Uniformity of Solutions to Diophantine Equations

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Uniformity of Diophantine Equations

Motivation Understand the set of solutions to specific Diophantine equations and what we can say about them.

Problem There is no general method or algorithm for finding sets of solutions to Diophantine equations

Methods Continued Fractions, Analyzing Growth of Fundamental Solutions, Thue Equation, and Plane Curves.

Continued Fraction Algorithm

```
def contFrac(n):
    '''
    Given a number n, returns a list containing the integer parts at each step.
    E.g. [n1, n2, n3, n4, n5] = n1 + (1 / n2 + (1 / n3 + 1 / n4 + (1 / n5)))
    '''
    result = []
    result.append(floor(n))

    yield floor(n)

    cur_fraction = n - result[0]

    for i in xrange(10):            # stops after generating 10 fractions, so it doesnt go forever
        if cur_fraction != 0:       # we're done if current fraction part is 0
            cur_fraction = (cur_fraction)^(-1)
            new_int = floor(cur_fraction)
            yield new_int
            cur_fraction = cur_fraction - new_int
```

Growth of Fundamental Solution with d

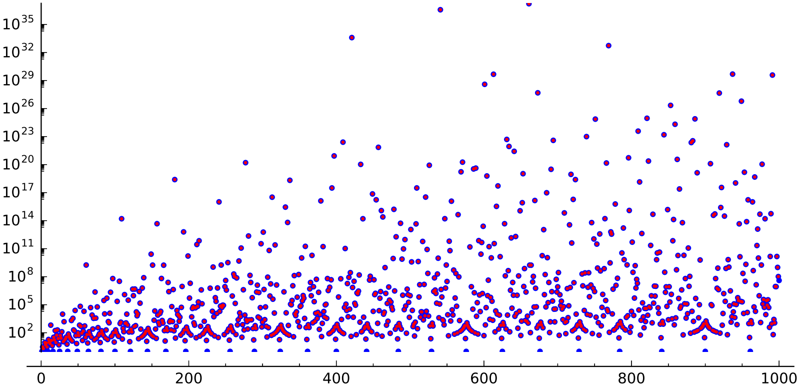

Figure: The x-axis is $d \in \mathbb{Z}$ and the y-axis is $\max(x, y)$ of $u = x + y\sqrt{d}$ where u is a fundamental solution of $x^2 - dy^2 = 1$

Thue Equation

$$f(x, y) = c$$

- $f(x, y)$ is a polynomial irreducible over $\mathbb{Q}$ with integer coefficients and of at least degree 3, with $c \in \mathbb{Q}$.
- Thue equations have a finite number of integer solutions (as opposed to the Pell equation).
- Algorithms exist to solve them.

Example:

$$x^3 + x^2y + 3xy^2 - y^3 = 17$$

Approximation for Thue Equations

Thue's approximation method is that an integral solution to $f(x, y) = c$ provides a good rational approximation $\frac{x}{y}$ to some root of $f(X, 1)$ where x, y is the solution to the Thue Equation.

Example:

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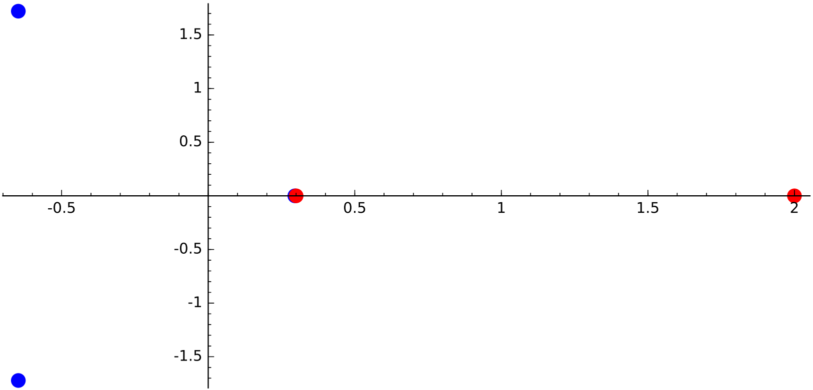

What's Worked

- Verifying our graph for growth of fundamental solutions with available data
- Finding continued fractions and approximating irrational numbers (CoCalc Algorithm)
- Understanding Thue equations, why we can approximate one of the roots, and why they have a finite set of solutions.
- Visualizing the closeness of approximations to solutions by graphing.

Future goals

Next steps

- Collecting a large amount of data for solutions
- Studying why Thue equations have a finite number of solutions

Challenges

- Understanding bounds of solution approximation
- Implementing the continued fraction algorithm