Large deviations and fluctuation exponents for some polymer models

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2 Large deviations

3 Fluctuation exponents
   - KPZ equation
   - Log-gamma polymer
Directed polymer in a random environment

simple random walk path \((x(t), t), t \in \mathbb{Z}_+\)
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\(\mathbb{P}\) probability distribution on \(\omega\), often \(\{\omega(x, t)\}\) i.i.d.
Key quantities again:

- Quenched measure $Q_n\{x(\cdot)\} = Z_n^{-1} \exp\left\{ \beta \sum_{t=1}^{n} \omega(x(t), t) \right\}$

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- Dependence on $\beta$ and $d$
**Question:** describe quenched limit $\lim_{n \to \infty} n^{-1} \log Z_n$ (ℙ-a.s.)
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Generalize: \( E_0 = \text{expectation under background RW } X_n \text{ on } \mathbb{Z}^\nu. \)
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Introduced shift \( (T_{x_k} \omega)_y = \omega_{x+y} \), steps \( Z_k = X_k - X_{k-1} \in \mathcal{R}, \)

\( Z_{1,\ell} = (Z_1, Z_2, \ldots, Z_\ell) \).
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\begin{align*}
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$g(\omega, z_{1,\ell})$ is a function on $\Omega_\ell = \Omega \times \mathcal{R}^\ell$. 
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\( S_z : (\omega, z_1, \ell) \rightarrow (T^z_1 \omega, z_2, \ell z) \).
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Defines kernel $p$ on $\Omega_\ell$: $p(\eta, S_z \eta) = |\mathcal{R}|^{-1}$. 

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Entropy

Let $\mu_0 = \Omega$-marginal of $\mu \in \mathcal{M}_1(\Omega_\ell)$. Define

$$H_\mathcal{P}(\mu) = \begin{cases} \inf \{ H(\mu \times q \mid \mu \times p) : \mu q = \mu \} & \text{if } \mu_0 \ll \mathcal{P} \\ \infty & \text{otherwise.} \end{cases}$$

Infimum over Markov kernels $q$ that fix $\mu$. 
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Inside the braces the familiar relative entropy

$$H(\mu \times q \mid \mu \times p) = \int_{\Omega_\ell} \sum_{z \in \mathcal{R}} q(\eta, S_z \eta) \log \frac{q(\eta, S_z \eta)}{p(\eta, S_z \eta)} \mu(d\eta).$$
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\]

\( H_\mathbb{P} \) is convex but not lower semicontinuous.
Assumptions.

- Environment $\{\omega_x\}$ IID under $\mathbb{P}$.
- $g$ local function on $\Omega_\ell$ s.t. $\mathbb{E}|g|^p < \infty$ for some $p > 2(\nu + 1)$.
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Theorem. (Rassoul-Agha, S, Yilmaz)  Deterministic limit

$$\Lambda(g) = \lim_{n \to \infty} n^{-1} \log E_0[e^{nR_n(g)}]$$

exists $\mathbb{P}$-a.s. and

$$\Lambda(g) = H^\#_{\mathbb{P}}(g) \equiv \sup_\mu \sup_{c > 0} \{E^\mu[g \wedge c] - H_\mathbb{P}(\mu)\}.$$
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\]

Remark. True more generally, e.g. ergodic \( \mathbb{P} \) if \( g \) is bounded.
Quenched weak LDP (large deviation principle) under $Q_n$.

$$Q_n(A) = \frac{1}{E_0[e^{nR_n(g)}]} E_0[e^{nR_n(g)}1_A(\omega, Z_1, \infty)]$$
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\[Q_n(A) = \frac{1}{E_0[e^{nR_n(g)}]} E_0[e^{nR_n(g)}1_A(\omega, Z_{1,\infty})]\]

Rate function \[I(\mu) = \sup_{\varphi \in \mathcal{U}(\Omega_\ell)} \{ E^\mu(\varphi) - \Lambda(\varphi + g) \} + \Lambda(g). \]
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**Theorem.** (RSY) Assumptions as above and $\Lambda(g)$ finite. Then $\mathbb{P}$-a.s. for compact $F \subseteq \mathcal{M}_1(\Omega_\ell)$ and open $G \subseteq \mathcal{M}_1(\Omega_\ell)$:

$$\lim_{n \to \infty} n^{-1} \log Q_n\{R_n \in F\} \leq - \inf_{\mu \in F} I(\mu),$$

$$\lim_{n \to \infty} n^{-1} \log Q_n\{R_n \in G\} \geq - \inf_{\mu \in G} I(\mu).$$
Return to $d + 1$ dim directed polymer in i.i.d. environment.
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**Question:** Is the path \( x(\cdot) \) diffusive or not, that is, does it scale like standard RW?
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**Early results:** diffusive behavior for $d \geq 3$ and small $\beta > 0$:

- 1988 Imbrie and Spencer: $n^{-1}E^Q(|x(n)|^2) \to c \quad \mathbb{P}$-a.s.
- 1989 Bolthausen: quenched CLT for $n^{-1/2}x(n)$. 

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In the opposite direction: if $d = 1, 2$, or $d \geq 3$ and $\beta$ large enough, then

$\exists \ c > 0 \ s.t.
\lim_{n \to \infty} \max_{z} Q_n\{x(n) = z\} \geq c \quad \mathbb{P}$-a.s.

(Carmona and Hu 2002, Comets, Shiga, and Yoshida 2003)
Definition of fluctuation exponents $\zeta$ and $\chi$
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- Conjecture for $d = 1$: $\zeta = 2/3$ and $\chi = 1/3$. 
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**Results:** these exact exponents for three particular $1+1$ dimensional models.
Earlier results for $d = 1$ exponents

Past rigorous bounds give $3/5 \leq \zeta \leq 3/4$ and $\chi \geq 1/8$: 
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(1) Log-gamma polymer: $\beta = 1$ and $e^{-\omega(x,t)} \sim \text{Gamma}$, plus appropriate boundary conditions
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(3) Continuum directed polymer, or Hopf-Cole solution of the Kardar-Parisi-Zhang (KPZ) equation:

   (i) Initial height function given by two-sided Brownian motion. (Joint with M. Balázs and J. Quastel.)

   (ii) Narrow wedge initial condition. (Amir, Corwin, Quastel)
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Next details on (3.i), then details on (1).
Hopf-Cole solution to KPZ equation

KPZ eqn for height function $h(t, x)$ of a 1+1 dim interface:

$$h_t = \frac{1}{2} h_{xx} - \frac{1}{2} (h_x)^2 + \dot{W}$$

where $\dot{W} = \text{Gaussian space-time white noise}$. 
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$Z = \exp(-h)$ satisfies $Z_t = \frac{1}{2} Z_{xx} - Z \dot{W}$ that can be solved.
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Bertini-Giacomin (1997): \( h \) can be obtained as a weak limit via a smoothing and renormalization of KPZ.
WASEP connection

\( \zeta_\varepsilon(t, x) \) height process of weakly asymmetric simple exclusion s.t.

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\zeta_\varepsilon(x + 1) - \zeta_\varepsilon(x) = \pm 1
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Jumps:

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\zeta_\varepsilon(x) \longrightarrow \begin{cases} 
\zeta_\varepsilon(x) + 2 & \text{with rate } \frac{1}{2} + \sqrt{\varepsilon} \text{ if } \zeta_\varepsilon(x) \text{ is a local min} \\
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\end{cases} \]

Initially:

\[ \zeta_{\varepsilon}(0, x + 1) - \zeta_{\varepsilon}(0, x) = \pm 1 \text{ with probab } \frac{1}{2}. \]
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\[ \zeta_\epsilon(x) \rightarrow \begin{cases} 
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\zeta_\epsilon(x) - 2 & \text{with rate } \frac{1}{2} \quad \text{if } \zeta_\epsilon(x) \text{ is a local max} 
\end{cases} \]

Initially: \[ \zeta_\epsilon(0, x + 1) - \zeta_\epsilon(0, x) = \pm 1 \quad \text{with probab } \frac{1}{2}. \]

\[ h_\epsilon(t, x) = \epsilon^{1/2} \left( \zeta_\epsilon(\epsilon^{-2} t, [\epsilon^{-1} x]) - \nu_\epsilon t \right) \]
**WASEP connection**

Jumps:

\[
\zeta_\varepsilon(x) \longrightarrow \begin{cases} 
\zeta_\varepsilon(x) + 2 & \text{with rate } \frac{1}{2} + \sqrt{\varepsilon} \text{ if } \zeta_\varepsilon(x) \text{ is a local min} \\
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\]

Initially: \( \zeta_\varepsilon(0, x + 1) - \zeta_\varepsilon(0, x) = \pm 1 \) with probablity \( \frac{1}{2} \).

\[
h_\varepsilon(t, x) = \varepsilon^{1/2} \left( \zeta_\varepsilon(\varepsilon^{-2} t, [\varepsilon^{-1} x]) - v_\varepsilon t \right)
\]

**Thm.** As \( \varepsilon \downarrow 0 \), \( h_\varepsilon \Rightarrow h \) (Bertini-Giacomin 1997).
Fluctuation bounds

From coupling arguments for WASEP

\[ C_1 t^{2/3} \leq \text{Var}(h_\varepsilon(t, 0)) \leq C_2 t^{2/3} \]
Fluctuation bounds

From coupling arguments for WASEP

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**Thm.** (Balázs-Quastel-S) For the Hopf-Cole solution of KPZ,

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The lower bound comes from control of rescaled correlations

\[ S_\varepsilon(t, x) = 4\varepsilon^{-1} \text{Cov}[\eta(\varepsilon^{-2} t, \varepsilon^{-1} x), \eta(0,0)] \]
Rescaled correlations:

\[ S_\varepsilon(t, x) = 4\varepsilon^{-1} \text{Cov} \left[ \eta(\varepsilon^{-2} t, \varepsilon^{-1} x), \eta(0, 0) \right] \]
Rescaled correlations:

\[ S_\varepsilon(t, x) = 4\varepsilon^{-1} \, \text{Cov}[\eta(\varepsilon^{-2} t, \varepsilon^{-1} x), \eta(0, 0)] \]

\[ S_\varepsilon(t, x)dx \Rightarrow S(t, dx) \text{ with control of moments:} \]

\[ \int |x|^m S_\varepsilon(t, x) \, dx \sim O(t^{2m/3}), \quad 1 \leq m < 3. \]

(A second class particle estimate.)
Rescaled correlations:

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\[ S(t, dx) = \frac{1}{2} \partial_{xx} \text{Var}(h(t, x)) \] as distributions.
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(A second class particle estimate.)

\[ S(t, dx) = \frac{1}{2} \partial_{xx} \text{Var}(h(t, x)) \text{ as distributions.} \]

With some control over tails we arrive at

\[ \text{Var}(h(t, 0)) = \int |x| S(t, dx) \sim O(t^{2/3}). \]
1+1 dimensional lattice polymer with log-gamma weights

Fix both endpoints.

![Diagram of a 1+1 dimensional lattice polymer with log-gamma weights.]
1+1 dimensional lattice polymer with log-gamma weights

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\[ \Pi_{m,n} = \text{set of admissible paths} \]
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$$Z_{m,n} = \sum_{x} \prod_{k=1}^{m+n} Y_{x_k}$$
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environment \( (Y_{i,j} : (i,j) \in \mathbb{Z}_+^2) \)

\[ Z_{m,n} = \sum_{x} \prod_{k=1}^{m+n} Y_{x_k} \]

quenched measure \( Q_{m,n}(x.) = Z_{m,n}^{-1} \prod_{k=1}^{m+n} Y_{x_k} \)

averaged measure \( P_{m,n}(x.) = \mathbb{E} Q_{m,n}(x.) \)
Weight distributions

- Parameters $0 < \theta < \mu$. 
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\[
\begin{array}{ccc}
0 & 1 & U_{i,0} \\
V_{0,j} & Y_{i,j} & \end{array}
\]
Weight distributions

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<table>
<thead>
<tr>
<th>$U_{i,0}$</th>
<th>$Y_{i,j}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{0,j}$</td>
<td></td>
</tr>
</tbody>
</table>

$U_{i,0}^{-1} \sim \text{Gamma}(\theta)$
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- **Gamma($\theta$) density**: $\Gamma(\theta)^{-1}x^{\theta-1}e^{-x}$ on $\mathbb{R}^+$
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\[
\begin{align*}
V_{0,j} & \sim \text{Gamma}(\theta) \\
Y_{i,j} & \sim \text{Gamma}(\mu - \theta) \\
Y_{i,j}^{-1} & \sim \text{Gamma}(\mu) \\
U_{i,0} & = Y_{i,0} \\
\end{align*}
\]

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- $\Psi_n(s) = (d^{n+1}/ds^{n+1})\log \Gamma(s)$
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\[
\begin{array}{c|c|c}
V_{0,j} & Y_{i,j} & U_{i,0} \\
0 & 1 & 0
\end{array}
\]

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- $\Psi_n(s) = (d^{n+1}/ds^{n+1}) \log \Gamma(s)$

- $\mathbb{E}(\log U) = -\Psi_0(\theta)$ and $\text{Var}(\log U) = \Psi_1(\theta)$
Variance bounds for log Z

With $0 < \theta < \mu$ fixed and $N \to \infty$ assume

$$| m - N\psi_1(\mu - \theta) | \leq CN^{2/3} \quad \text{and} \quad | n - N\psi_1(\theta) | \leq CN^{2/3}$$  \hspace{1cm} (1)
Variance bounds for log $Z$

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Theorem

For $(m, n)$ as in (1), $C_1 N^{2/3} \leq \text{Var}(\log Z_{m,n}) \leq C_2 N^{2/3}$. 
Variance bounds for log Z

With $0 < \theta < \mu$ fixed and $N \nearrow \infty$ assume

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For $(m, n)$ as in (1), $C_1 N^{2/3} \leq \text{Var}(\log Z_{m,n}) \leq C_2 N^{2/3}$.

**Theorem**

Suppose $n = \psi_1(\theta)N$ and $m = \psi_1(\mu - \theta)N + \gamma N^\alpha$ with $\gamma > 0$, $\alpha > 2/3$. Then

$$N^{-\alpha/2} \left\{ \log Z_{m,n} - \mathbb{E}(\log Z_{m,n}) \right\} \Rightarrow \mathcal{N}(0, \gamma \psi_1(\theta))$$
Fluctuation bounds for path

\( v_0(j) = \text{leftmost}, \ v_1(j) = \text{rightmost point of } x. \text{ on horizontal line:} \)

\[
v_0(j) = \min\{ i \in \{0, \ldots, m\} : \exists k : x_k = (i, j) \}
\]

\[
v_1(j) = \max\{ i \in \{0, \ldots, m\} : \exists k : x_k = (i, j) \}
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\end{align*}
\]

**Theorem**

Assume \((m, n)\) as previously and \(0 < \tau < 1\). Then

(a) \( P \left\{ v_0(\lfloor \tau n \rfloor) < \tau m - bN^{2/3} \quad \text{or} \quad v_1(\lfloor \tau n \rfloor) > \tau m + bN^{2/3} \right\} \leq \frac{C}{b^3} \)

(b) \( \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that} \)

\[
\lim_{N \to \infty} P \left\{ \exists k \text{ such that } |x_k - (\tau m, \tau n)| \leq \delta N^{2/3} \right\} \leq \varepsilon.
\]
Results for log-gamma polymer summarized

With reciprocals of gammas for weights, both endpoints of the polymer fixed and the right boundary conditions on the axes, we have identified the one-dimensional exponents

\[ \zeta = \frac{2}{3} \quad \text{and} \quad \chi = \frac{1}{3}. \]
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Next step is to

- eliminate the boundary conditions and
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In both scenarios we have the upper bounds for both \( \log Z \) and the path.
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In both scenarios we have the upper bounds for both log $Z$ and the path. But currently do not have the lower bounds.

Next some key points of the proof.
Burke property for log-gamma polymer with boundary

Given initial weights \((i, j \in \mathbb{N})\):

\[
U_{i,0}^{-1} \sim \text{Gamma}(\theta), \quad V_{0,j}^{-1} \sim \text{Gamma}(\mu - \theta)
\]

\[
Y_{i,j}^{-1} \sim \text{Gamma}(\mu)
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\]

Compute \(Z_{m,n}\) for all \((m, n) \in \mathbb{Z}_+^2\) and then define

\[
U_{m,n} = \frac{Z_{m,n}}{Z_{m-1,n}}, \quad V_{m,n} = \frac{Z_{m,n}}{Z_{m,n-1}}, \quad X_{m,n} = \left(\frac{Z_{m,n}}{Z_{m+1,n}} + \frac{Z_{m,n}}{Z_{m,n+1}}\right)^{-1}
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\]

For an undirected edge \(f\):

\[
T_f = \begin{cases} 
U_x & f = \{x - e_1, x\} \\
V_x & f = \{x - e_2, x\}
\end{cases}
\]
down-right path \((z_k)\) with edges \(f_k = \{z_{k-1}, z_k\}, k \in \mathbb{Z}\)

- interior points \(\mathcal{I}\) of path \((z_k)\)
• down-right path \((z_k)\) with edges \(f_k = \{z_{k-1}, z_k\}, k \in \mathbb{Z}\)

• interior points \(I\) of path \((z_k)\)

**Theorem**

Variables \(\{T_{f_k}, X_z : k \in \mathbb{Z}, z \in I\}\) are independent with marginals

\[U^{-1} \sim \text{Gamma}(\theta), \quad V^{-1} \sim \text{Gamma}(\mu - \theta),\]

and \(X^{-1} \sim \text{Gamma}(\mu)\).
down-right path \((z_k)\) with edges \(f_k = \{z_{k-1}, z_k\}, k \in \mathbb{Z}\)

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**Theorem**

Variables \(\{T_{f_k}, X_z : k \in \mathbb{Z}, z \in \mathcal{I}\}\) are independent with marginals \(U^{-1} \sim\Gamma(\theta), V^{-1} \sim \Gamma(\mu - \theta),\) and \(X^{-1} \sim \Gamma(\mu).\)

“Burke property” because the analogous property for last-passage is a generalization of Burke’s Theorem for M/M/1 queues, via the last-passage representation of M/M/1 queues in series.
Proof of Burke property

Induction on $\mathcal{I}$ by flipping a growth corner:

\[ U' = Y(1 + U/V) \quad V' = Y(1 + V/U) \]
\[ X = (U^{-1} + V^{-1})^{-1} \]
Proof of Burke property

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Lemma. Given that $(U, V, Y)$ are independent positive r.v.’s, $(U', V', X) \overset{d}{=} (U, V, Y)$ iff $(U, V, Y)$ have the gamma distr’s.

Proof. “if” part by computation, “only if” part from a characterization of gamma due to Lukacs (1955). \(\square\)
Proof of Burke property

Induction on $I$ by flipping a growth corner:

\[ \begin{align*}
V & \quad U \quad Y \\
\quad X & \quad V' \\
\end{align*} \]

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**Proof.** “if” part by computation, “only if” part from a characterization of gamma due to Lukacs (1955). □

This gives all $(z_k)$ with finite $I$. General case follows.
Recall that

\[
\begin{cases}
  n = \psi_1(\theta)N \\
  m = \psi_1(\mu - \theta)N + \gamma N^\alpha
\end{cases}
\]

\[\gamma > 0, \; \alpha > 2/3.\]
Proof of off-characteristic CLT

Recall that

\[
\begin{cases}
  n = \Psi_1(\theta)N \\
  m = \Psi_1(\mu - \theta)N + \gamma N^\alpha
\end{cases}
\]

\(\gamma > 0, \alpha > 2/3.\)

Set \(m_1 = \lfloor \Psi_1(\mu - \theta)N \rfloor.\)
Proof of off-characteristic CLT

Recall that
\[
\begin{cases}
  n = \Psi_1(\theta)N \\
  m = \Psi_1(\mu - \theta)N + \gamma N^\alpha
\end{cases}
\]

Set \( m_1 = \lfloor \Psi_1(\mu - \theta)N \rfloor \). Since \( Z_{m,n} = Z_{m_1,n} \cdot \prod_{i=m_1+1}^{m} U_{i,n} \)
Proof of off-characteristic CLT

Recall that

\[
\begin{aligned}
  n &= \psi_1(\theta) N \\
  m &= \psi_1(\mu - \theta) N + \gamma N^\alpha
\end{aligned}
\]

Set \( m_1 = [\psi_1(\mu - \theta) N] \). Since \( Z_{m,n} = Z_{m_1,n} \cdot \prod_{i=m_1+1}^{m} U_{i,n} \)

\[
N^{-\alpha/2} \log Z_{m,n} = N^{-\alpha/2} \log Z_{m_1,n} + N^{-\alpha/2} \sum_{i=m_1+1}^{m} \log U_{i,n}
\]

First term on the right is \( O(N^{1/3-\alpha/2}) \rightarrow 0 \). Second term is a sum of order \( N^\alpha \) i.i.d. terms. \( \Box \)
Variance identity

Exit point of path from $x$-axis

\[ \xi_x = \max\{ k \geq 0 : x_i = (i, 0) \text{ for } 0 \leq i \leq k \} \]
Exit point of path from $x$-axis

$\xi_x = \max\{k \geq 0 : x_i = (i, 0) \text{ for } 0 \leq i \leq k\}$

For $\theta, x > 0$ define positive function

$$L(\theta, x) = \int_0^x (\psi_0(\theta) - \log y)x^{-\theta}y^{\theta-1}e^{x-y} \, dy$$
Variance identity

Exit point of path from x-axis

\[ \xi_x = \max\{ k \geq 0 : x_i = (i, 0) \text{ for } 0 \leq i \leq k \} \]

For \( \theta, x > 0 \) define positive function

\[
L(\theta, x) = \int_0^x (\psi_0(\theta) - \log y) x^{-\theta} y^{\theta-1} e^{x-y} \, dy
\]

**Theorem.** For the model with boundary,

\[
\text{Var}[\log Z_{m,n}] = n\psi_1(\mu - \theta) - m\psi_1(\theta) + 2 E_{m,n} \left[ \xi_x \sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1}) \right]
\]
Variance identity, sketch of proof

\[ N = \log Z_{m,n} - \log Z_{0,n} \]

\[ W = \log Z_{0,n} \]

\[ E = \log Z_{m,n} - \log Z_{m,0} \]

\[ S = \log Z_{m,0} \]
Variance identity, sketch of proof

\[ N = \log Z_{m,n} - \log Z_{0,n} \]
\[ W = \log Z_{0,n} \]
\[ E = \log Z_{m,n} - \log Z_{m,0} \]
\[ S = \log Z_{m,0} \]

\[ \text{Var} \left[ \log Z_{m,n} \right] = \text{Var}(W + N) \]
\[ = \text{Var}(W) + \text{Var}(N) + 2 \text{Cov}(W, N) \]
\[ = \text{Var}(W) + \text{Var}(N) + 2 \text{Cov}(S + E - N, N) \]
\[ = \text{Var}(W) - \text{Var}(N) + 2 \text{Cov}(S, N) \quad (E, N \text{ ind.}) \]
\[ = n\Psi_1(\mu - \theta) - m\Psi_1(\theta) + 2 \text{Cov}(S, N). \]
To differentiate w.r.t. parameter $\theta$ of $S$ while keeping $W$ fixed, introduce a separate parameter $\rho (= \mu - \theta)$ for $W$.

$$-\text{Cov}(S, N) = \frac{\partial}{\partial \theta} \mathbb{E}(N)$$
To differentiate w.r.t. parameter $\theta$ of $S$ while keeping $W$ fixed, introduce a separate parameter $\rho (= \mu - \theta)$ for $W$.

\[-\text{Cov}(S, N) = \frac{\partial}{\partial \theta} \mathbb{E}(N) = \tilde{\mathbb{E}} \left[ \frac{\partial}{\partial \theta} \log Z_{m,n}(\theta) \right] \]
To differentiate w.r.t. parameter \( \theta \) of \( S \) while keeping \( W \) fixed, introduce a separate parameter \( \rho (= \mu - \theta) \) for \( W \).

\[
-\text{Cov}(S, N) = \frac{\partial}{\partial \theta} \mathbb{E}(N) = \tilde{\mathbb{E}} \left[ \frac{\partial}{\partial \theta} \log Z_{m,n}(\theta) \right]
\]

when

\[
Z_{m,n}(\theta) = \sum_{x. \in \Pi_{m,n}} \xi_{x} \prod_{i=1}^{\xi_{x}} H_{\theta}(\eta_{i})^{-1} \cdot \prod_{k=\xi_{x}+1}^{m+n} Y_{x_{k}} \quad \text{with}
\]

\[
\eta_{i} \sim \text{IID Unif}(0,1), \quad H_{\theta}(\eta) = F_{\theta}^{-1}(\eta), \quad F_{\theta}(x) = \int_{0}^{x} \frac{y^{\theta-1}e^{-y}}{\Gamma(\theta)} \, dy.
\]
To differentiate w.r.t. parameter $\theta$ of $S$ while keeping $W$ fixed, introduce a separate parameter $\rho \ (= \mu - \theta)$ for $W$.

$$-\text{Cov}(S, N) = \frac{\partial}{\partial \theta} \mathbb{E}(N) = \tilde{\mathbb{E}} \left[ \frac{\partial}{\partial \theta} \log Z_{m,n}(\theta) \right]$$

when $Z_{m,n}(\theta) = \sum_{x. \in \Pi_{m,n}} \prod_{i=1}^{\xi_x} H_{\theta}(\eta_i)^{-1} \cdot \prod_{k=\xi_x+1}^{m+n} Y_{x_k}$ with

$$\eta_i \sim \text{IID Unif}(0, 1), \quad H_{\theta}(\eta) = F_{\theta}^{-1}(\eta), \quad F_{\theta}(x) = \int_0^x y^{\theta-1} e^{-y} \frac{1}{\Gamma(\theta)} \, dy.$$

Differentiate:

$$\frac{\partial}{\partial \theta} \log Z_{m,n}(\theta) = - E_{Q_{m,n}} \left[ \sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1}) \right].$$
Together:

$$\text{Var} \left[ \log Z_{m,n} \right] = n\psi_1(\mu - \theta) - m\psi_1(\theta) + 2 \text{Cov}(S, N)$$

$$= n\psi_1(\mu - \theta) - m\psi_1(\theta) + 2 E_{m,n} \left[ \sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1}) \right].$$

This was the claimed formula. □
Sketch of upper bound proof

The argument develops an inequality that controls both $\log Z$ and $\xi_x$ simultaneously. Introduce an auxiliary parameter $\lambda = \theta - bu/N$. 
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Since $H_\lambda(\eta) \leq H_\theta(\eta)$,

$$Q^{\theta,\omega} \{\xi_x \geq u\} = \frac{1}{Z(\theta)} \sum_x 1\{\xi_x \geq u\} W(\theta) \leq \frac{Z(\lambda)}{Z(\theta)} \cdot \prod_{i=1}^{\lfloor u \rfloor} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)}.$$
For $1 \leq u \leq \delta N$ and $0 < s < \delta$,

$$
P\left[ Q^\omega \{ \xi_x \geq u \} \geq e^{-su^2/N} \right] \leq P\left\{ \prod_{i=1}^{\left\lfloor u \right\rfloor} \frac{H_{\lambda}(\eta_i)}{H_{\theta}(\eta_i)} \geq \alpha \right\}$$

$$+ P\left( \frac{Z(\lambda)}{Z(\theta)} \geq \alpha^{-1} e^{-su^2/N} \right).$$
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+ \mathbb{P}\left( \frac{Z(\lambda)}{Z(\theta)} \geq \alpha^{-1} e^{-su^2/N} \right).
$$

Choose $\alpha$ right. Bound these probabilities with Chebychev which brings $\text{Var}(\log Z)$ into play. In the characteristic rectangle $\text{Var}(\log Z)$ can be bounded by $E(\xi_x)$. The end result is this inequality:

$$
\mathbb{P}\left[ Q^\omega \{ \xi_x \geq u \} \geq e^{-su^2/N} \right] \leq \frac{CN^2}{u^4} E(\xi_x) + \frac{CN^2}{u^3}
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Handle $u \geq \delta N$ with large deviation estimates. In the end, integration gives the moment bounds.
For $1 \leq u \leq \delta N$ and $0 < s < \delta$,

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$$

$$
+ \mathbb{P}\left( \frac{Z(\lambda)}{Z(\theta)} \geq \alpha^{-1} e^{-su^2/N} \right).
$$

Choose $\alpha$ right. Bound these probabilities with Chebychev which brings $\mathbb{V}ar(\log Z)$ into play. In the characteristic rectangle $\mathbb{V}ar(\log Z)$ can be bounded by $E(\xi_x)$. The end result is this inequality:

$$
\mathbb{P}\left[ Q^\omega \{ \xi_x \geq u \} \geq e^{-su^2/N} \right] \leq \frac{CN^2}{u^4} E(\xi_x) + \frac{CN^2}{u^3}
$$

Handle $u \geq \delta N$ with large deviation estimates. In the end, integration gives the moment bounds. **END.**
Polymer in a Brownian environment

**Environment:** independent Brownian motions $B_1, B_2, \ldots, B_n$

**Partition function (without boundary conditions):**

$$Z_{n,t}(\beta) = \int_{0<s_1<\ldots<s_{n-1}<t} \exp[\beta(B_1(s_1) + B_2(s_2) - B_2(s_1) +$$

$$+ B_3(s_3) - B_3(s_2) + \cdots + B_n(t) - B_n(s_{n-1})]) \, ds_{1,n-1}$$