WXML Final Report: Number Theory and Noise

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1 Introduction

Three undergraduates participated in this project during Spring quarter 2017: Penny Espinoza, Emily Flanagan, and Hannah Van Wyk. They investigated integer sequences via their representations as sound. The following are their reports on their work during this quarter.

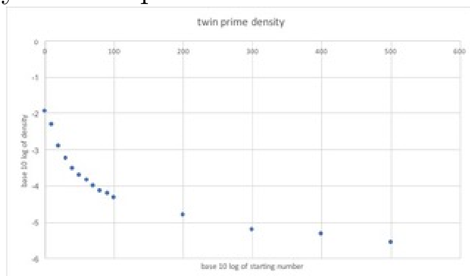
2 Progress

2.1 P. Espinoza's commentary

2.1.1 The long view on primes (A000040) and twin primes (A001097)

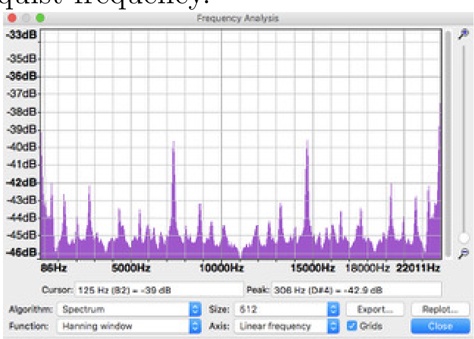
An interesting perspective on the sparsity of the primes was achieved by listening to sounds generated from sequences of large prime numbers. For example, one can generate a sound that represents primes between 10^{50} and $10^{50}+1,000,000$ by generating the sequence, subtracting 10^{50} from each such prime, and then creating a sound from the offsets. Sequences beginning as high as 10^{600} were generated.

A similar technique was used for twin primes. A graph of twin primes in the first 15,000,000 integers following powers of 10 from 10^{10} through 10^{100} shows the density of twin primes at various points, with only 712 twin primes between 10^{100} and $10^{100} + 15000000$. Later sequences showed that there are only 42 twin primes in the fifteen million integers starting with 10^{500}.



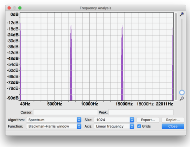
2.1.2 What is in a spectrogram?

A frequency analysis for sequence A181172 (primes whose base 4 representation does not contain a 0) showed near-perfect symmetry about half the Nyquist frequency.

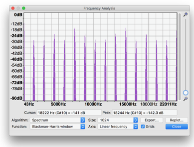


Further exploration revealed that this was true for any subset of the primes. This led to questions of how frequencies result from a sequence. To gain traction on answering this question, the sequence of composite numbers was used, as its waveform is the inverse of the prime number sequence over the same range of integers, and consequently includes the same frequencies. The advantage of using composite numbers is that the sequence can be built up gradually, by starting with only multiples of 2, then adding multiples of 3 (actually composites congruent to 3 mod 6, since any multiple of 6 was already included in the multiples of 2), then new multiples of 5, etc. The frequency analyses for multiples of the first few primes is shown below, followed by a frequency analysis of all composites between 1 and 1,000,000. Note the scale changes in these graphs.

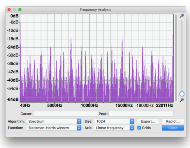
multiples of 2 or 3



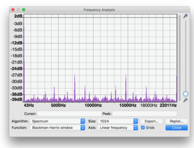
multiples of 2, 3, or 5



multiples of 2, 3, 5, or 7

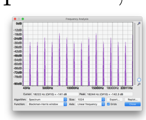


all composites

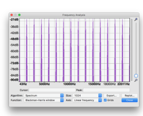


In an attempt to figure out exactly how frequencies come to be, sounds were also created for strictly periodic sequences that included other subsets of composite numbers. For example, adding multiples of 5 to the composites actually only adds composites congruent to 5 or 25 mod 30 (all other multiples of 5 are already present because they are multiples of 2 or 3). Below are frequency analyses for composites with multiples of 2, 3, or 5, alongside the purely periodic sequence for numbers congruent to 5 mod 30, and for just those integers added when multiples of 5 are included in the composite sequence. Note that all frequencies for 5 mod 30 are of equal strength.

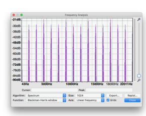
multiples of 2, 3, or 5



5 mod 30



5 or 25 mod 30



Further understanding may be gained by trying to answer the following questions:

- Why are the frequencies symmetric about half the Nyquist frequency?
- How can the amplitudes for each frequency be determined?

2.2 E. Flanagan's commentary

2.2.1 Beatty Sequence of Sqrt 2

A Beatty Sequence is defined as the floor of some irrational number times a rational number. When making sounds, the Beatty sequences we are interested in are those in which the irrational number is greater than 1, as to create a strictly increasing sequence. The Beatty sequence we worked with the most this quarter was $n\sqrt{2}$, with n being an integer. More specifically, we investigated various approximations of the irrational $\sqrt{2}$.

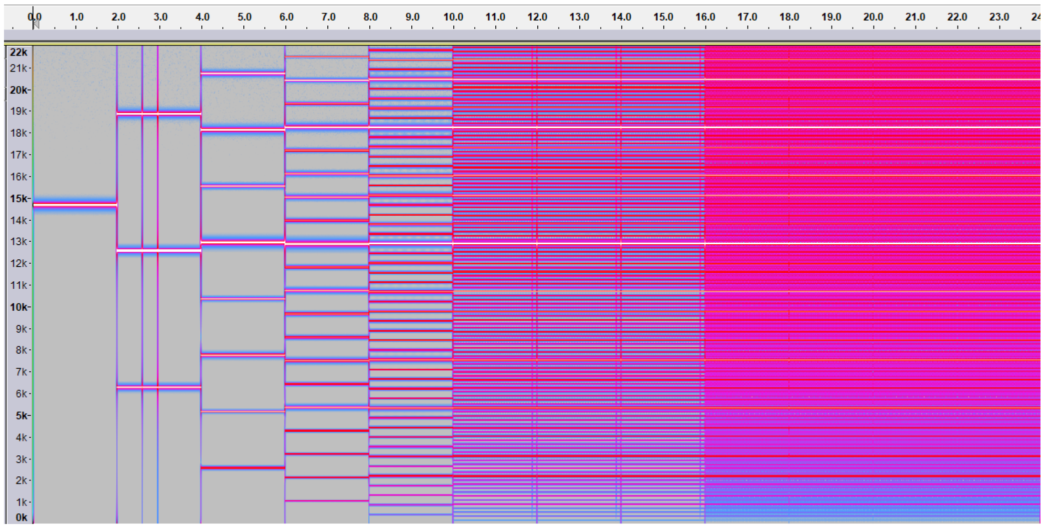
One way of approximating an irrational number is the method is that of continued fractions. Continued fractions are an expression of the form

$$a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \ldots}}}$$

For $\sqrt{2}$, the approximation is of the form

$$1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ldots}}}$$

This yields a sequence of integers beginning $1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots$. The use of a continuted fraction is advantageous because they produce the "closest" value to the irrational number, and thus converge faster to the irrational than truncation of decimals. Using these approximations of $\sqrt{2}$, we generated a Beatty sequence, and converted it into a sound. Spectrogram below demonstrates the convergence of these sequences to the "true" sound of the Beatty sequence of $n\sqrt{2}$. Every two seconds, the next approximation is used. The last 2 second block of the spectrogram is the "true" sequence. It is important to note that the "true" sequence is in itself an approximation, because it would be impossible to compute the sequence using all the infinite digits of an irrational number.



We can see that as the continued fraction approximation gets more precise, the fundamental frequency decreases, causing the sound to become closer to the "true" Beatty sequence. Interestingly, after 12 seconds, or 6 iterations of the continued fraction, the change in frequencies caused by the change in approximation is not detected by the human ear.

2.3 Hannah Van Wyk's commentary

At the beginning of the quarter, I focused on using Fourier analysis to make basic sequences such as number congruent to 0 (mod 10), or numbers congruent to 0,3 or 6 (mod 10). After making these sequences and looking at the spectrograms and waveforms, I found the sequences made from the integer sequences and the sequences made from Fourier analysis were almost completely similar (figures 1 and 2). However, the waveforms looked very different when zoomed into (figures 3 and 4).

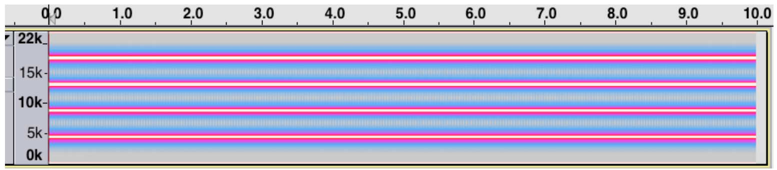


Figure 1: Spectrogram of integers congruent to 0 (mod 10). Frequency plotted against time.

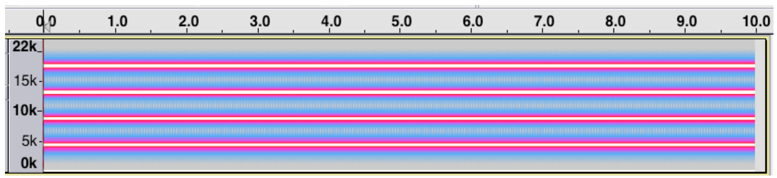


Figure 2: Spectrogram of integers congruent to 0 (mod 10) made from Fourier analysis

Overall, I was surprised by how similar the sounds made from Fourier were to the sounds made from using the actual files of the numbers. Both the sound files and spectrograms were undistinguishable from each other.

Throughout the quarter, I continued researching the sound of $\lfloor n \log(n) \rfloor$. I found a sound that had a very similar spectrogram, which was $\lfloor n^{1.01} \rfloor$ (Figure 6).

One thing I observed that was interesting about these spectrograms, is that at certain points, the pink lines in the background seem to converge/cross. For example, this can be seen at 8.6, and 21.4 seconds (Figure 7).

I now have concluded that at these instants, the sound of floor(n log(n)) will sound like the sound of the multiples of some integer n. I also zoomed in the waveform of this sequence, and concluded that the terms in the sequence

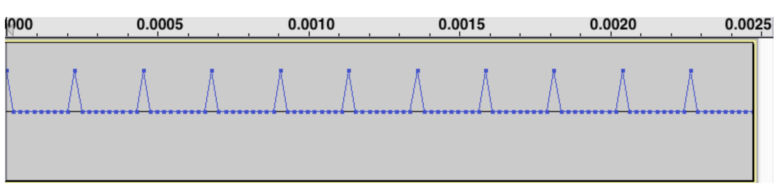


Figure 3: Waveform of integers congruent to 0 (mod 10)

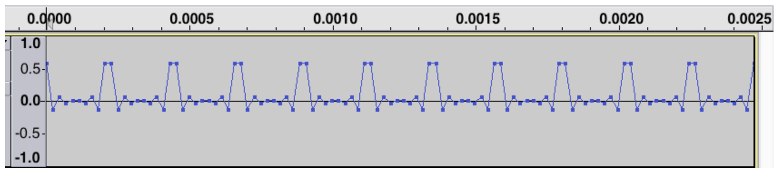


Figure 4: Waveform of integers congruent to 0 (mod 10) made from Fourier analysis

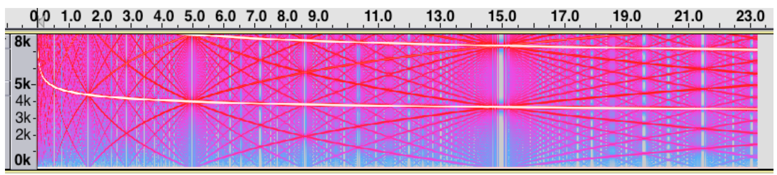


Figure 5: Spectrogram of $\lfloor n \log n \rfloor$

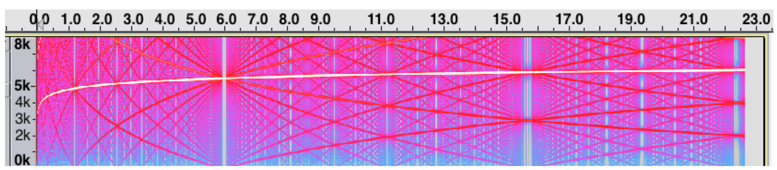


Figure 6: Spectrogram of $\lfloor n^{1.01} \rfloor$

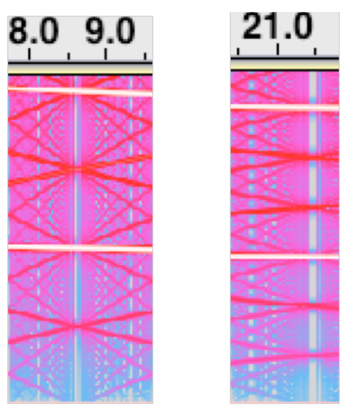


Figure 7: Spectrogram of $\lfloor n \log n \rfloor$ at 8.6 and 21.4 seconds

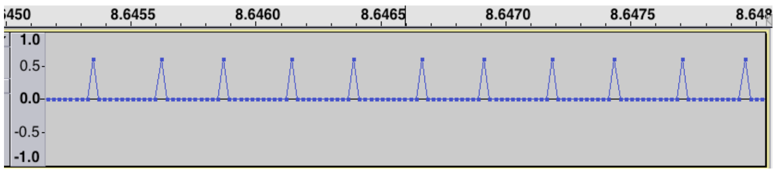


Figure 8: Waveform of $\lfloor n \log n \rfloor$ near 8.6 seconds

indeed are increasing by the same amount when these points of convergence occur. For example, at 8.6 seconds each term is increasing by 12. We can see this both by zooming in on the waveform at this instant, and we see in figure 8, that each term is 12 more than the last:

However, since this increase of 12 between each term only happens briefly, floor(n log(n)) will only momentarily sound like the sequence of the multiples of 12. In the future, I would like to still go more in depth with the sequence $\lfloor n \log n \rfloor$. Perhaps we would be able to combine what I learned about Fourier analysis this quarter with my discoveries from the spectrogram of $\lfloor n \log n \rfloor$ to be able to reach a concrete answer as to why sequences like $\lfloor n \log n \rfloor$ and $\lfloor n^{1.01} \rfloor$ have such a unique sound.