WXML Final Report: Prime Spacings

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1 Introduction

1.1 The initial problem

In mathematics, prime gaps (defined as $r = p_n - p_{n-1}$) are widely studied due to the potential insights they may reveal about the distribution of the primes, as well as methods they could reveal about obtaining large prime numbers.

Our research of prime spacings focused on studying prime intervals, which is defined as $p_n - p_{n-1} - 1$, where p_n is the n^{th} prime. Specifically, we looked at what we defined as prime-prime intervals, that is primes having $p_n - p_{n-1} - 1 = r$ where r is prime. We have also researched how often different values of r show up as prime gaps or prime intervals, and why this distribution of values of r occurs the way that we have experimentally observed it to.

1.2 New directions

The idea of studying the primality of prime intervals had not been previously researched, so there were many directions to take from the initial proposed problem. We saw multiple areas that needed to be studied: most notably, the prevalence of prime-prime intervals in the set of all prime intervals, and the distribution of the values of prime intervals.

2 Progress

2.1 Computational

The first step taken was to see how many prime intervals up to the first n^{th} prime were prime. Our original thoughts were that the amount of prime-prime intervals would grow at a rate of $\pi(\pi(x))$, since the function π tells us how many primes there are up to some number x. However, our findings suggest that the ratio of prime-intervals that are prime is close to $\frac{2}{3}\pi(x)$ up to $\approx$ 100,000,000 and continues to drop as we take the intervals between higher and higher primes.

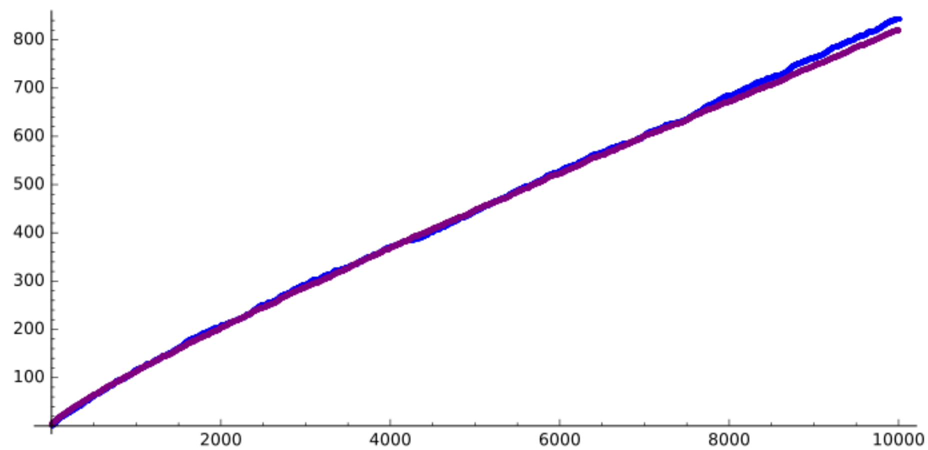
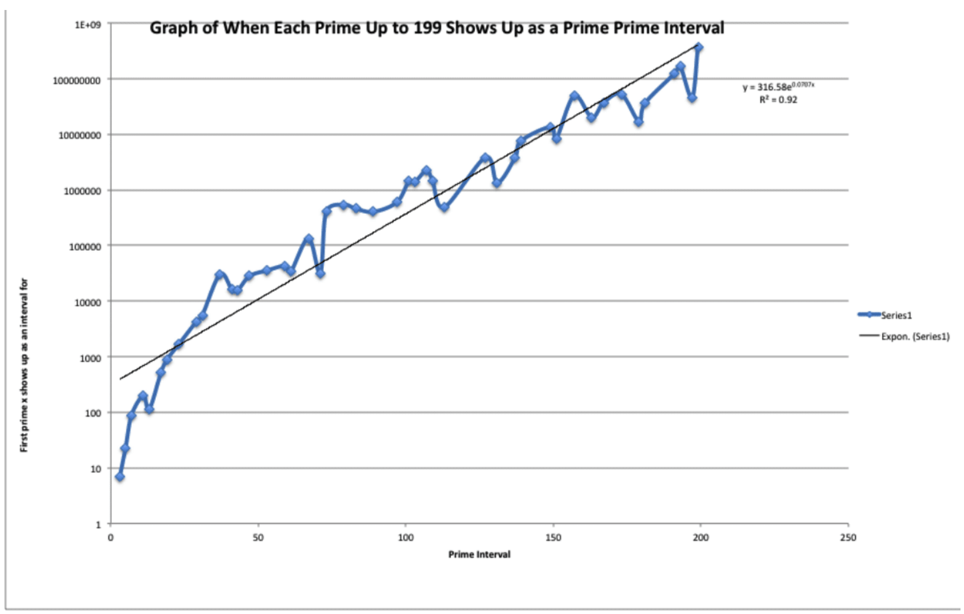


Figure 1: In purple we have $\pi(x)$, in blue we have $I(x)$

The decreasing behavior of this ratio below can be explained by the increasing sparsity of prime numbers as $x \to \infty$. Thus the intervals between them get larger and these larger numbers are less likely to be prime.

Primes up to x (in billions)	Proportion of Primes Intervals that are Prime
0.1	0.666412043485543
0.2	0.664083386339321
0.3	0.662695584
0.43	0.661420428
1	0.658453603669354
2	0.656046311
3	0.654667505
4	0.65371121

2.2 Computations Part 2



This graph shows when each prime-prime interval up to 199 first shows up. As we can see, there is somewhat of an order and this makes sense because as primes get larger, they get farter apart, so the intervals between them will begin to get larger.

2.3 Theoretical

We developed no mathematical proofs that related to prime-prime intervals. However, we have hypothesized that there are an infinite number of prime intervals that are prime. We also have evidence to believe that every prime shows up as a prime prime interval at least once. Since primes are infinite, there is a strong possibility that the prime intervals will continue to increase infinitely as well.

We also looked at when primes up to 199 initially show up as prime intervals. In doing so, we observed that 199 took an especially long time to appear as a prime interval, not doing so until approximately 378 million. Although this could be explained by the fact that 199 was the largest prime we looked at - thus it would require a larger integer to appear as a prime interval than other primes which are smaller in magnitude - we were enticed to explore why this might be, since other primes close in magnitude to 199 took much less time to appear. To answer our question, with the help of the Bateman-Horn conjecture, we looked at the density of primes produced by the polynomial $(x)(x+k)$, where k is a prime gap.

$$P(x) \sim \frac{C}{D} \int_2^x \frac{dt}{(\log t)^m},$$

$$C = \prod_p \frac{1 - N(p)/p}{(1 - 1/p)^m}$$

Figure 2: The Bateman-Horn conjecture. Top: the statement of the conjecture, where $P(x)$ is defined to be the number of primes less than x produced by some $f(x)$, a polynomial. Bottom: The formula for C, where $N(p)$ denotes the number of solutions to $f(x) \equiv 0 \mod p$.

On the whole, Bateman-Horn conjectured densities of prime gaps were larger than what we found experimentally, particularly for larger prime gaps.

Since we have primarily looked at the range of integers up to 430 million when gathering our experimental data, where the average prime-prime interval is about 16.8, it is possible that we have too small of a sample size to draw conclusions with our experimental data for larger prime-prime intervals.

In further researching prime gaps, we came across previous research on the subject of the frequency of prime gaps. In the paper, entitled Jumping Champions, Odlyzko, et al. present a heuristic argument in support of the idea that primorials, or prime gaps whose prime factorizations consist of distinct consecutive primes (e.g. $30 = 2 * 3 * 5$), appear the most often. In the paper, Odlyzko, et al. argue that 6 is the most common prime gap until approximately $1.7427 * 10^{35}$.

Noting this idea, we explored how often prime gaps appear using both the Bateman-Horn conjecture and experimental methods. Using the Bateman-Horn conjecture, in order to increase the density of primes produced by a polynomial $f(x) = (x)(x + k)$, the number of solutions to $f(x) \mod p$ for all primes p must be minimized. One way to do this is for k to have a prime factorization consisting of distinct consecutive primes, particularly smaller primes, since those influence the calculation of the conjectured density the most. At each prime p that is an element of the factorization of k, $(x)(x+k)$ will have one solution mod p and two solutions at every other prime. Hence the more distinct consecutive primes in the factorization of k, the smaller $N(p)$ will be.

A consequence of this idea would be that prime gaps whose factorizations skip primes appear less often relative to prime gaps similar in magnitude that have more consecutive primes in their factorization. For example, 127 (a Mersenne prime) appears less often as a prime interval than other primes similar in magnitude. Its corresponding gap, $128 = 2^7$, does not have a prime factorization consisting of distinct consecutive primes. In relation to 199, we noted that $200 = 2^3 * 5^2$ has a prime factorization which does not consist of distinct consecutive primes.

3 Future directions

In the future we would like to develop a hypothetical proportion of prime prime intervals up to infinity using similar methods to how the twin prime constant was derived. We would also like to continue gathering evidence for our hypothesis that each prime appears infinitely often as a prime interval.

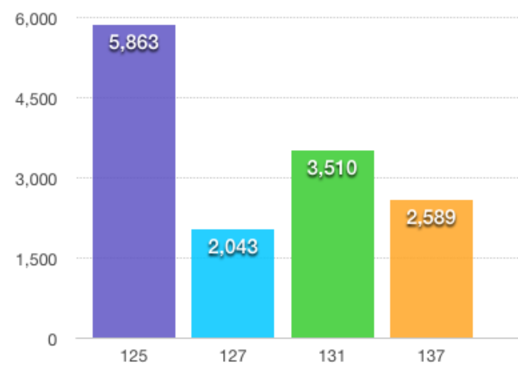


Figure 3: The number of primes up to 1 billion that have an interval of 125, 127, 131 and 137.

Another future goal is to verify the claims of Odlyzko, et al., with regards to the idea that 30 is the most frequent prime gap after a certain threshold around 10^{35}. We would like to further explore previous research on these related topics.

More specifically, we would like to narrow down the conditions that cause a prime interval to appear more or less often than other prime intervals. Although one way to ensure a higher conjectured density for $f(x) = (x)(x + k)$ is for k to have a prime factorization consisting of distinct consecutive primes, is there any other way to increase the density of primes produced by polynomials of this type?

References

[1] The Sage Developers. SageMath, the Sage Mathematics Software System (Version 6.10), 2016. http://www.sagemath.org.

[2] A. Odlyzko, M. Rubenstein, and M. Wolf. Jumping Champions. *Experimental Mathematics*, 8, 1999.