WXML Final Report: Large Scale Behavior in Graphs

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1 Introduction

Properties of infinite graphs (like a 2-dimensional infinite lattice) can often be approximated by large finite subgraphs (like a large square lattice). A key example is that of spanning trees, where the growth rate of the answer for finite graphs converges to a well-known quantity called Mahler measure that appears in many areas of math. The project seeks to understand how this happens, its connections to other parts of math (like random surfaces), and new variations of these phenomena to explore.

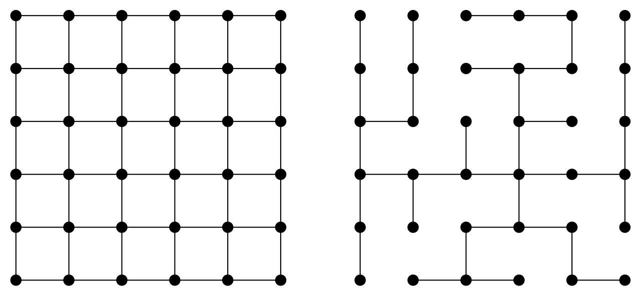
1.1 The initial problem

The problem for Spring 2016 was to determine the growth rate for the number of spanning trees on a family of graphs that look the same at every vertex or have translational symmetry.
Below is some terminology that will be used in the following sections.

Definition 1 *A* **tree** *is a connected graph with no cycles. Equivalently, it is a graph in which there is exactly one path from any vertex to any other.*

Definition 2 *A subgraph $T(V, E')$ of a graph G(V,E) is a* **spanning tree** *if it is a tree that contains every vertex in V.*

Example 1 (Spanning Tree) *Here we see a graph on the left, and one of its spanning trees on the right.*



The motivating question: if we consider a family of finite graphs which approximate an infinite one, how does the number of spanning trees grow as the graphs get bigger?

To approach this, we need to know what we mean by approximating an infinite graph by a finite one. There are a couple of ways of doing this:

Truncated Boundary Conditions: In this approach, we simply take a family of finite subgraphs of the infinite graph.

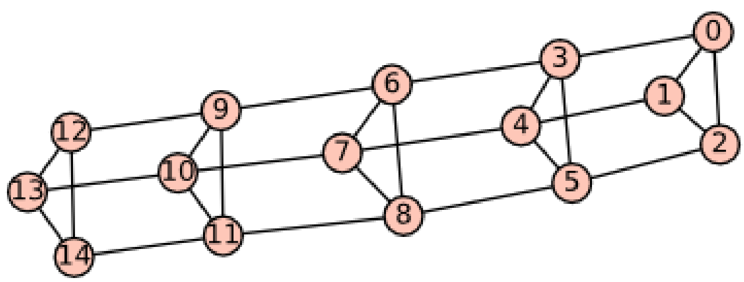


Figure 1: A triangle tube with truncated boundary.

Periodic Boundary Conditions: In this approach, we take a family of finite subgraphs, but identify the ends of the graph, where it was cut out. This preserves more of the graph's symmetry (since there is no longer a distinguished cutoff point).

It has been shown in fairly general circumstances that the growth rate of the number of spanning trees is independent of the boundary conditions [4]. Truncated boundary conditions facilitate inductive methods of proof, by giving an end to start from. Periodic boundary conditions, because of their symmetry, lend themselves to algebraic techniques.

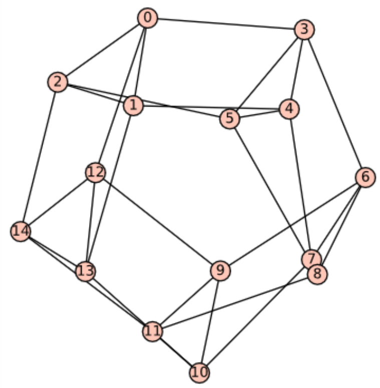


Figure 2: Triangle tube graph with periodic boundary.

1.2 New directions

As the quarter proceeded, we focused on certain "1-dimensional" graphs, which we referred to as **generalized ladder graphs**. Such a graph has its vertices organized into a grid of fixed-height columns, and there is a rule specifying the edges between each column and its neighbors, giving the graph some translational symmetry.

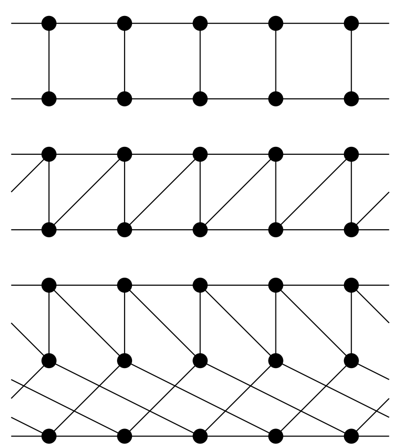


Figure 3: From top to bottom: the prototypical ladder graph, a slightly more complicated example, and a much more complicated example.

In some cases, these ladder graphs (with periodic boundary conditions) turn out to be Cayley graphs of abelian groups, which have a straightforward answer (described in section 2.3, below). In a few other cases, the Laplacian matrix (see definition 3) has a nice enough form that the same sort of analysis works. However, the case of general ladder graphs is still open.

2 Progress

2.1 Computational

The Matrix Tree Theorem gives us a way to compute the number of spanning trees in a given graph. However, this does not tell us about the growth rate. The Matrix Tree Theorem was used to experimentally compute how spanning trees grow.

Definition 3 (Graph Laplacian) *Given a simple graph G with n vertices, its Laplacian matrix L is defined as the $n \times n$ matrix $L = D - A$, where D is the degree matrix and A is the adjacency matrix of the graph.*
The entries of L are given by:

$$L_{i,j} = \begin{cases} \deg(v_i) & \textit{if } i = j, \\ -1 & \textit{if } i \neq j \textit{ and } v_i \textit{ is adjacent to } v_j \\ 0 & \textit{otherwise.} \end{cases}$$

Theorem 1 (Kirchoff's Matrix Tree Theorem, 1847 [3]) *If $G(V, E)$ is an undirected graph and L is its graph Laplacian, then the number $t(G)$ of spanning trees contained in G is given by the following computation.*

(1) Choose a vertex v_j and eliminate the jth row and column from L to get a new matrix $\hat{L}_j$

(2) Compute
$$t(G) = det(\hat{L}_J).$$

It is often helpful to express determinants in terms of eigenvalues. Although this is not trivial when the matrix is missing a row and column as above, we can still obtain such an expression by formally differentiating the characteristic polynomial of L. This gives the following variation.

Theorem 2 (Kirchoff's Matrix Tree Theorem) *For a given connected graph G with n labeled vertices, let $\lambda_1, \lambda_2, ..., \lambda_{n-1}$ be the non-zero eigenvalues of its graph Laplacian. Then the number of spanning trees of G is*

$$t(G) = \frac{1}{n}\lambda_1\lambda_2 \cdots \lambda_{n-1}.$$

Both versions of the Matrix Tree Theorem can be used to easily compute the number of spanning trees of very large graphs. Using this we can create a log plot of the number of spanning trees for each graph in a particular family of graphs and experimentally estimate the growth rate by taking a regression.

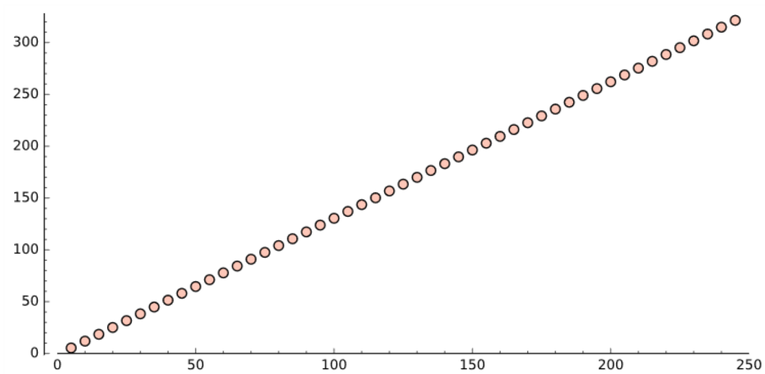


Figure 4: Log plot of the number of spanning trees of truncated ladder graphs.

For example, we experimentally computed that the growth rate for the ladder graphs (Figure 3) with truncated boundary conditions is approximately 1.316957. In fact, we actually found that the exact growth rate is $\log(2+\sqrt{3})$ using the nice symmetry of the ladder graph.

Comparing to our experimental growth rate, we can see this is a reasonable computation. We were able to run these computations for several other families of graphs to get an estimated decimal approximation of growth rates. However, since we want to get exact values, these approximations are more useful for verifying if exact computations are reasonable rather than finding the exact growth rates.

We also created plots (see Figures 5 and 6) of the eigenvalues of various graphs to look for patterns that may be useful combined with Theorem 2.

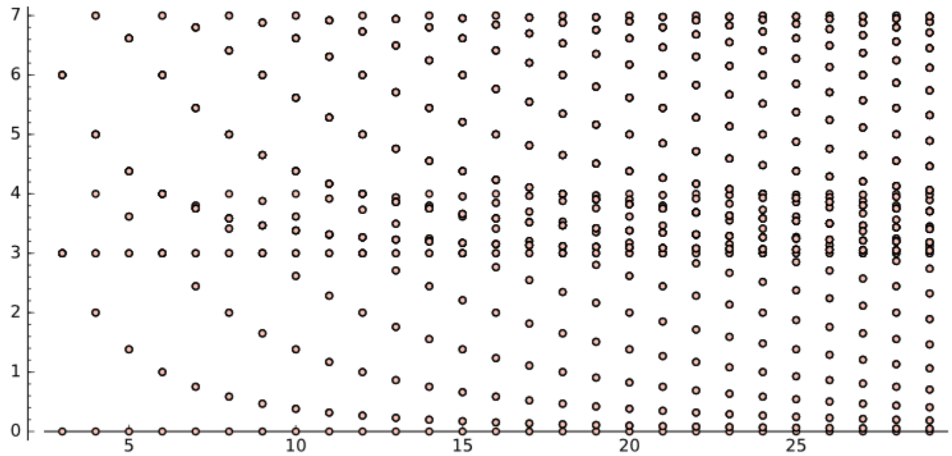


Figure 5: Plot of eigenvalues of Laplacian for triangle tube graph with periodic boundary conditions of length n. The horizontal axis is n; the vertical axis depicts the eigenvalues. In this case, the eigenvalues are exactly $2 - \omega_n - \omega_n^{-1}$ and (with multiplicity 2) $5 - \omega_n - \omega_n^{-1}$, for ω_n any nth root of unity.

2.2 Software

The software used in experimentation for this project is `Sage`, a free open-source mathematics software that builds on top of many other existing open-source packages [1]. Additionally, it is not necessary to download any software on a computer to use `Sage`. Instead, anyone can use the SageMathCloud at `https://sagemathcloud.com`.

2.3 Theoretical

One class of highly symmetrical, homogeneous graphs is given by Cayley graphs.

Definition 4 *[5] Let G be a group, and let $S = \{g_1, \ldots, g_m\}$ be a generating set of this group. The* ***Cayley graph*** *associated to G and S has vertex set G, and has an edge between g and h if and only if $g = hg_i$ for some g_i.*

Theorem 3 *Let G be an abelian group of order n, and let $\{g_1, \ldots, g_m\}$ be a generating set. Then the eigenvalues of the Laplacian of the resulting Cayley* *graph are given by*

$$2m - \left(\sum_{i=1}^{m} \rho(g_i) + \rho(g_i^{-1}) \right)$$

for any group homomorphism $\rho : G \to \mathbb{C}^\times$.

In fact, this formula gives an eigenvalue when we start with any group; however, it is only when G is abelian that there are enough homomorphisms to give all of the eigenvalues. In this case, we get a fairly explicit formula for the number of spanning trees by applying the second form of Kirchhoff's theorem:

Corollary 1 *With the above notation, the number of spanning trees on the Cayley graph is given by*

$$\frac{1}{n} \prod_{\rho \neq 1} \left(2m - \left(\sum_{i=1}^{m} \rho(g_i) + \rho(g_i^{-1}) \right) \right)$$

where the product is taken over all homomorphisms $\rho : G \to \mathbb{C}^\times$ except for the trivial one sending all elements to 1.

These homomorphisms, also known as **characters** or **one-dimensional representations** of the group, are easily expressed in terms of roots of unity. For example, for the periodic triangle tube of length n, the number of spanning trees is

$$t(n) = \frac{3}{n} \prod_{k=1}^{n-1} (2 - \omega_n^k - \omega_n^{-k})(5 - \omega_n^k - \omega_n^{-k})^2 \qquad \omega_n = e^{\frac{2\pi i}{n}}$$

We should be able to obtain the growth rate from this, by treating the expression $(1/n) \log t(n)$ as a Riemann sum approaching an integral:

$$\lim_{n \to \infty} \frac{1}{n} \log t(n) \stackrel{?}{=} \int_0^{2\pi} \log \left| (2 - e^{ix} - e^{-ix})(5 - e^{ix} - e^{-ix})^2 \right| dx$$

Slight difficulties come from the presence of a singularity at $x = 0$, making the integral improper. We did not focus on resolving this point, so this reasoning is incomplete.

This conjectured integral is called the **Mahler measure** of the polynomial $(2-x-x^{-1})(5-x-x^{-1})^2$. A nice feature of this measure is that we have

$$\int_0^{2\pi} \log|f(e^{ix})|dx = \log\left(|a_d|\prod_{i=1}^{d}\max\{1,|\alpha_i|\}\right)$$

where d denotes the degree of f, a_d is its leading coefficient, and the α_i are its roots (Lemma 1.8 in [2]). This gives a conjectural growth rate for the triangle tube of

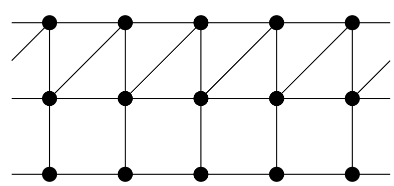
$$2\log\left(\frac{5+\sqrt{21}}{2}\right) \approx 3.136$$

which is borne out by experimental evidence.

3 Future directions

The current problem is to understand the growth rates of spanning tree counts for ladder graphs. More generally, it would be good to understand the limiting distribution of eigenvalues of the Kirchhoff matrix for these families. We have a couple of conjectures for the growth rates of specific graphs:

- The growth rate for the width-3 ladder with 1 cross rung appears to be the log of a root of $x^4-25x^3+69x^2-25x+1$.



- The growth rate for the width-2 ladder with one length-3 cross rung appears to be the log of a root of $x^4-7x^3-15x^2-7x+1$. (See Figure 6 for the graph.)

These graphs are not particularly special on their own, but represent some of the simplest ladder graphs for which we don't have an explicit formula for the number of spanning trees.

More generally, we conjecture that the growth rate for a general ladder graph should be the log of an algebraic number.

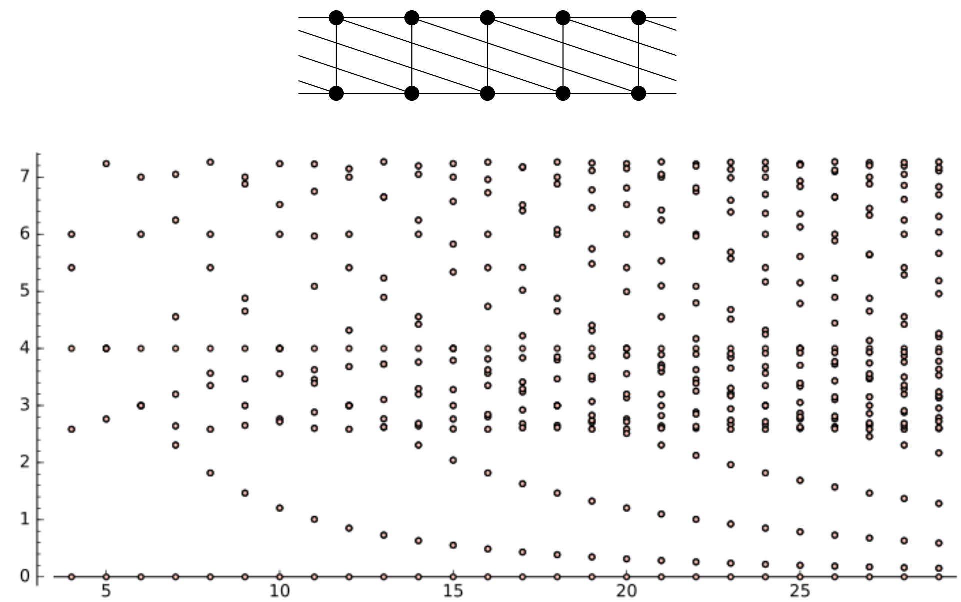


Figure 6: Plot of eigenvalues of Laplacian for the general ladder graph depicted above. Though there is no known formula for the eigenvalues, some of the curving patterns which were associated with roots of unity in the plot for the triangle tube persist.

References

[1] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version x.y.z)*, 2016. `http://www.sagemath.org`.

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[3] Dieter Jungnickel. *Graphs, Networks and Algorithms.* Springer, 3rd edition, 2008.

[4] Russell Lyons. Asymptotic enumeration of spanning trees. *Combinatorics, Probability and Computing*, 14:491–522, July 2005.

[5] John Meier. *Groups, Graphs and Trees: An Introduction to the Geometry of Infinite Groups.* Cambridge University Press, 2008.