Large deviations and fluctuation exponents for some polymer models

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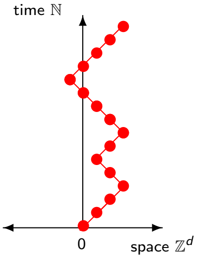
Directed polymer in a random environment



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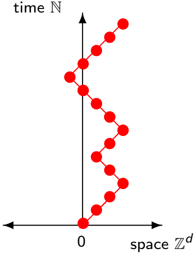

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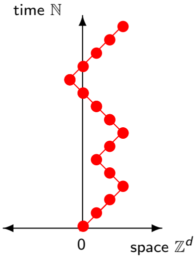

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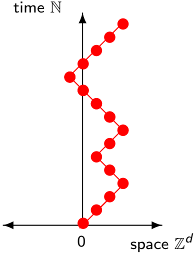

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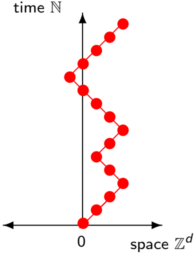

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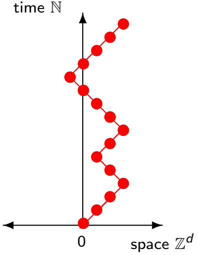

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$\mathbb{P}$ probability distribution on ω, often $\{\omega(x,t)\}$ i.i.d.

Key quantities again:

- Quenched measure $Q_n\{x(\cdot)\} = Z_n^{-1} \exp\Big\{\beta \sum_{t=1}^{n} \omega(x(t), t)\Big\}$
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- Dependence on β and d

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Introduced shift $(T_x\omega)_y = \omega_{x+y}$, steps $Z_k = X_k - X_{k-1} \in \mathcal{R}$,

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$g(\omega, z_{1,\ell})$ is a function on $\boldsymbol{\Omega}_\ell = \Omega \times \mathcal{R}^\ell$.

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Defines kernel p on $\mathbf{\Omega}_\ell$: $p(\eta, S_z\eta) = |\mathcal{R}|^{-1}$.

Entropy

Let $\mu_0 = \Omega$-marginal of $\mu \in \mathcal{M}_1(\mathbf{\Omega}_\ell)$. Define

$$H_{\mathbb{P}}(\mu) = \begin{cases} \inf\left\{ H(\mu \times q \,|\, \mu \times p) : \mu q = \mu \right\} & \text{if } \mu_0 \ll \mathbb{P} \\ \infty & \text{otherwise.} \end{cases}$$

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$H_{\mathbb{P}}$ is convex but not lower semicontinuous.

Assumptions.

- Environment $\{\omega_x\}$ IID under $\mathbb{P}$.
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Theorem. (Rassoul-Agha, S, Yilmaz) Deterministic limit

$$\Lambda(g) = \lim_{n\to\infty} n^{-1} \log E_0\big[e^{nR_n(g)}\big]$$

exists $\mathbb{P}$-a.s. and

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Remark. True more generally, e.g. ergodic $\mathbb{P}$ if g is bounded.

Quenched weak LDP (large deviation principle) under Q_n.

$$Q_n(A) = \frac{1}{E_0\big[e^{nR_n(g)}\big]}\, E_0\big[e^{nR_n(g)}\mathbf{1}_A(\omega, Z_{1,\infty})\big]$$

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Theorem. (RSY) Assumptions as above and $\Lambda(g)$ finite. Then $\mathbb{P}$-a.s. for compact $F \subseteq \mathcal{M}_1(\mathbf{\Omega}_\ell)$ and open $G \subseteq \mathcal{M}_1(\mathbf{\Omega}_\ell)$:

$$\overline{\lim_{n\to\infty}} n^{-1} \log Q_n\{R_n \in F\} \leq - \inf_{\mu \in F} I(\mu),$$

$$\underline{\lim_{n\to\infty}} n^{-1} \log Q_n\{R_n \in G\} \geq - \inf_{\mu \in G} I(\mu).$$

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Early results: diffusive behavior for $d \geq 3$ and small $\beta > 0$:

1988 Imbrie and Spencer: $n^{-1} E^Q(|x(n)|^2) \to c \quad \mathbb{P}$-a.s.

1989 Bolthausen: quenched CLT for $n^{-1/2} x(n)$.

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In the opposite direction: if $d = 1, 2$, or $d \geq 3$ and β large enough, then $\exists\ c > 0$ s.t.

$$\varlimsup_{n\to\infty} \max_z Q_n\{x(n) = z\} \geq c \quad \mathbb{P}\text{-a.s.}$$

(Carmona and Hu 2002, Comets, Shiga, and Yoshida 2003)

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Results: these exact exponents for three particular 1+1 dimensional models.

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- Licea, Newman, Piza 1995-96: corresponding results for first passage percolation

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(3) Continuum directed polymer, or Hopf-Cole solution of the Kardar-Parisi-Zhang (KPZ) equation:

   (i) Initial height function given by two-sided Brownian motion. (Joint with M. Balázs and J. Quastel.)

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Next details on (3.i), then details on (1).

Hopf-Cole solution to KPZ equation

KPZ eqn for height function $h(t,x)$ of a 1+1 dim interface:

$$h_t = \frac{1}{2}\,h_{xx} - \frac{1}{2}\,(h_x)^2 + \dot{W}$$

where $\dot{W}$ = Gaussian space-time white noise.

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Bertini-Giacomin (1997): h can be obtained as a weak limit via a smoothing and renormalization of KPZ.

WASEP connection

$\zeta_\varepsilon(t, x)$ height process of weakly asymmetric simple exclusion s.t.

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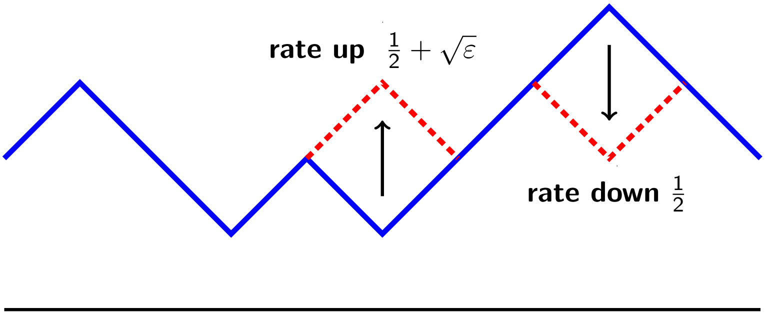

WASEP connection

Jumps:

$$\zeta_\varepsilon(x) \longrightarrow \begin{cases} \zeta_\varepsilon(x) + 2 & \text{with rate } \frac{1}{2} + \sqrt{\varepsilon} \quad \text{if } \zeta_\varepsilon(x) \text{ is a local min} \\ \zeta_\varepsilon(x) - 2 & \text{with rate } \frac{1}{2} \quad \text{if } \zeta_\varepsilon(x) \text{ is a local max} \end{cases}$$

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Thm. As $\varepsilon \searrow 0$, $h_\varepsilon \Rightarrow h$ (Bertini-Giacomin 1997).

Fluctuation bounds

From coupling arguments for WASEP

$$C_1 t^{2/3} \leq \mathrm{Var}(h_\varepsilon(t,0)) \leq C_2 t^{2/3}$$

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The lower bound comes from control of rescaled correlations

$$S_\varepsilon(t,x) = 4\varepsilon^{-1}\,\mathrm{Cov}\big[\eta(\varepsilon^{-2}t,\,\varepsilon^{-1}x)\,,\,\eta(0,0)\big]$$

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With some control over tails we arrive at

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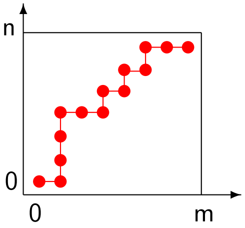
1+1 dimensional lattice polymer with log-gamma weights

Fix both endpoints.



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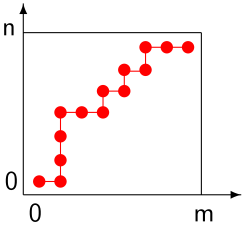

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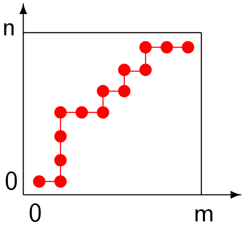

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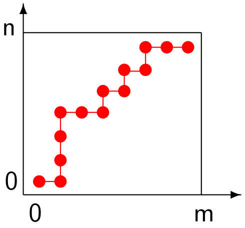

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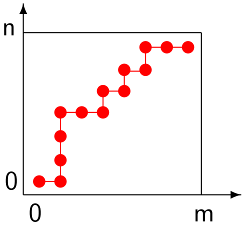

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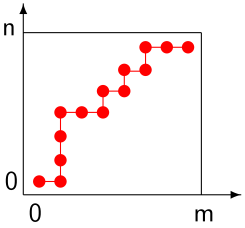

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averaged measure $P_{m,n}(x_\cdot) = \mathbb{E} Q_{m,n}(x_\cdot)$

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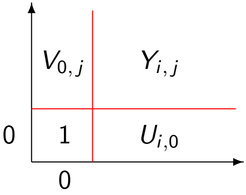

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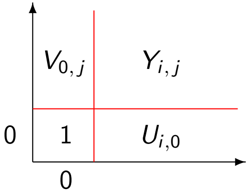

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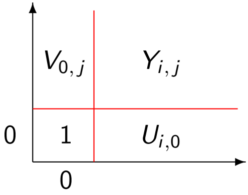

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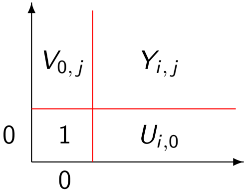

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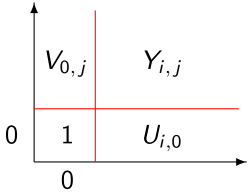

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- $\mathbb{E}(\log U) = -\Psi_0(\theta)$ and $\mathbb{V}\mathrm{ar}(\log U) = \Psi_1(\theta)$

Variance bounds for log Z

With $0 < \theta < \mu$ fixed and $N \nearrow \infty$ assume

$$|\,m - N\Psi_1(\mu - \theta)\,| \le CN^{2/3} \quad \text{and} \quad |\,n - N\Psi_1(\theta)\,| \le CN^{2/3} \tag{1}$$

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Theorem

Suppose $n = \Psi_1(\theta)N$ and $m = \Psi_1(\mu - \theta)N + \gamma N^{\alpha}$ with $\gamma > 0$, $\alpha > 2/3$. Then

$$N^{-\alpha/2}\Big\{\log Z_{m,n} - \mathbb{E}\big(\log Z_{m,n}\big)\Big\} \Rightarrow \mathcal{N}\big(0, \gamma\Psi_1(\theta)\big)$$

Fluctuation bounds for path

$v_0(j)$ = leftmost, $v_1(j)$ = rightmost point of $x_\bullet$ on horizontal line:

$$v_0(j) = \min\{i \in \{0, \ldots, m\} : \exists k : x_k = (i, j)\}$$

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Theorem

Assume (m, n) as previously and $0 < \tau < 1$. Then

(a) $P\Big\{ v_0(\lfloor \tau n \rfloor) < \tau m - bN^{2/3} \;\text{ or }\; v_1(\lfloor \tau n \rfloor) > \tau m + bN^{2/3} \Big\} \le \dfrac{C}{b^3}$

(b) $\forall \varepsilon > 0 \; \exists \delta > 0$ such that

$$\varlimsup_{N\to\infty} P\{\, \exists k \text{ such that } |\, x_k - (\tau m, \tau n)| \le \delta N^{2/3} \,\} \le \varepsilon.$$

Results for log-gamma polymer summarized

With reciprocals of gammas for weights, both endpoints of the polymer fixed and the right boundary conditions on the axes, we have identified the one-dimensional exponents

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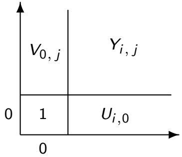
Next some key points of the proof.

Burke property for log-gamma polymer with boundary



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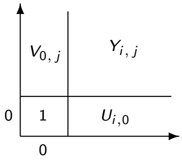

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Compute $Z_{m,n}$ for all $(m, n) \in \mathbb{Z}_+^2$ and then define

$$U_{m,n} = \frac{Z_{m,n}}{Z_{m-1,n}} \qquad V_{m,n} = \frac{Z_{m,n}}{Z_{m,n-1}} \qquad X_{m,n} = \left(\frac{Z_{m,n}}{Z_{m+1,n}} + \frac{Z_{m,n}}{Z_{m,n+1}} \right)^{-1}$$

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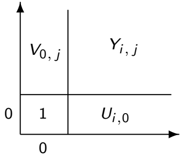

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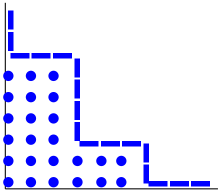
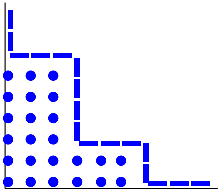
For an undirected edge f: $\quad T_f = \begin{cases} U_x & f = \{x - e_1, x\} \\ V_x & f = \{x - e_2, x\} \end{cases}$



down-right path (z_k) with edges $f_k = \{z_{k-1}, z_k\}$, $k \in \mathbb{Z}$

interior points $\mathcal{I}$ of path (z_k)



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Variables $\{T_{f_k}, X_z : k \in \mathbb{Z},\, z \in \mathcal{I}\}$ are independent with marginals $U^{-1} \sim \text{Gamma}(\theta)$, $V^{-1} \sim \text{Gamma}(\mu - \theta)$,
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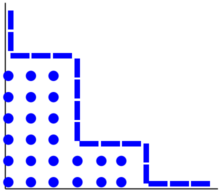

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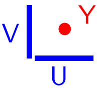
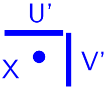
"Burke property" because the analogous property for last-passage is a generalization of Burke's Theorem for M/M/1 queues, via the last-passage representation of M/M/1 queues in series.

Proof of Burke property

Induction on $\mathcal{I}$ by flipping a growth corner:





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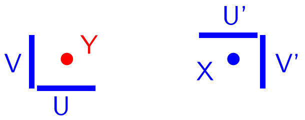

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Lemma. Given that (U, V, Y) are independent positive r.v.'s, $(U', V', X) \stackrel{d}{=} (U, V, Y)$ iff (U, V, Y) have the gamma distr's.

Proof. "if" part by computation, "only if" part from a characterization of gamma due to Lukacs (1955). $\square$

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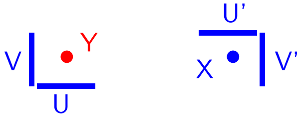

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Lemma. Given that (U, V, Y) are independent positive r.v.'s, $(U', V', X) \stackrel{d}{=} (U, V, Y)$ iff (U, V, Y) have the gamma distr's.

Proof. "if" part by computation, "only if" part from a characterization of gamma due to Lukacs (1955). $\square$

This gives all (z_k) with finite $\mathcal{I}$. General case follows.

Proof of off-characteristic CLT

Recall that
$$\begin{cases} n = \Psi_1(\theta)N \\ m = \Psi_1(\mu-\theta)N + \gamma N^{\alpha} \end{cases} \qquad \gamma > 0,\ \alpha > 2/3.$$

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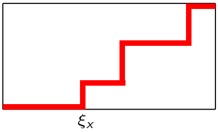
$$N^{-\alpha/2}\,\overline{\log Z_{m,n}} \;=\; N^{-\alpha/2}\,\overline{\log Z_{m_1,n}} \;+\; N^{-\alpha/2}\sum_{i=m_1+1}^{m}\overline{\log U_{i,n}}$$

First term on the right is $O(N^{1/3-\alpha/2}) \to 0$. Second term is a sum of order N^α i.i.d. terms. □

Variance identity



Exit point of path from x-axis

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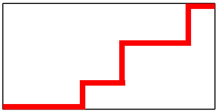

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For $\theta, x > 0$ define positive function

$$L(\theta, x) = \int_0^x \big(\Psi_0(\theta) - \log y\big) x^{-\theta} y^{\theta-1} e^{x-y}\, dy$$

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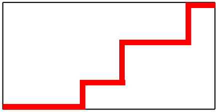

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Theorem. For the model with boundary,

$$\mathbb{V}\mathrm{ar}\big[\log Z_{m,n}\big] = n\Psi_1(\mu - \theta) - m\Psi_1(\theta) + 2\, E_{m,n}\Big[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\Big]$$

Variance identity, sketch of proof

$$N = \log Z_{m,n} - \log Z_{0,n}$$

$$W = \log Z_{0,n} \qquad E = \log Z_{m,n} - \log Z_{m,0}$$

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$$\begin{aligned}
\mathbb{V}\mathrm{ar}\big[\log Z_{m,n}\big] &= \mathbb{V}\mathrm{ar}(W+N) \\
&= \mathbb{V}\mathrm{ar}(W) + \mathbb{V}\mathrm{ar}(N) + 2\,\mathbb{C}\mathrm{ov}(W,N) \\
&= \mathbb{V}\mathrm{ar}(W) + \mathbb{V}\mathrm{ar}(N) + 2\,\mathbb{C}\mathrm{ov}(S+E-N,N) \\
&= \mathbb{V}\mathrm{ar}(W) - \mathbb{V}\mathrm{ar}(N) + 2\,\mathbb{C}\mathrm{ov}(S,N) \qquad (E, N \text{ ind.}) \\
&= n\Psi_1(\mu-\theta) - m\Psi_1(\theta) + 2\,\mathbb{C}\mathrm{ov}(S,N).
\end{aligned}$$

To differentiate w.r.t. parameter θ of S while keeping W fixed, introduce a separate parameter ρ $(= \mu - \theta)$ for W.

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Differentiate: $\quad \displaystyle \frac{\partial}{\partial\theta} \log Z_{m,n}(\theta) \;=\; -\, E^{Q_{m,n}}\Big[\, \sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1}) \Big].$

Together:

$$\mathbb{V}\mathrm{ar}\big[\log Z_{m,n}\big] = n\Psi_1(\mu-\theta) - m\Psi_1(\theta) + 2\,\mathbb{C}\mathrm{ov}(S,N)$$

$$= \; n\Psi_1(\mu-\theta) - m\Psi_1(\theta) + 2\,E_{m,n}\Bigg[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\Bigg].$$

This was the claimed formula. □

Sketch of upper bound proof

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Since $H_\lambda(\eta) \le H_\theta(\eta)$,

$$Q^{\theta,\omega}\{\xi_x \ge u\} = \frac{1}{Z(\theta)} \sum_{x_{\cdot}} \mathbf{1}\{\xi_x \ge u\} W(\theta) \;\le\; \frac{Z(\lambda)}{Z(\theta)} \cdot \prod_{i=1}^{\lfloor u \rfloor} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)}.$$

For $1 \le u \le \delta N$ and $0 < s < \delta$,

$$\mathbb{P}\Big[Q^{\omega}\{\xi_x \ge u\} \ge e^{-su^2/N}\Big] \le \mathbb{P}\Big\{\prod_{i=1}^{\lfloor u \rfloor} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)} \ge \alpha\Big\} + \mathbb{P}\Big(\frac{Z(\lambda)}{Z(\theta)} \ge \alpha^{-1} e^{-su^2/N}\Big).$$

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Choose α right. Bound these probabilities with Chebychev which brings $\mathbb{V}\mathrm{ar}(\log Z)$ into play. In the characteristic rectangle $\mathbb{V}\mathrm{ar}(\log Z)$ can be bounded by $E(\xi_x)$. The end result is this inequality:

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Polymer in a Brownian environment

Environment: independent Brownian motions $B_1, B_2, \ldots, B_n$

Partition function (without boundary conditions):

$$Z_{n,t}(\beta) = \int_{0<s_1<\cdots<s_{n-1}<t} \exp\Big[\beta\Big(B_1(s_1) \;+\; B_2(s_2) - B_2(s_1) +$$

$$+\; B_3(s_3) - B_3(s_2) \;+\; \cdots +\; B_n(t) - B_n(s_{n-1})\Big)\Big]\, ds_{1,n-1}$$