WXML Final Report: Limiting Behavior in Graphs

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1 Introduction

Properties of infinite graphs can often be approximated by large finite subgraphs. An example of this phenomenon occurs with spanning trees, where the growth rate of the number of spanning trees on finite approximations converges to a well-known quantity called Mahler measure. Specifically Mahler measure gives the entropy of a dynamical system involving spanning forests of the infinite graph [4]. This project seeks to understand how this happens, its connections to other areas of math, and new variations of these phenomena to explore.

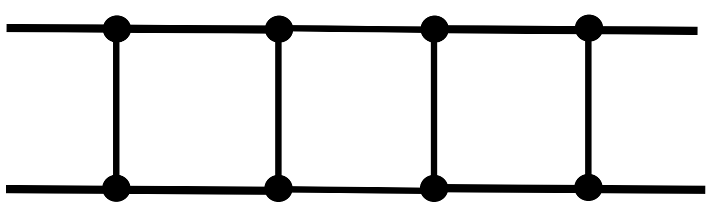

Figure 1.1: A ladder graph. This illustrates the infinite ladder by considering the infinite graph that goes to infinity in both directions.

1.1 Background

This project was a continuation of the "Large Scale Behavior in Graphs" WXML project from Spring 2016[1] [2]. We are interested in determining the growth rate of the number of spanning trees on families of graphs that look the same at every vertex or have translational symmetry. This gives a way to measure the "complexity" of the infinite structure.

Below is some terminology that will be used in the following sections.

Definition 1.1. *A* **tree** *is a connected graph with no cycles. Equivalently, it is a graph in which there is exactly one path from any vertex to any other.*

Definition 1.2. *A subgraph $T(V, E')$ of a graph $G(V, E)$ is a* **spanning tree** *if it is a tree that contains every vertex in V.*

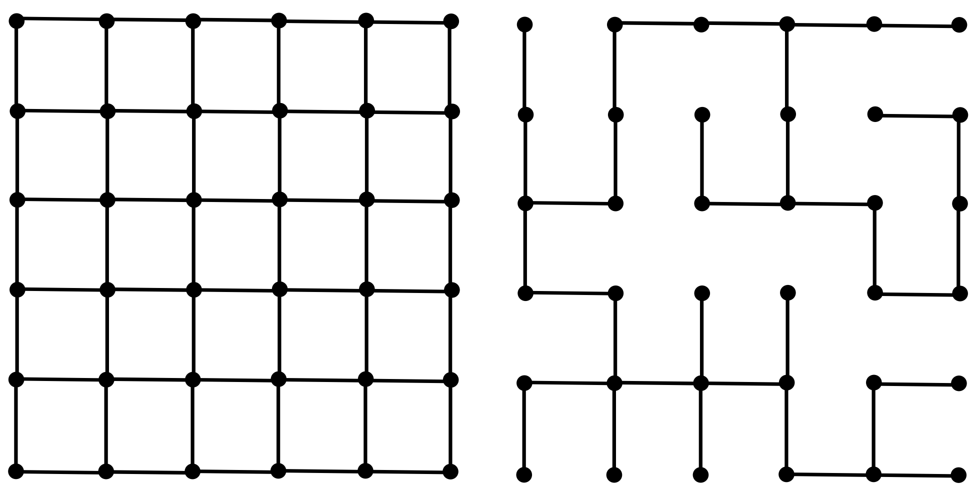

Figure 1.2: A grid graph on the left and example of a spanning tree of the grid graph on the right.

1.2 Project Goals

Our motivating question is as follows. If we consider a family of finite graphs which approximate an infinite one, how does the number of spanning trees

[1]The project report for the Spring 2016 project is available at `http://www.math.washington.edu/wxml/Trees.pdf`

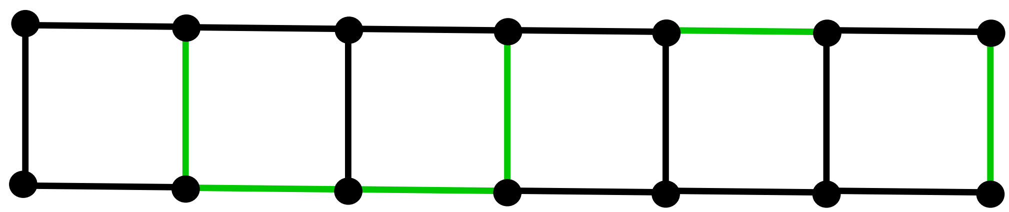

Figure 1.3: An example of a spanning tree (in black) on a truncated ladder graph.

grow as the graphs get bigger? To approach this question we must first determine what it means to approximate an infinite graph, which is described in section 1.3.

1.3 Boundary Conditions

In this section we will discuss different finite approximations, called boundary conditions. In section 2.3 we will discuss why boundary conditions do not affect the limiting numbers for the graphs we have been investigating.

All these methods of approximation start with taking a nested family of finite subgraphs of the infinite graph. If we stop there, we are working with **truncated boundary conditions**. However, we can also make modifications at the **boundary vertices**, that is, the vertices which connect to a vertex outside of the subgraph.

If we identify opposite boundary vertices, we have **periodic boundary conditions**. This preserves more of the graph's symmetry since there is no longer a distinct cutoff point as in the truncated boundary.

Alternatively, under **wired boundary conditions**, one "wires" the boundary vertices to an additional vertex "at infinity" by adding an edge between each of the boundary vertices and this new vertex. Examples of the ladder graph with these various boundary conditions are given in figures 1.4, 1.5, and 1.6.

We have been focusing on periodic boundary because of the symmetry it provides, which is discussed in section 2.4. We found that boundary conditions do not affect the growth rate for typical graphs, which is discussed in section 2.3.

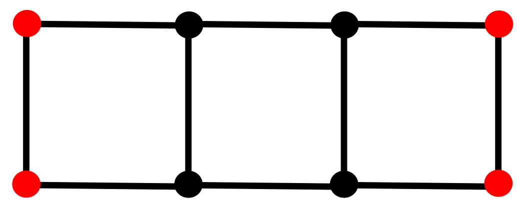

Figure 1.4: This is an example of the ladder graph with truncated boundary.

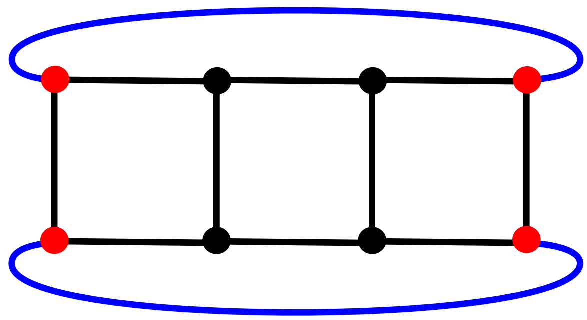

Figure 1.5: This is an example of the ladder graph with periodic boundary where one identifies the ends of the graph.

2 Progress

2.1 Computational

Definition 2.1 (Graph Laplacian)**.** *Given a simple graph G with n vertices, its Laplacian matrix L is defined as the $n \times n$ matrix $L = D - A$, where D is the degree matrix and A is the adjacency matrix of the graph.*
The entries of L are given by:

$$L_{i,j} = \begin{cases} \deg(v_i) & \textit{if } i = j, \\ -1 & \textit{if } i \neq j \textit{ and } v_i \textit{ is adjacent to } v_j \\ 0 & \textit{otherwise.} \end{cases}$$

Theorem 2.2 (Kirchoff's Matrix Tree Theorem)**.** *For a given connected graph G with n labeled vertices, let $\lambda_1, \lambda_2, ..., \lambda_{n-1}$ be the non-zero eigenvalues of its graph Laplacian. Then the number of spanning trees of G is*

$$t(G) = \frac{1}{n}\lambda_1\lambda_2 \cdots \lambda_{n-1}.$$

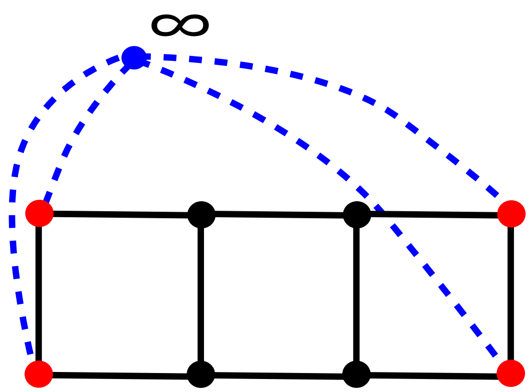

Figure 1.6: This is an example of the ladder graph with wired boundary where one wires the boundary vertices to a vertex at infinity.

The Matrix Tree Theorem gives us a way to compute the number of spanning trees in a given graph. However, this does not tell us about the growth rate. Given an increasing sequence of finite subgraphs, we applied the Matrix Tree Theorem to each graph to compute the number of spanning trees. In the examples we considered, this generally grew exponentially in the number of vertices, so we used a linear regression on a logarithmic scale to approximate the growth rate of the number of spanning trees. If τ_n denotes the number of spanning trees of the nth graph in the sequence, the quantity we are interested in is

$$\lim_{n\to\infty} \frac{1}{n} \log \tau_n$$

For example, in spring 2016 we computed the growth rate for the ladder graph with truncated boundary conditions, as shown in figure 1.4, which was approximately 1.316957. However, this decimal approximation does not tell you much about the graph, nor does it give the exact growth rate. In fact, we proved that the exact growth rate is $\log(2+\sqrt{3})$ using the convenient symmetry of the ladder graph. We are interested in finding exact growth rates for more graphs.

Comparing to our experimental growth rate, we can see this is a reasonable computation. We were able to run these computations for several other families of graphs to get an estimated decimal approximations of growth rates. However, since we want to get exact values, these approximations are mainly used to verify our exact computations rather than to find the exact growth rates.

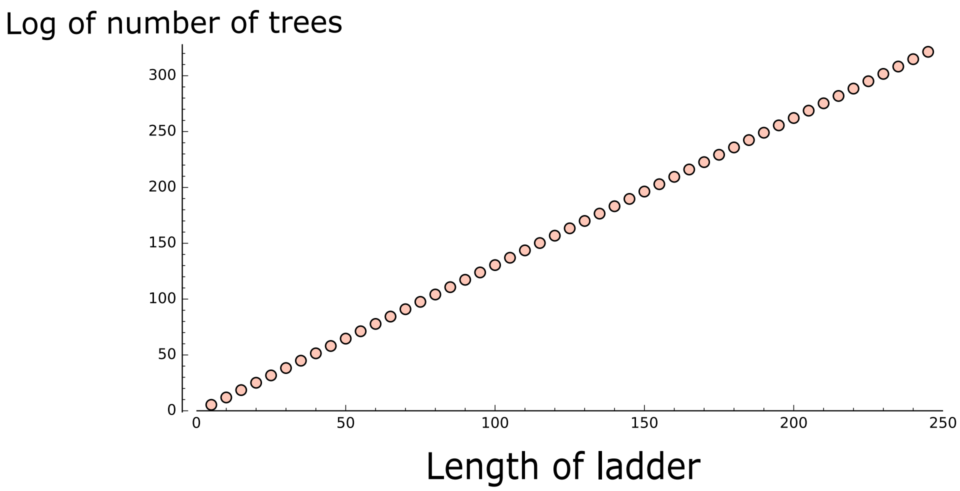

Figure 2.1: Log plot of the number of spanning trees of truncated ladder graphs.

2.2 Software

The software used in experimentation for this project is `Sage`, a free open-source mathematics software that builds on top of many other existing open-source packages [5]. Additionally, it is not necessary to download any software on a computer to use `Sage`. Instead, anyone can use the SageMathCloud at `https://sagemathcloud.com`.

2.3 Boundary Conditions and Growth Rate

Given the different boundary conditions discussed in section 1.3, a natural question is whether or not the growth rate of the number of spanning trees stays constant under different boundary conditions. Intuitively, as long as most of the graph is not boundary in any given finite approximation, the growth rate should be the same. This quarter, we showed a result that made this precise in all the cases we have been working with.

First, we can consider how adding a single edge to a graph affects the number of spanning trees with the following proposition.

Proposition 2.3. *Let $G = (V, E)$ be a connected graph, and let $G' = (V, E')$ be obtained from G by adding a single edge, e. Let $\tau(G)$ denote the number* *of spanning trees of G. Then*

$$\tau(G) \leq \tau(G') \leq |E'|\tau(G).$$

While this is a weak bound, especially when applied repeatedly to account for adding multiple edges we still get our result since the number of trees usually grows exponentially. This bound is sufficient in many cases, as described in the following theorem:

Theorem 2.4. *Let $\{G_n = (V_n, E_n)\}$ be a sequence of connected graphs. Let $\{G'_n = (V_n, E'_n)\}$ be a sequence of graphs obtained by adding some finite number of edges, m_n, to each G_n, such that*

$$\lim_{n\to\infty} \frac{m_n \log |E'_n|}{|V_n|} = 0.$$

Then if one of the limits

$$\lim_{n\to\infty} \left(\frac{1}{|V_n|} \log \tau(G_n)\right), \ \lim_{n\to\infty} \left(\frac{1}{|V_n|} \log \tau(G'_n)\right)$$

exists, the other one also exists and is equal to it.

The condition

$$\lim_{n\to\infty} \frac{m_n \log |E'_n|}{|V_n|} = 0$$

is not a difficult one to satisfy. In the example of the ladder graphs, the number of edges added in passing from truncated to periodic boundary conditions is constant ($m_n = 2$), while the number of vertices (V_n) and edges (E'_n) both grow linearly in n; hence the limit is 0. This example is illustrated in figure 2.2.

Additionally, while Theorem 2.4 only directly describes adding edges, it is flexible. We can also add a vertex with a single incident edge (which does not change the number of spanning trees by itself) to each graph in a sequence, and then add more edges with Theorem 2.4. This allows us to consider wired boundary conditions.

2.4 Cayley Graphs

A **Cayley graph**[2] encodes the algebraic structure of a group in a graph using a set of generating elements of the group that act as coordinates. The vertices

[2]For more information on Cayley graphs we recommend [3].

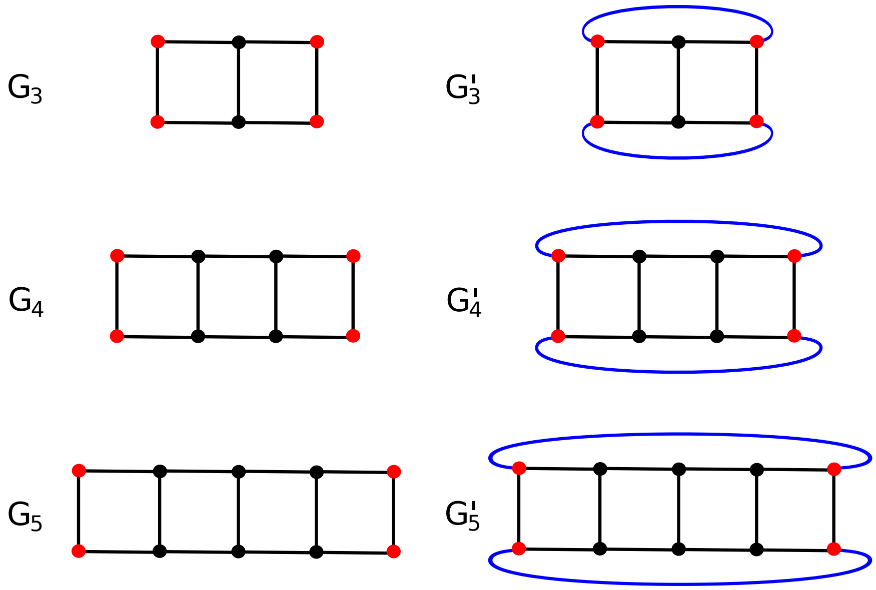

Figure 2.2: Applying Theorem 2.4 to the truncated and periodic ladder graphs. In this example, $m_n = 2$, $V_n = 2n$, $E'_n = 3n$.

of the Cayley graph correspond to group elements and an edge corresponds to adding one of the generators.

In Spring 2016, we found a formula for counting spanning trees on a Cayley graph of a finite abelian group.

Theorem 2.5. *Let G be a finite abelian group of order n, and let $\{g_1, \ldots, g_m\}$ be a set of generators. Then the number of spanning trees on the Cayley graph is given by*

$$\frac{1}{n}\prod_{\rho\neq 1}\left(2m - \left(\sum_{i=1}^{m}\rho(g_i) + \rho(g_i^{-1})\right)\right)$$

where the product is taken over all homomorphisms $\rho : G \to \mathbb{C}^\times$ except for the trivial one sending all elements to 1.

This quarter, we stepped from finite to infinite graphs to obtain growth rates for Cayley graphs of the groups $F\times\mathbb{Z}$, where F is a finite abelian group. The finite approximations to these graphs with periodic boundary conditions are themselves Cayley graphs, of the groups $F \times \mathbb{Z}/(n)$.

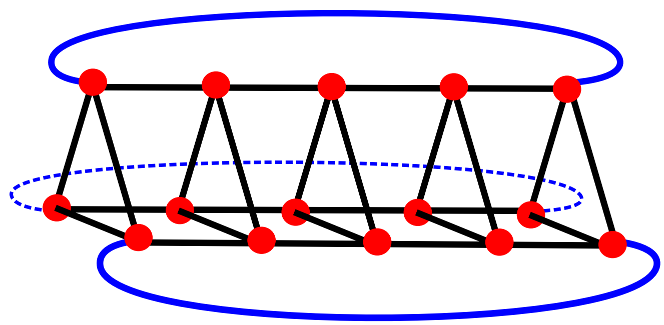

Figure 2.3: An example of the triangle tube graph with periodic boundary.

Theorem 2.6. *Let F be a finite abelian group, and let $\{(f_1, k_1), \ldots, (f_m, k_m)\}$ be a set of generators of $F \times \mathbb{Z}$. Let $\tau(n)$ denote the number of spanning trees of the Cayley graph of $F \times \mathbb{Z}/(n)$ given by this generating set. For a homomorphism $\rho' : F \to \mathbb{C}^\times$, define the polynomial*

$$g(\rho', x) = 2m - \rho'(f_1)x^{k_1} - \rho'(f_1^{-1})x^{-k_1} - \ldots - \rho'(f_m)x^{k_m} - \rho'(f_m^{-1})x^{-k_m}$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \log \tau(n) = \sum_{\rho'} \int_0^1 \log |g(\rho', e^{2\pi i\theta})| \; d\theta$$

where the sum is taken over all homomorphisms $\rho' : F \to \mathbb{C}^\times$.

The integral in the statement of Theorem 2.6 is also known as the **Mahler measure** of the polynomial $g(\rho', x)$. Mahler measure has some nice properties. In particular, it also equals the log of the absolute value of the product of the polynomial's leading coefficient and its roots of modulus greater than 1[1]. This allows us to get simple exact values for some growth rates.

Example. The Cayley graph of $\mathbb{Z}/(3) \times \mathbb{Z}$ with generators $(1, 0), (0, 1)$ is a triangular tube. Finite approximations of this graph, Cayley graphs of $\mathbb{Z}/(3) \times \mathbb{Z}/(n)$, are illustrated in figures 2.3 and 2.4. If $\omega = e^{2\pi i/3}$ denotes a third root of unity, the homomorphisms $\rho_0, \rho_1, \rho_2 : \mathbb{Z}/(3) \to \mathbb{C}^\times$ are given by $\rho_0(1) = 1$, $\rho_1(1) = \omega$, $\rho_2(1) = \omega^2 = \omega^{-1}$. Then

$$\begin{aligned} g(\rho_0, x) &= 4 - 1 - 1 - x - x^{-1} = 2 - x - x^{-1} \\ g(\rho_1, x) &= 4 - \omega - \omega^{-1} - x - x^{-1} = 5 - x - x^{-1} \\ g(\rho_2, x) &= 4 - \omega^{-1} - \omega - x - x^{-1} = 5 - x - x^{-1} \end{aligned}$$

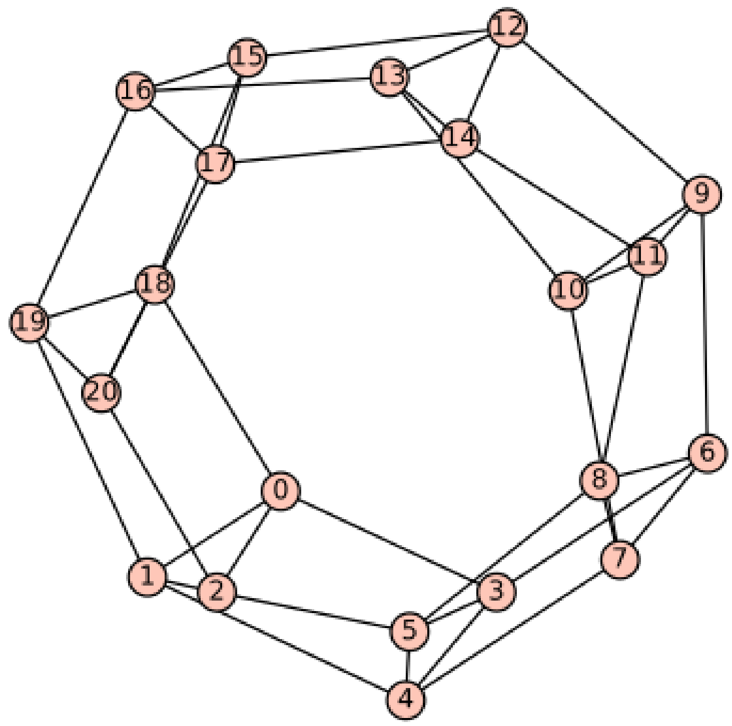

Figure 2.4: The triangle tube graph with periodic boundary. This image shows how periodic boundary makes the end of the approximation ambiguous.

The alternative characterization of Mahler measure gives that

$$\int_0^1 \log|2 - e^{2\pi i\theta} - e^{-2\pi i\theta}|\, d\theta = 0$$
$$\int_0^1 \log|5 - e^{2\pi i\theta} - e^{-2\pi i\theta}|\, d\theta = \log\frac{5+\sqrt{21}}{2}$$

thus the growth rate of the number of spanning trees is

$$2\log\frac{5+\sqrt{21}}{2}.$$

3 Future directions

We plan to continue this project and investigate growth rates of spanning trees particularly in more complicated graphs than the Cayley graphs. The following sections describe some questions we have started as well as what we plan to investigate.

3.1 Breaking Symmetry

A natural next step is to investigate graphs that do not have the nice symmetry of the Cayley graphs investigated. Figure 3.1 is where we started this quarter as it loses the vertical symmetry but still has the translational symmetry of the Cayley graphs. This translational symmetry should allow us to use some of the same techniques.

Given a graph on the vertex set $\{1,\ldots,n\} \times \mathbb{Z}$ with translational symmetry (such that (i,s) is adjacent to $(j,s+k)$ if and only if $(i,0)$ is adjacent to (j,k)), define $n \times n$ matrices M_k by

$$(M_k)_{ij} = \begin{cases} 1 & (i,0) \text{ is adjacent to } (j,k) \\ 0 & \text{otherwise} \end{cases}$$

Additionally, define a diagonal matrix D by

$$D_{ii} = \deg((i,0))$$

In the example of Figure 3.1 (labeling the rows 1, 2, 3 from top to bottom), we have

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad M_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and all other M_k are 0.

Conjecture 3.1. *Using the notation described above, the growth rate is*

$$\int_0^1 \log|\det(D - M_0 - M_1 e^{2\pi i\theta} - M_1^T e^{-2\pi i\theta} - M_2 e^{4\pi i\theta} - M_2^T e^{-4\pi i\theta} - \ldots)|d\theta$$

As above, this integral is the Mahler measure of a polynomial. This gives us an impressive conjectural growth rate for the graph in figure 3.1:

$$\lim_{n\to\infty} \frac{1}{n} \log \tau(n) = \log \left(\frac{1}{4} \left(25 + \sqrt{357} + 2\sqrt{\frac{483 + 25\sqrt{357}}{2}} \right) \right)$$
$$\approx 3.0866.$$

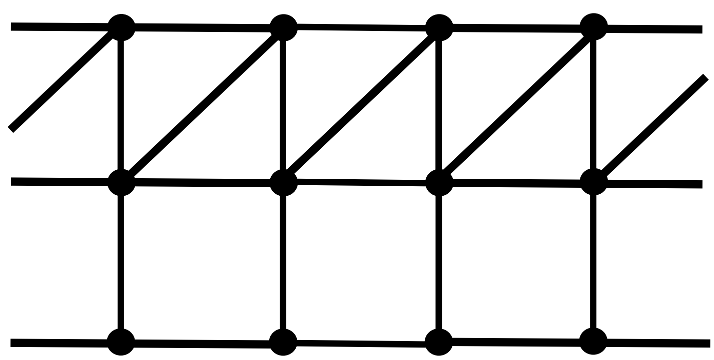

Figure 3.1: An example of the graph we investigated that begins to break the symmetry of the ladder graph. Note that we still have the translational symmetry of the ladder graph.

3.2 Spanning Forests

Definition 3.2. *A* **forest** *is a graph that contains no cycles.*

Hence a forest is like a tree, but need not be connected.

Definition 3.3. *A* **spanning forest** *is a forest that contains all the vertices of a graph $G(V, E)$.*

When taking a finite approximation of a spanning tree on an infinite graph the approximation is not necessarily a spanning tree on that subgraph, but will be a spanning forest. An example in shown in figure 3.2. As such, in investigating the interaction between infinite graphs and finite subgraphs, it is in some ways more natural to consider spanning forests, rather than trees, on finite approximations.

References

[1] G. Everest and T. Ward, *Heights of Polynomials and Entropy in Algebraic Dynamics*, Springer, 1999.

[2] D. Lind, J. Morrow, A. Levy, W. Dana, C. Litterell, and M. Stadt, *WXML final report: Large scale behavior in graphs*. `http://www.math.washington.edu/wxml/Trees.pdf`.

[3] J. Meier, *Groups, Graphs and Trees: An Introduction to the Geometry of Infinite Groups*, Cambridge University Press, 2008.

[4] R. Solomyak, *On coincidence of entropies for two classes of dynamical systems*, Ergodic Theory and Dynamical Systems, 18 (1998), pp. 731–738.

[5] The Sage Developers, *SageMath: The Sage Mathematics Software System (Version 7.3)*, 2016. `http://www.sagemath.org`.

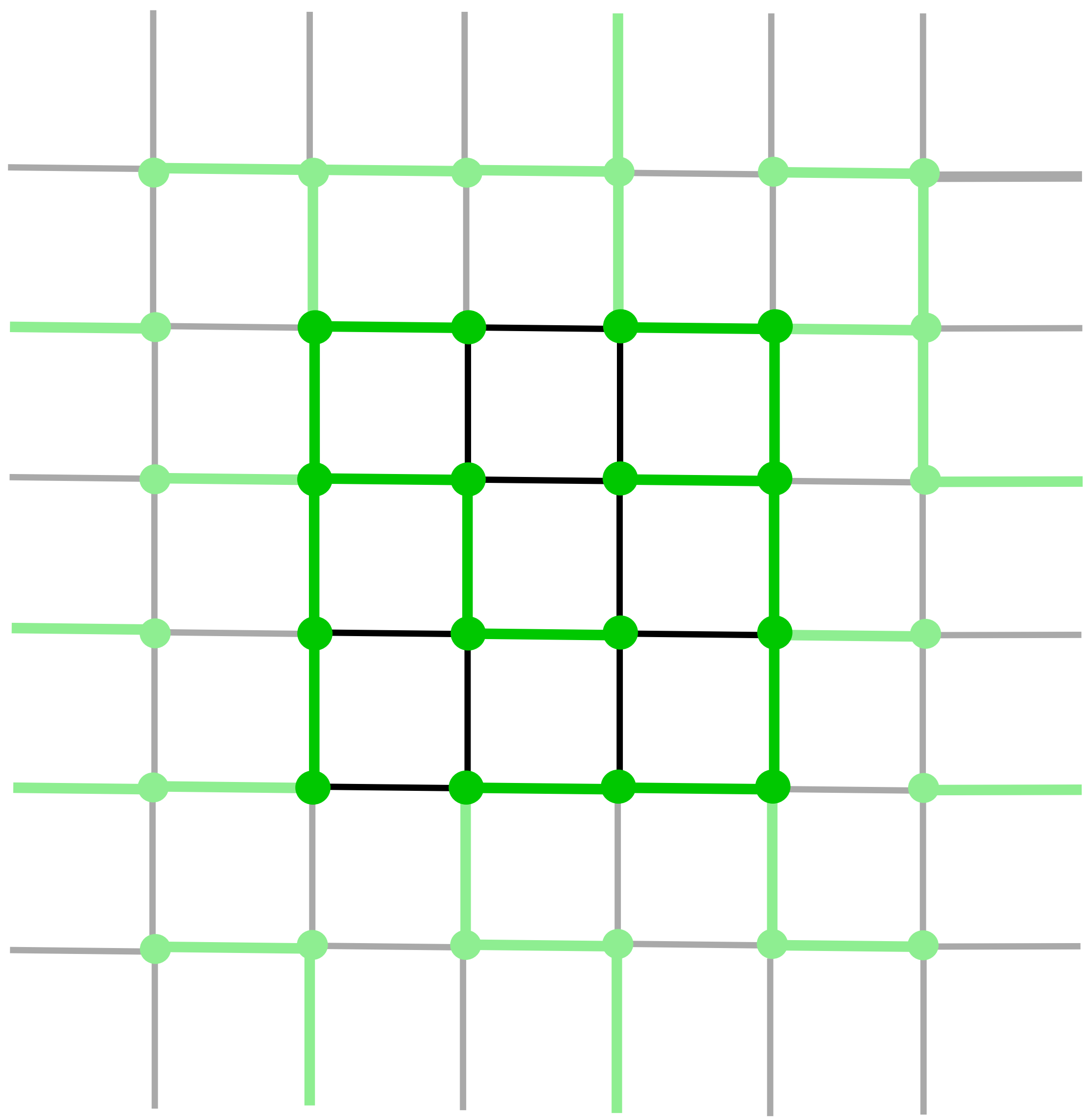

Figure 3.2: An illustration of a spanning tree (in green) on the infinite grid graph showing the darker part as a finite approximation of the grid graph which gives a spanning forest.