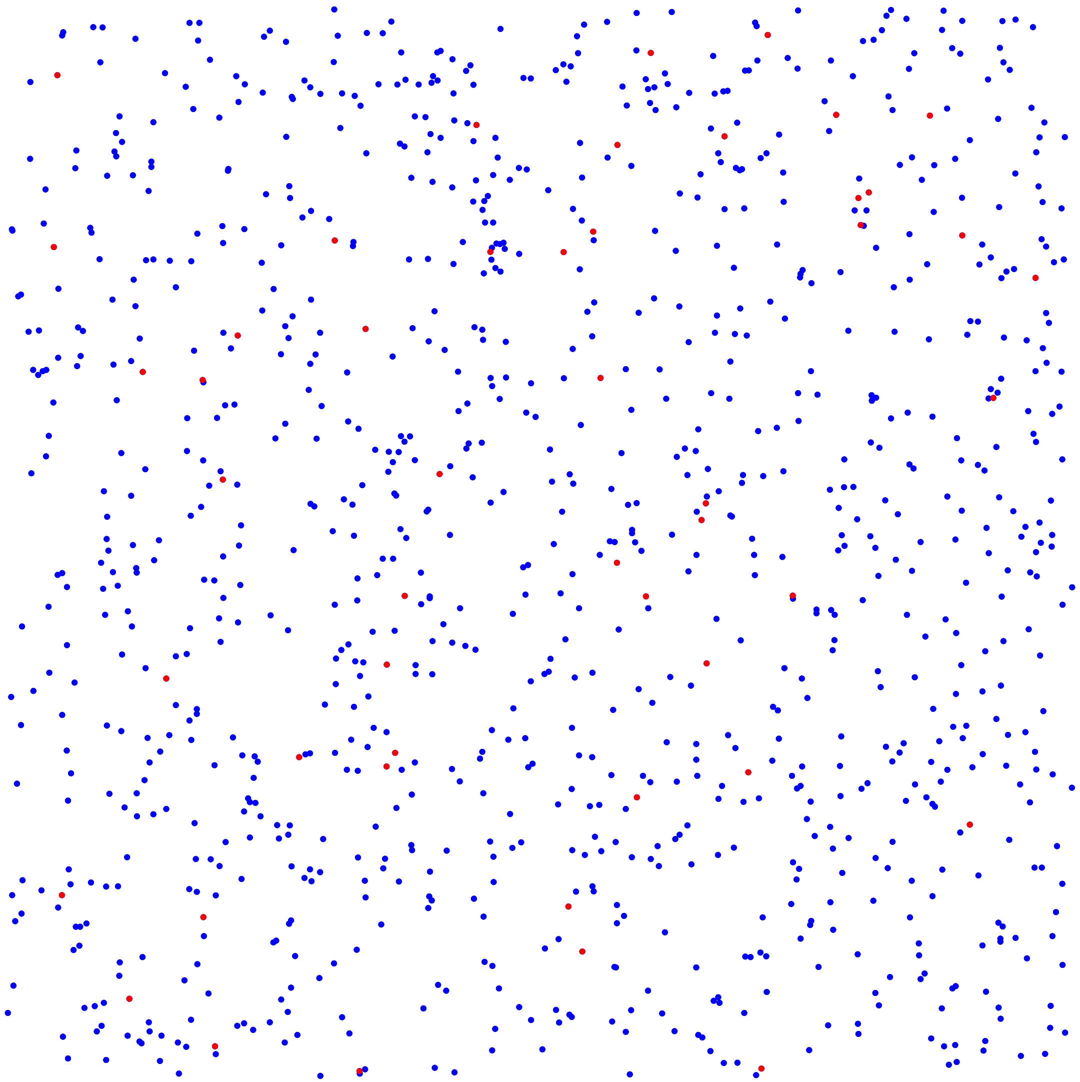


Figure 4: Poisson process.

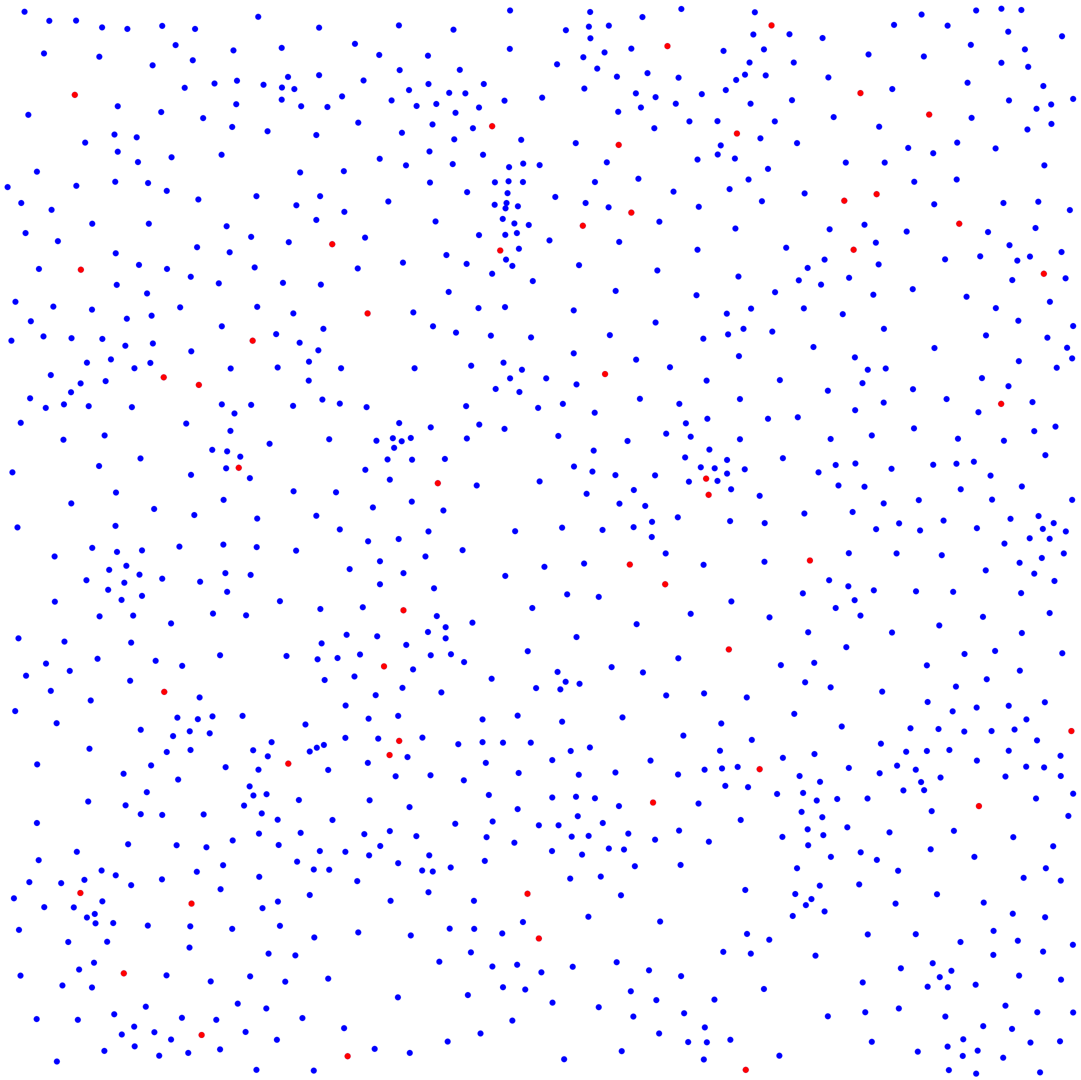


Figure 5: Harmonization of the above Poisson process.

Background

Harmonic deformation graph constructed in two settings:

Percolation clusters:

Berger and Biskup (2007) (our motivation)

Mathieu and Piatnitski (2007)

Barlow and Deuschel (2010)

Poisson process with energy marks:

Caputo, Faggionato and Prescott (2010)

Both approaches use static methods.

Surfaces are functions $\eta : \Xi_1 \to \mathbb{R}$, where

$$\Xi_1 := \{(s, S) \in \mathbb{R}^d \times \mathcal{N} : s \in S\}.$$

Laplacian of a surface η:

$$\Delta\eta(s) = \frac{1}{a(s)} \sum_{s' \in S} a(s, s')[\eta(s') - \eta(s)]$$

A surface h is **harmonic** if for all $s \in S$

$$\Delta h(s) = 0$$

Coordinates of a harmonic graph are harmonic surfaces:

Let $H : S \to \mathbb{R}^d$ and $h_1(s), \ldots, h_d(s)$ the coordinates of $H(s)$.

Graph H is harmonic <u>iff</u> $h_1, \ldots, h_d$ are harmonic surfaces.

Surface Inclination.

$u \in \mathbb{R}^d$ unit vector. Surface η has *inclination* $\mathcal{I}_u(\eta)$ in the direction u if the following limit exists and does not depend on s

$$\mathcal{I}_u(\eta) := \lim_{K\to\infty} \frac{\eta(\mathrm{Cen}(s+Ku)) - \eta(s)}{K} \qquad \mathbb{P}\text{-a.s.} \tag{1}$$

where $\mathrm{Cen}(x)$ is the point in S closest to $x \in \mathbb{R}$.

Let $H = (h_1, \ldots, h_d)$. Then

Graph $(H(S), \widetilde{\mathcal{E}})$ is a **sub-linear** perturbation of S
<u>iff</u> coordinate h_i has **inclination 1** in the direction e_i for all i.

Harness process.

Let $M_s\eta$ the surface obtained by substituting the value of $\eta(s)$ with the average of the heights at the neighbors of s:

$$(M_s\eta)(v) = \begin{cases} \frac{1}{a(s)} \sum_{s' \in S \setminus \{s\}} a(s,s')\,\eta(s'), & \text{if } v = s \\ \eta(v), & \text{if } v \neq s \end{cases}$$

The harness process η_t is the Markov process with generator

$$Lf(\eta) = \sum_{s \in S} [f(M_s\eta) - f(\eta)]$$

At rate 1, the height at s is updated to the average of the heigths at the neighbors of s.

Construction of the harness process

Enumerate the points of S in a point-translation invariant way (Holroyd-Peres).

Associate to each point $s \in S$ a (time) one-dimensional Poisson process or rate 1.

These processes are independent.

Use these times to update the corresponding site.

Use the same notation $\mathbb{P}$ and $\mathbb{E}$ for the product of the law of S and the time Poisson processes.

More definitions

Fields are functions $\zeta : \Xi_2 \to \mathbb{R}$ where

$$\Xi_2 = \{(s, s', S) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{N} : s, s' \in S\}.$$

Will drop dependence on S.

Gradient of a surface η is the field $\nabla\eta$ defined by

$$\nabla\eta(s, s') \;=\; (\eta(s') - \eta(s))$$

A field $\zeta : \Xi_2 \to \mathbb{R}$ is **covariant** if

$$\zeta(s' - s, s'' - s, S) \;=\; \zeta(s', s'', \tau_s S)$$

Theorem 1. (with Rafael Grisi and Pablo Groisman)
(a) if $\eta_0(s) = s_1$ *where* s_1 *is the first coordinate of* s*, then* η_t *converges in* L_2 *to a surface* $h : \Xi_1 \to \mathbb{R}$*:*

$$\lim_{t\to\infty} \mathbb{E}[(\eta_t(s) - \eta_t(0)) - h(s)]^2 \;=\; 0$$

(b) The limit h *is harmonic, has covariant gradient and inclination* 1 *in the direction* e_1*,* $\mathbb{P}$*-a.s..*

Percolation: Berger-Biskup, Mathieu-Piatnitski;

Poisson + energy marks: Caputo-Faggionato-Prescott.

Application to random walk in Delaunay triangulation.

Y_t^S: random walk in the Delaunay triangulation with generator

$$L_S f(s) = \sum_{s' \in S} a(s, s')[f(s') - f(s)]$$

Since the graph H is harmonic, $H(Y_t^S)$ is a martingale and so satisfies the invariance principle $\mathbb{P}$-a.s..

To show the invariance principle for Y_t^S it suffices uniform sub-linearity of the corrector $H(s) - s$ (but seems too much).

(OK in $d = 2$ à la BB, or Heat Kernel Estimates à la Barlow)

Positive diffusion easy.

Berger-Biskup, Mathieu-Piatnitski, Sidoravicius-Snitzman, Caputo-Faggionato-Prescott and many others.

The space of fields as a Hilbert space

Recall

$$\Xi_2 = \{(s, s', S) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{N} : s, s' \in S\}.$$

S is the Palm realization of a Poisson process in $\mathbb{R}^d$ with law $\mathbb{P}$, $\mathbb{E}$.

For a field $\zeta : \Xi_2 \to \mathbb{R}$ define

$$\mathcal{C}(\zeta) = \mathbb{E}\left[\sum_{s \in S} a(0, s)\zeta(0, s)\right].$$

Hilbert space $\mathcal{H} := L^2(\Xi_2, \mathbb{R}, \mathcal{C})$.

Inner product: for fields ζ and ζ' in $\mathcal{H}$:

$$\mathcal{C}(\zeta \cdot \zeta') = \mathbb{E}\left[\sum_{s \in S} a(0, s)\zeta(0, s)\zeta'(0, s)\right].$$

Cesàro limit of covariant fields

Let $\zeta \in \mathcal{H}$ be a covariant field and define

$$C(\zeta) := \lim_{\Lambda \nearrow \mathbb{R}^d} \frac{1}{2|\Lambda|} \sum_{s \in S \cap \Lambda,\, s' \in S} a(s,s')\zeta(s,s').$$

Since S is ergodic, by the Point Ergodic Theorem we have that almost surely

$$C(\zeta) = \mathcal{C}(\zeta)$$

Inclination as inner product

For Voronoi neighbors $s, s' \in S$ define:

$b(s, s')$:= $(d-1)$-dimensional common side of cells of s and s'

$b_u(s, s')$:= $(d-1)$-dimensional Lebesgue measure of the projection of $b(s, s')$ over the hiperplane perpendicular to u.

s_u := $u\langle s, u\rangle$ (projection of s over the line determined by u).

Define the field κ_u by

$$\kappa_u(s, s') := \frac{1}{2}\,\mathrm{sg}(s'_u - s_u)\, b_u(s, s')\, a(s, s')$$

Remark: $\kappa_u \in \mathcal{H}$ is covariant.

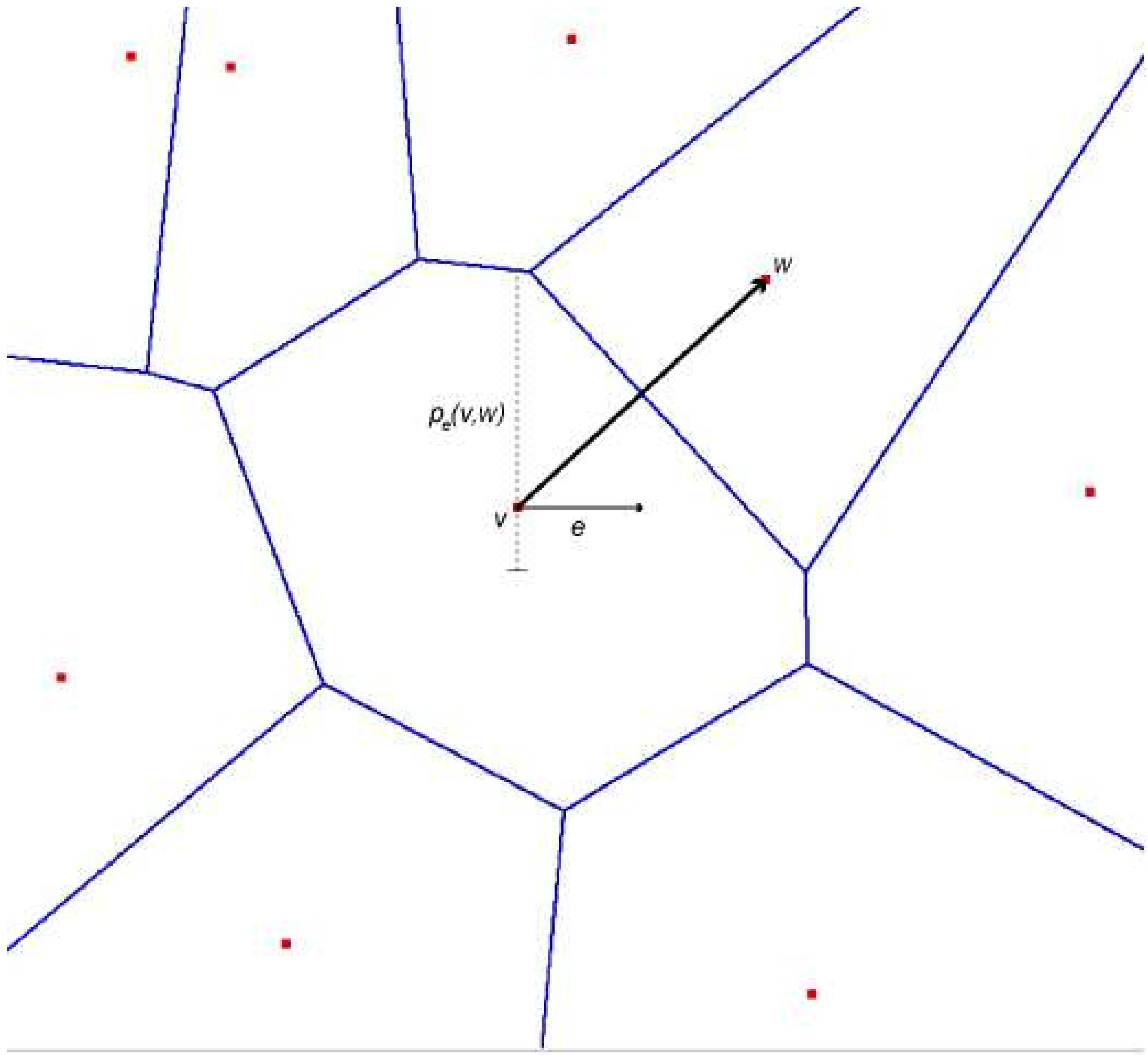


Figure 6: Definition of the field κ_u with $u = (1, 0)$.

From the definition:

$$\sum_{s' \in S} \kappa_u(0, s') \;=\; \frac{1}{2} \sum_{s'} a(0, s') \, \mathrm{sg}(s'_u) \, b_u(0, s') \;=\; 0$$

the projections of the "negative" sides has the same area as the projections of the "positive" sides.

By covariance, for all $s \in S$:

$$\sum_{s' \in S} \kappa_u(s, s') \;=\; 0$$

Second definition of inclination:

$$\mathcal{J}_u(\eta) \;:=\; \mathcal{C}(\nabla\eta \cdot \kappa_u)$$

$$\begin{aligned}
\mathcal{J}_u(\eta) \quad &= \quad \frac{1}{2}\,\mathbb{E}\sum_{s'\in S} a(0,s)\,(\eta(s)-\eta(0))\,\kappa_u(0,s) \\
&= \quad \lim_{\Lambda\nearrow\mathbb{R}^d}\;\frac{1}{2|\Lambda|}\;\sum_{s\in S\cap\Lambda,\, s'\in S} a(s,s')\,(\eta(s')-\eta(s))\,\kappa_u(s,s').
\end{aligned}$$

Proposition 2. Let η be a surface with covariant $\nabla\eta \in \mathcal{H}$. Then

$$\mathcal{I}_u(\eta) = \mathcal{J}_u(\eta) \qquad \mathbb{P}\text{-a.s.}$$

Proof.

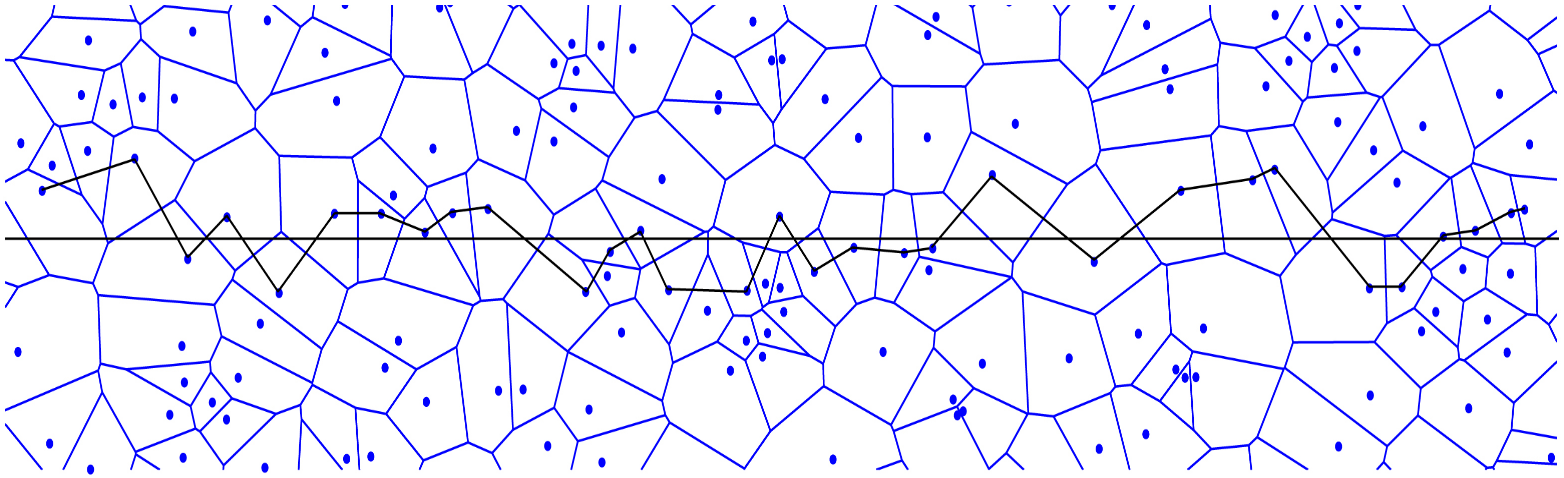


Figure 7: Points contributing to the inclination along the line $y = 0$.

Inclination is invariant for the dynamics:

$$\mathcal{J}_u(\eta_t) \;=\; \mathcal{J}_u(\eta_0)$$

Why? Updating the origin gives zero contribution: Let $\tilde\eta = M_0\eta$

$$\begin{aligned}
\mathcal{J}_u(\eta) - \mathcal{J}_u(\tilde\eta) \;&=\; \mathbb{E}\sum_{s'} \kappa_u(0,s')[\nabla\eta(0,s') - \nabla\tilde\eta(0,s')] \\
&=\; \mathbb{E}\Big[(\tilde\eta(0) - \eta(0))\sum_{s'} a(0,s')\,\mathrm{sg}(s'_u)\,b_u(0,s')\Big] \;=\; 0
\end{aligned}$$

because the $(d-1)$-Lebesgue measure of the projections with negative contribution coincides with the one of the projections with positive contribution.

The contributions of the updating of neighbors of the origin are also zero by translation invariance and covariance of the fields involved.

Proof of (a) and (b) of the Theorem

(a) Existence of a harmonic surface with inclination 1

will be a consequence of

(b) Convergence of the harness process to a harmonic surface.

To show (b) we show:

(1) the gradients of the harness process starting with a hyperplane converge to a field in $L_2(\mathcal{C})$.

(2) the limit field is the gradient of a harmonic surface with inclination 1.

Ingredients:

1) Integration by parts formula:

$\zeta\colon \Xi_2 \to \mathbb{R}$ covariant field

$\psi\colon \Xi_1 \to \mathbb{R}$ translation invariant surface $(\psi(v, S) = \psi(0, \tau_v S))$

such that $\nabla\psi, \zeta \in \mathcal{H}$. Then

$$\mathcal{C}(\nabla\psi \cdot \zeta) \;=\; -\mathbb{E}[\psi(0)\,\mathrm{div}\zeta(0)]$$

where the *divergence* is given by

$$\mathrm{div}\zeta(s) = \sum_{s' \in S} \zeta(s, s')$$

Used for $\psi_t = \eta_t - \eta_0$.

Write $\eta_t = \eta_0 + \psi_t$, where η_0 is a "hyperplane" and ψ_t is translation invariant.

2) Square of gradients decrease: For all $t > 0$

$$\frac{d}{dt}\mathcal{C}(|\nabla \eta_t|^2) \;=\; -2\mathbb{E}\left[\frac{|\Delta \eta_t(0)|^2}{a(0)}\right],$$

3) Laplacian converges almost surely to 0

$$\infty \;>\; \mathcal{C}(|\nabla \eta_0|^2) \;\geq\; \lim_{t\to\infty} \mathcal{C}(|\nabla \eta_t|^2) \;=\; 2\int_0^{\infty} \mathbb{E}\left[\frac{|\Delta \eta_t(0)|^2}{a(0)}\right] dt,$$

4) Weak convergence of $\nabla \eta_t$ to ζ_∞ by subsequences:

By (2) there exists a subsequence $\{t_k\}$ and a field $\zeta_\infty \in \mathcal{H}$ such that

$$\lim_{k\to\infty} \mathcal{C}(\nabla \eta_{t_k} \cdot \zeta) \;=\; \mathcal{C}(\zeta_\infty \cdot \zeta),$$

for all $\zeta \in \mathcal{H}$.

5) Uniqueness of the limit:

Using integration by parts,

$$\mathcal{C}(\nabla \eta_0 \cdot \zeta_\infty) = \mathcal{C}(|\zeta_\infty|^2).$$

$$\mathcal{C}(\tilde{\zeta}_\infty \cdot \zeta_\infty) = \mathcal{C}(|\zeta_\infty|^2) = \mathcal{C}(|\tilde{\zeta}_\infty|^2).$$

6) Convergence in L_2 and a.s. along subsequences.

Using Holder and convergence of Laplacian to zero,

$$\lim_{t \to \infty} \mathcal{C}(|\nabla \eta_t - \zeta_\infty|^2) = 0.$$

7) Limit ζ_∞ is covariant.

Follows from the covariance of $\nabla \eta_t$ for each t and a.s. convergence along subsequences.

8) The limiting field has zero divergence. Hölder:

$$\lim_{t\to\infty} \mathbb{E}(a(0)^{-2}|\Delta\eta_t - \mathrm{div}\zeta_\infty|^2) \;\le\; \lim_{t\to\infty} \mathcal{C}(|\nabla\eta_t - \zeta_\infty|^2) \;=\; 0.$$

implies

$$\mathrm{div}(\zeta_\infty) = 0 \qquad \text{a.s.}$$

9) The limit is a gradient field

Convergence in L_2 implies there exists a subsequence converging almost surely. This sequence must satisfy the cocycle property.

10) The limit is the gradient of a harmonic surface

Follows from (8) and (9).

11) The limit has the same inclination as η_0

This is because the inclination $\mathcal{J}$ is invariant for the dynamics:

$$\mathcal{J}_u(\eta_t) = \mathcal{C}(\nabla\eta_0 \cdot \kappa_u) = \mathcal{C}(\nabla\eta_t \cdot \kappa_u) \longrightarrow \mathcal{C}(\nabla\eta_\infty \cdot \kappa_u)$$

Generalization

Theorem holds if S is the Palm version of a stationary point process in $\mathbb{R}^d$ and

A1 The law of S is mixing. (To get one-dimensional LLN)

A2 For every ball $B \subset R^d$, $|S \cap \partial B| < d + 2$.

A3 $\mathbb{E} \exp(\beta a(0, S)) < \infty$ for some positive constant β.

A4 $\mathcal{C}(\omega_u^2) < \infty$ for every $u \in \mathbb{R}^d$.

A5 S aperiodic, meaning that $\mathbb{P}(\exists x \in \mathbb{R}^d \setminus \{0\} : \tau_x S = S) = 0$.

A6 $\mathbb{E}[\sum_{s \in S} a(0, s)|s|^r] < \infty$ for some $r > 4$.

A7 $S = \{s_n ; n \in \mathbb{Z}\}$, and $\tau_{s_n} S \stackrel{law}{=} S$ for every $n \in \mathbb{Z}$.

A8 $\mathbb{E}[\ell(\mathrm{Vor}(0, S))^2] < \infty$.

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