WXML Final Report: Number Theory and Noise

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1 Introduction

Three undergraduates participated in this project during Winter quarter 2017: Penny Espinoza, Emily Flanagan, and Jesse Rivera. The following are their reports on their work during this quarter.

2 Progress

2.1 P. Espinoza's commentary

2.1.1 Analyzing the sequences of differences between terms of a sequence (A153777)

A153777 is a sequence S such that 1 is in S, and if x is in S, then $5x-1$ and $5x+1$ are in S. Thus, the primary definition of the sequence is generational in nature, with each generation adding twice as many terms as the previous generation.

- First generation only: 1
- Add second generation, where $4 = 5*1-1$ and $6 = 5*1+1$: $1, 4, 6$
- Add third generation, where $19 = 5*4-1$ and $21 = 5*4+1$, and $29 = 5*6-1$ and $31 = 5*6+1$: $1, 4, 6, 19, 21, 29, 31$

- Add fourth generation: $1, 4, 6, 19, 21, 29, 31, 94, 96, 104, 106, 144, 146, 154, 156$
- $\ldots$

However, one can take another perspective on the sequence by considering the differences between terms of the sequence. If the first 31 terms of the sequence, representing the first five generations, are as follows:

$$1, 4, 6, 19, 21, 29, 31, 94, 96, 104, 106, 144, 146, 154, 156, 469, 471,$$

$$479, 481, 519, 521, 529, 531, 719, 721, 729, 731, 769, 771, 779, 781$$

Then the sequence representing the differences between consecutive terms of the sequence is shown below:

$$3, 2, 13, 2, 8, 2, 63, 2, 8, 2, 38, 2, 8, 2, 313, 2, 8, 2, 38, 2, 8, 2, 188, 2, 8, 2, 38, 2, 8, 2$$

The difference sequence has a definite pattern than can be composed of sequences within sequences.

The first sequence, which here will be called A, is defined as $A(1) = 2$, and $A(n) = 5 * A(n-1) - 2$.

$$2, 8, 38, 188, \ldots$$

There is then a sequence of sequences B such that $B(1) = A(1)$ and $B(n) = [B(n-1)A(n)B(n-1)]$. B represents the differences between terms in a single generation of the main sequence.

$$[2], [2, 8, 2], [2, 8, 2, 38, 2, 8, 2], [2, 8, 2, 38, 2, 8, 2, 188, 2, 8, 2, 38, 2, 8, 2], \ldots$$

A third sequence C is defined such that $C(1) = 3$ and $C(n) = C(n-1) + 10 * 5^{n-2}$. This sequence represents the gap between generations of the main sequence.

$$3, 13, 63, 313, \ldots$$

Finally, the entire difference sequence D can be defined as $D(n) = [C(n)B(n)]$.

$$[3, 2], [13, 2, 8, 2], [63, 2, 8, 2, 38, 2, 8, 2], [383, 2, 8, 2, 38, 2, 8, 2, 188, 2, 8, 2, 38, 2, 8, 2], \ldots$$

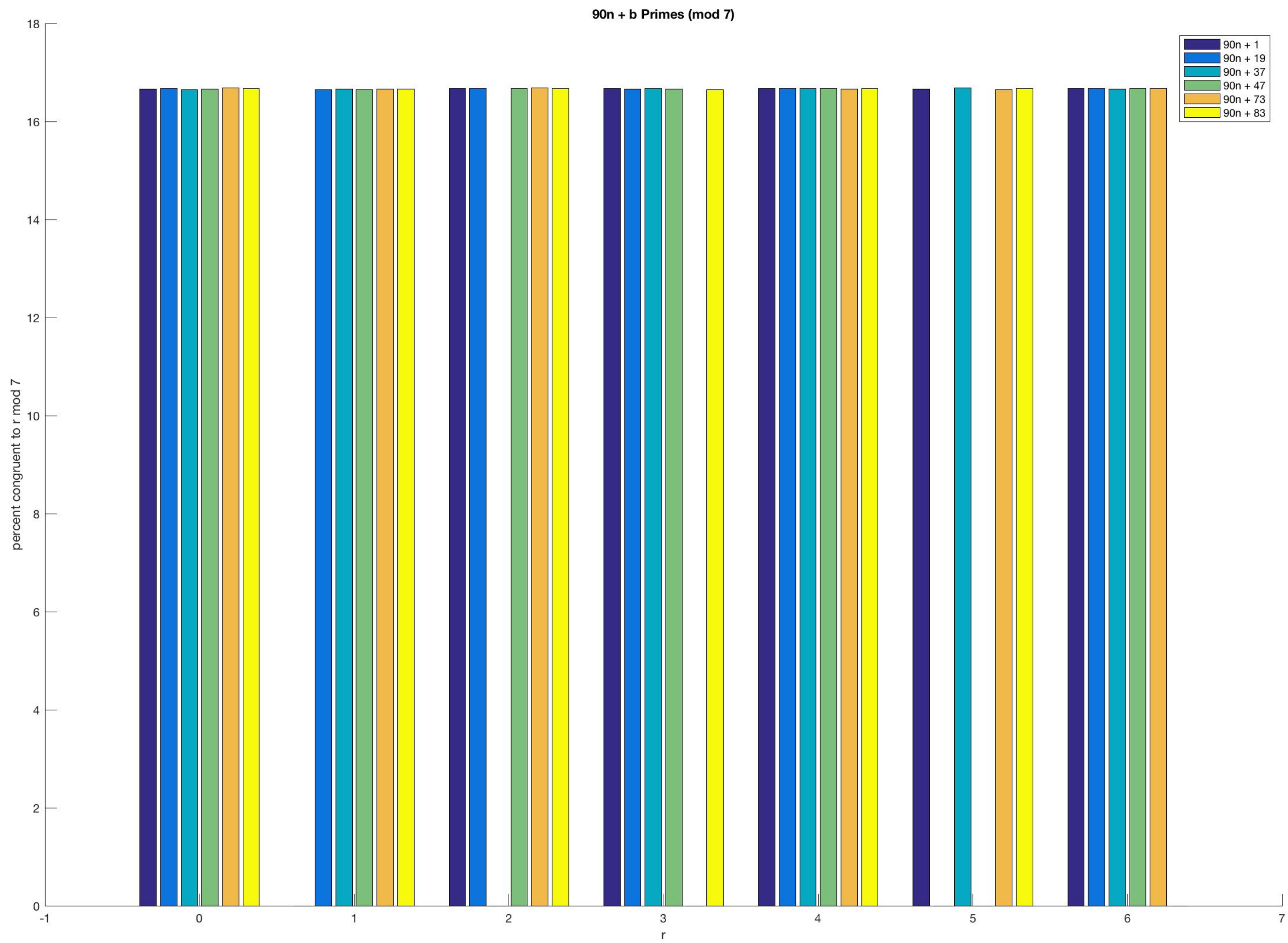
2.1.2 Exploring Dirichlet's Theorem on Primes in Arithmetic Progression

Dirichlet's Theorem on Primes in Arithmetic Progression states that if a and b are relatively prime, there will be an infinite number of primes of the form $an + b$ where n is any non-negative integer.

The OEIS contains many sequences that exploit this theorem. Below is a group of sequences that always uses $a = 90$, but varies the value of b.

Sequences of integers n such that $90n + b$ is prime	
b = 1	A181732
b = 19	A196000
b = 37	A198382
b = 47	A201734
b = 73	A195993
b = 83	A196007

An evaluation of the distribution of these sequences reveals that each sequence has lacks any members in one of the seven mod 7 residue classes:



Sequence $90n + 1$ has no members congruent to 1 mod 7.
Sequence $90n + 19$ has no members congruent to 5 mod 7.
Sequence $90n + 37$ has no members congruent to 2 mod 7.
Sequence $90n + 47$ has no members congruent to 5 mod 7.
Sequence $90n + 73$ has no members congruent to 3 mod 7.
Sequence $90n + 83$ has no members congruent to 6 mod 7.

One might wonder why certain residue classes are missing, and how they are dependent on b. Furthermore, a quantifiable answer to this question might reveal a way to identify classes of numbers that will never generate primes. Such an answer is given below.

For any prime p that is not a factor of a, let x be the $mod\ p$ multiplicative inverse of a (i.e., $ax \equiv 1(mod\ p)$). Then if $r \equiv -xb(mod\ p)$ and k is any non-negative integer, any

$$n = pk + r$$

will yield no primes in the arithmetic progression $an + b$, since $a(pk + r) + b$ will be divisible by p. (unless when $k = 0$ and $ar + b = p$, which *is* prime) Note that r is not uniquely specified, but represents a congruence class $mod\ p$. For simplicity in the following example, the lowest possible integer that fits the criteria is used.

Consider the sequence comprising prime numbers of form $90n + 19$. Then $a = 90$ and $b = 19$, and since this started with an analysis of residue classes mod 7, let $p = 7$. The mod 7 multiplicative inverse of 90 is 6.

$$90(6) \equiv 1(mod\ 7)$$

and $r \equiv (6)(-19)(mod 7) \equiv 5\ (\equiv 12 \equiv 19 \ldots)$.
In this case, all terms with $n = 7k + 5$ have the form

$$90(7k + 5) + 19 = 630k + 469 = 7(90k + 67)$$

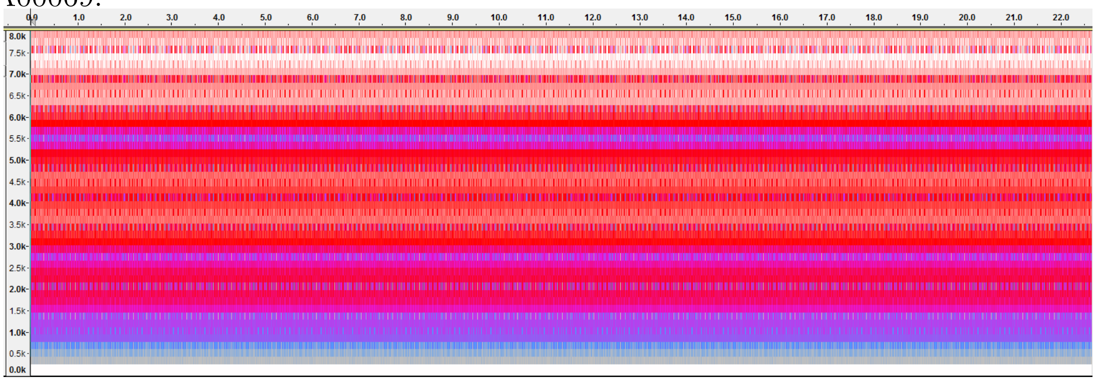
so are not prime.

A few further questions are suggested by this analysis:

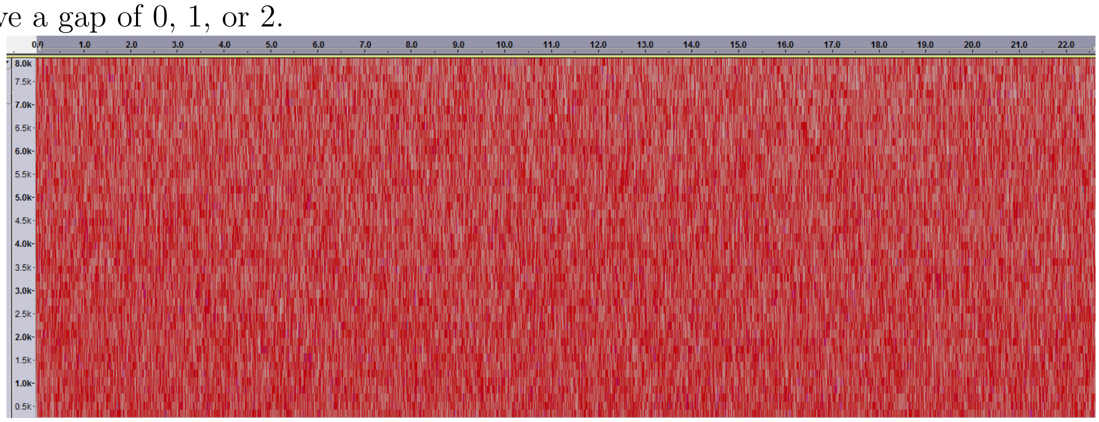
- Does this cover all circumstances where the arithmetic progression does not result in a prime number?
- It is possible for $ar + b = p$ - consider $a = 2$, $b = 1$, $p = 3$: $2(2) \equiv 1(mod\ 3)$ and $r = 1$. Then $ar + b = 2(1) + 1 = 3$, which is prime. Is it possible to give a general characterization of such circumstances?

2.2 Emily Flanagan's Commentary

Of all the sequences I worked with this quarter, the one that I spent the most time investigating was A001969, or the "Evil Numbers." Evil numbers are those that have an even number of 1's in their binary expansion. The compliment of this sequence, A000069, or numbers with an odd number of 1's in their binary or base 2 expansion, are called "Odious Numbers." The sounds of these two sequences are almost identical. Below is the spectrogram of A00069:



For comparative purposes, below is a spectrogram of a similar sequence to A000069 and A001969. To generate this sequence, we first noticed that the gaps between numbers included in the sequence was always 0, 1, or 2 with the same break never repeating more than once in a row. The breaks of 0, 1, and 2 occur with the same frequency, meaning no one break occurs more frequently than another. Thus, the spectrogram of the sequence below was created by randomly deciding if the next number in the sequence should have a gap of 0, 1, or 2.



The spectrogram for the Odious Numbers is obviously more organized and has specific frequencies that are represented, whereas the randomly generated sequence's spectrogram lacks any notion of a solid band. When listening to these sounds, the randomly generated sequence sounds like pure static. On the other hand, the Odious and Evil numbers, while static sounding, lack the grittyness that the randomly generated sequence has.

Later in the quarter, we decided to extend the definition of an Evil Number and created what we call "Bvil Numbers." Bvil numbers are numbers such that the sum of their digits in some base b is divisible by the base b. Our group was then able to prove that the density for bvil numbers in any base is $\frac{1}{b}$. We did so by proving the following theorem:

Theorem 1. *Let k be any integer representing the sum of digits in base b of some number n. Exactly one of the set $\{k,\ k-1,\ \ldots,\ k+b-1\}$ is divisible by b.*

2.3 J. Rivera's commentary

2.3.1 Beatty Sequences

Definition. Beatty sequences are those defined by $\{a_n\} = \{\lfloor n\alpha \rfloor\}$ where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. We are interested in those with $\alpha > 1$.

Sound. Beatty sequences produce sounds with many different frequencies. Each frequency can be heard in a constant tone, with some frequencies being stronger than others.

Properties Beatty Sequences are interesting for a number of reasons. The complement of a Beatty sequence is another Beatty sequence $\{b_n\} = \{\lfloor n\beta \rfloor\}$ where $\beta = \frac{\alpha}{\alpha-1}$ (note that this satisfies $\frac{1}{\alpha} + \frac{1}{\beta} = 1$).

Consequently, for every pair of Beatty sequences with irrational coefficients $\alpha, \beta > 1$ it will always be the case that either $\alpha \in (1,2)$ or $\beta \in (1,2)$ but not both.

Theorem 2. *For a pair of Beatty sequences with irrational coefficients $\alpha, \beta > 1$, exactly one of α, β is less than 2.*

Proof. Consider a Beatty sequence $\{a_n\}$ with irrational coefficient α, and its complement (with respect to $\mathbb{N}$) $\{b_n\}$ with irrational coefficient β. It is either the case that $\beta < 2$ or $\beta > 2$.

1. If $\beta < 2$ then

$$\begin{aligned}\beta = \frac{\alpha}{\alpha - 1} &< 2 \Leftrightarrow \\ \alpha &< 2(\alpha - 1) \Leftrightarrow \\ \alpha &< 2\alpha - 2 \Leftrightarrow \\ -\alpha &< -2 \Leftrightarrow \\ \alpha &> 2\end{aligned}$$

2. If $\beta > 2$ then

$$\begin{aligned}\beta = \frac{\alpha}{\alpha - 1} &> 2 \Leftrightarrow \\ \alpha &> 2(\alpha - 1) \Leftrightarrow \\ \alpha &> 2\alpha - 2 \Leftrightarrow \\ -\alpha &> -2 \Leftrightarrow \\ \alpha &< 2\end{aligned}$$

That is, $\beta < 2 \implies \alpha > 2$ and $\beta > 2 \implies \alpha < 2$ so it will always be the case that exactly one of α, β is less than 2 and the other greater than 2. $\square$

Since a sequence and its complement produce identical sounds, this means that the sound of any given Beatty sequence can be generated with an irrational coefficient between 1 and 2.

The first difference of a Beatty sequence will always be constrained to two values, $\lfloor \alpha \rfloor$ and $\lfloor \alpha \rfloor + 1$ (if we consider two sequences to be equivalent if they produce the same sound, then these two values are 1 and 2). Rather than the gaps themselves, it is the distribution of these gaps that results in the variety of frequencies we hear in the sound. The fact that α is irrational ensures that the resulting sound is not periodic; to add to this, the randomness introduced by the floor function also contributes to the complexity of the sound.

2.3.2 Sequence A051913

Definition. A positive integer n is in the sequence if and only if $\frac{\varphi(n)}{\varphi(\varphi(n))} = 3$ where φ is Euler's totient function.

Sound. Sequence A051913 has a very unique sound. It begins with a medium-pitched tone that gradually decreases as the sequence progresses. The sequence grows more sparse and the sound becomes more noisy with time. Perhaps the most interesting feature of this sequence is the cyclic pattern that can be heard throughout the sound. Much like the other features of the sound, as the sequence progresses this pattern becomes slower and more drawn out.

Properties. The following are several properties of Euler's totient function that are relevant to sequence A051913. While they do not fully explain the phenomena we hear in the sound, they give us some insight that may help us better understand the sequence and the sound that it produces.

Theorem 3. *$\varphi(n) = \frac{n}{3} \iff n = 2^s 3^t$ with $s, t \geq 1$*

Proof. Suppose $n = 2^s 3^t$ with $s, t \geq 1$. Then

$$\begin{aligned} \varphi(n) &= 2^s 3^t \prod_{p|n} \left(\frac{p-1}{p} \right) \\ &= 2^s 3^t \cdot \frac{2-1}{2} \cdot \frac{3-1}{3} \\ &= 2^s 3^t \cdot \frac{1}{3} \\ &= \frac{n}{3} \end{aligned}$$

Now suppose $\varphi(n) = \frac{n}{3}$. Then

$$\prod_{p|n} \left(\frac{p-1}{p} \right) = \frac{1}{3}$$

Since each $(p-1)$ in the numerator is less than the largest p in the denominator, the largest prime factor of n must be one of the prime factors of the denominator of the resulting fraction. Then $\varphi(n) = \frac{n}{3}$ implies that 3 is largest prime factor of n (since 3 is the only prime factor of the denominator of $\frac{1}{3}$). In the case where $n = 3^t$ we have

$$\varphi(n) = n \prod_{p|n} \left(\frac{p-1}{p} \right) = \frac{2n}{3} \neq \frac{n}{3}$$

So we conclude that n must have prime factors $2, 3$. $\square$

Corollary 1. *If $n = 3^a$ with $a \geq 2$ then $\frac{\varphi(n)}{\varphi(\varphi(n))} = 3$.*

Proof. Suppose $n = 3^a$ where $a \geq 2$. Then

$$\begin{aligned}\varphi(n) &= 3^a \prod_{p|n} \left(\frac{p-1}{p}\right) \\ &= 3^a \left(\frac{2}{3}\right) \\ &= 2^1 3^{a-1}\end{aligned}$$

$a - 1 \geq 1$ so by Theorem 3 we have that $\frac{\varphi(n)}{\varphi(\varphi(n))} = 3$. □

From this result we see that sequence A051913 has infinitely many terms.

Theorem 4. *If $n = 2^s 3^t p_1 \cdots p_k$ with $s \geq 1$, $t \geq 2$ and where p_k are distinct primes greater than 3 and $(p_k - 1)$ is 3-smooth for each k, then $\frac{\varphi(n)}{\varphi(\varphi(n))} = 3$.*

Proof. Suppose n is of the form described above. Then

$$\begin{aligned}\varphi(n) &= 2^s 3^t p_1 \cdots p_k \prod_{p|n} \left(\frac{p-1}{p}\right) \\ &= 2^s 3^t p_1 \cdots p_k \cdot \left(\frac{2-1}{2}\right)\left(\frac{3-1}{3}\right)\left(\frac{p_1-1}{p_1}\right)\cdots\left(\frac{p_k-1}{p_k}\right) \\ &= 2^s 3^t \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)(p_1-1)\cdots(p_k-1) \\ &= 2^s 3^{t-1}(p_1-1)\cdots(p_k-1)\end{aligned}$$

Since each $(p_k - 1)$ is 3-smooth, this expression can be written as $\varphi(n) = 2^q 3^r$ where $q \geq s$ and $r \geq t-1$. Then $q, r \geq 1$ so by Theorem 3 we have that $\varphi(\varphi(n)) = \frac{\varphi(n)}{3}$, which is equivalent to $\frac{\varphi(n)}{\varphi(\varphi(n))} = 3$.

□

Theorem 5. *If $\frac{\varphi(n)}{\varphi(\varphi(n))} = 3$ then n is of the form $2^a 3^b p_1 \cdots p_k$ where $a, b \geq 0$ and p_k are distinct primes with $(p_k - 1)$ 3-smooth for each k.*

Proof. Suppose $\frac{\varphi(n)}{\varphi(\varphi(n))} = 3$. Then $\varphi(\varphi(n)) = \frac{\varphi(n)}{3}$, so by Theorem 3 $\varphi(n) = 2^s 3^t$ with $s, t \geq 1$. That is,

$$\begin{aligned} 2^s 3^t &= n \prod_{p|n} \left(\frac{p-1}{p} \right) \\ &= n \left(\frac{p_1 - 1}{p_1} \right) \cdots \left(\frac{p_k - 1}{p_k} \right) \end{aligned}$$

Rearranging this equation we get

$$n = \frac{2^s 3^t p_1 \cdots p_k}{(p_1 - 1) \cdots (p_k - 1)}$$

Writing n as its prime factorization we have

$$2^a 3^b p_1^{e_1} \cdots p_k^{e_k} = \frac{2^s 3^t p_1 \cdots p_k}{(p_1 - 1) \cdots (p_k - 1)}$$

where $a, b \geq 0$. For this equality to be true it must be the case that $s \geq a$, $t \geq b$, and $e_k = 1$ for each k. Furthermore, $(p_1 - 1) \cdots (p_k - 1)$ must have prime factorization $2^{s-a} 3^{t-b}$, implying that each $(p_k - 1)$ is 3-smooth. □