WXML Final Report

Uniformity of Solutions of Diophantine Equations

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1 Introduction

A Diophantine Equation is a polynomial equation, usually in two or more unknowns with integer coefficients, in which we look for integer solutions. The simplest case is where we look for integral solutions (x, y) in the linear equation:

$$ax + by = c$$

where a, b, c are known integers.

The study of Diophantine Equations and their analyses dates back to the year AD 300, when a Greek mathematician, Diophantus of Alexandria, wrote about them in his series of books, *Arithmetica*. It would be a problem in Diophantus' *Arithmetica* that led Pierre de Fermat in 1637 to annotate within the margins of his own copy, famously known as Fermat's Last Theorem which would go on unresolved for the next 4 centuries.

As for our study of Diophantine Equations, we investigate the special cases of the Pell Equation and Thue Equation, with the following questions in mind: Are there any solutions to these equations? If so, how many are there? Along the way, we also investigate rational approximations to irrational numbers, a method known as Diophantine Approximation.

1.1 The initial problem

Initially we found that Diophantine Equations look "easy to solve" but are actually very difficult to find solutions for. This brought up questions surrounding understanding the set of solutions to specific Diophantine equations, and how solutions to some equations provide approximations to certain algebraic numbers. In general, no method or algorithm exists for finding sets of solutions to all Diophantine equations, so we study specific cases in an attempt to find some sort of pattern. (Hilbert's 10th problem)

2 Progress

2.1 Theoretical

2.1.1 Diophantine Approximation

Our initial approach to investigate solutions of Diophantine Equations began with learning about Diophantine Approximation in order to understand how these solutions are represented.

Diophantine Approximation is the theory of approximations of real numbers by rational numbers. But of course, any real number can be approximated by a rational number with arbitrary accuracy, so the field of study is rather to describe and determine how "good" approximations can be.

We discuss an important theorem in Diophantine Approximation:

Theorem. ***(Dirichlet Approximation Theorem)*** *Let $\xi \in \mathbb{R}$ and let $Q > 0$ be a positive integer. Then there exists p,$q \in \mathbb{Z}$ such that $(p, q) = 1$ and*

$$0 < q \leq Q, \qquad \left|\xi - \frac{p}{q}\right| \leq \frac{1}{q(Q+1)}.$$

The implication of Dirichlet's result is that if we are trying to approximate the irrational number, ξ with the rational number $\frac{p}{q}$, then the best approximation we can achieve is determined by how small or large we allow our denominator, q, to be. And a corollary of this result is that ξ would have an infinite sequence of p, q that approximate it.

As we will discuss later, continued fractions provide the best rational approximation to ξ.

2.1.2 Pell Equation

A Pell Equation is any equation of the form $x^2 - dy^2 = 1$ where d is a non-square integer. If d were a perfect square then the solution to the Pell Equation would be trivial. When d is not a perfect square, the solutions to $(x + \sqrt{d}y)(x - \sqrt{d}y) = 1$ are infinitely many, and correspond to units in the quadratic field $\mathbb{Q}[\sqrt{d}]$. It's interesting to note that the Pell Equation represents a hyperbola.

Examining solutions to the Pell Equation through the lens of Diophantine Approximation we can approximate $\sqrt{d}$. We find that the error term on the right side of the inequality is inversely-proportional to the square of y, limiting our approximation even more. This leads to the following corollary to our theorem on approximation:

Theorem. ***(Corollary to Dirichlet Approx. Theorem)*** *There are infinitely many fractions $\frac{x}{y} \in \mathbb{Q}$, x, y coprime integers, such that*

$$\left| \frac{x}{y} - \sqrt{d} \right| \leq \frac{1}{2\sqrt{d}y^2}.$$

This implies that the fraction $\frac{x}{y}$ provides a "good" approximation to the algebraic number $\sqrt{d}$ (where good means within the bounds of the error term), and in fact that there exist infinitely many "good" approximations (which will be expanded upon later with the discussion of the continued fraction algorithm).

2.1.3 Thue Equation

A Thue equation is of the form $f(x, y) = c$ such that $f(x, y)$ is a homogeneous irreducible polynomial over $\mathbb{Q}$, of degree at least 3, with integer coefficients, and c being an integer. Unlike Pell equations, Thue equations have a finite number of integer solutions. Apart from comparing their behavior to Pell equations, we studied Thue equations because algorithms do exists to solve them as well as a very convenient solution approximation method.

The claim behind Thue Approximation is that an integral solution to $f(x,y) = c$ gives a good rational approximation, $\frac{x}{y}$, to some root of the associated polynomial $f(X,1)$ where (x,y) is a solution to the Thue Equation. If the roots are distinct, $\frac{x}{y}$ can approximate only one of the roots which implies that the approximation must be very good. We know that these approximations are "good" by the following proposition proposition: Let η be the minimum distance between roots, $\eta = min_{i \neq j}|\xi_i - \xi_j|$

Proposition. *If $f(x,y) = c$ for integers $x, y \neq 0$, there exists a root ξ of $f(X,1)$ with $|\xi - \frac{x}{y}| \leq B|y|^d$, where $B = |c|(\frac{2}{\eta})^{d-1}$ depends only on f and c.*

This proposition shows that for solutions of a Thue equation with degree three or higher ($d \geq 3$) we actually have very good approximations even if the size of the denominator, $|y|$, is large, meaning that the accuracy of the approximation does not depend on y.

2.2 Computational

2.2.1 Continued Fractions

The importance of Diophantine Approximation in finding solutions to Diophantine Equations presented us with the need to implement an algorithm that would compute the continued fraction expansion for a number. Figure 1 represents the algorithm we implemented in Sage/CoCalc.

Continued fractions are useful to us because an irrational number expressed as a infinite continued fraction provides an initial segment which we can use as a rational approximation; these rational numbers are called the *convergents* of the continued fraction.

Our algorithm works by taking the integer part at each step, a_i, subtracting that from the number at the beginning of the step, taking the reciprocal of the remainder and repeating the process. This yields a list of integers $a_0, a_1, a_2, \cdots$ which are represented in a continued fraction expansion of an irrational number, ξ as,

$$\xi = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}.$$

```
def contFrac(n):
    '''
    Given a number n, returns a list containing the integer parts at each step.
    E.g. [n1, n2, n3, n4, n5] = n1 + (1 / n2 + (1 / n3 + 1 / n4 + (1 / n5)))
    '''
    result = []
    result.append(floor(n))

    yield floor(n)

    cur_fraction = n - result[0]

    for i in xrange(10):           # stops after generating 10 fractions, so it doesnt go forever
        if cur_fraction != 0:       # we're done if current fraction part is 0
            cur_fraction = (cur_fraction)^(-1)
            new_int = floor(cur_fraction)
            yield new_int
            cur_fraction = cur_fraction - new_int
```

Figure 1: Our Sage implementation to compute the continued fraction expansion of a number

2.2.2 Thue Approximation

In order to solve Thue equations and see how accurately the solutions were approximated, we created a CoCalc algorithm using Pari GP functions specific to solving Thue Equations.

```
def thue_graph_finder(thue_input, c):
    R.<x,y> = QQbar[]
    ts = []
    thue_eq = thue_input(y=1)
    thue_eq2 = thue_eq.univariate_polynomial()
    thue_eq_roots = thue_eq2.roots()
    eq_roots = []
    for p in thue_eq_roots:
        eq_roots.append(CC(p[0]))
    th_init = gp.thueinit(thue_eq2,1)
    sols = gp.thue(th_init,c)

    for p in sols:
        if p[1] != 0 and p[2] != 0:
            t = p[1] / p[2]
            ts.append(CC(t))

    approx_plot = plot([0,0])
    if len(ts) > 0:
        approx_plot = point(ts, pointsize=100, color = "red")
    root_plot = point(eq_roots, pointsize=100)
    (root_plot + approx_plot).save("approx.eps")
    show(root_plot + approx_plot)
```

Figure 2: Our function to find and graph solutions to a given Thue Equation as well as its root approximations

2.3 Results

2.3.1 Fundamental Unit Growth

We looked at the growth of the fundamental solutions to the Pell Equations by looking at the maximum value between the (x, y) solution. As we can see in the figure below, the fundamental solutions get larger as d increases. Since these values correspond to fundamental units of $\mathbb{Q}[\sqrt{d}]$, we see that the fundamental units grow.

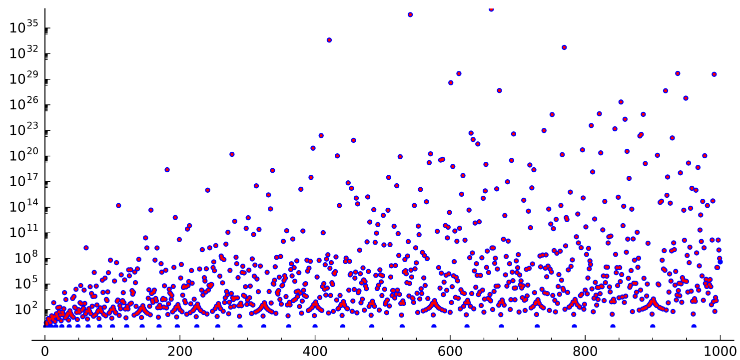

Figure 3: The x-axis is $d \in \mathbb{Z}$ and the y-axis is $\max(x, y)$ of $u = x + y\sqrt{d}$ where u is a fundamental solution of $x^2 - dy^2 = 1$

2.3.2 Thue Approximation

Below are the plotted solutions and approximations for an example Thue equation. The red dots represent the solutions to the Thue equation as $\frac{x}{y}$ and the blue dots are the roots to $f(X, 1)$.

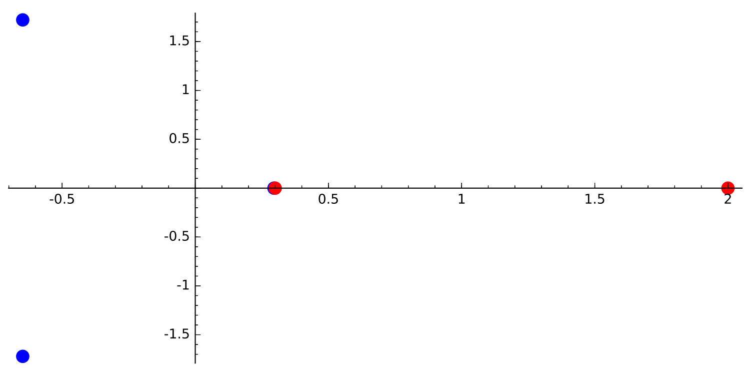

Figure 4: Complex plotting of solutions and approximations $\frac{x}{y}$ to the Thue equation $f(x, y) = x^3 + x^2y + 3xy^2 - y^3 = 17$ with associated polynomial $f(X, 1) = X^3 + X^2 + 3X - 1$

3 Future directions

We conclude our research for the quarter on the topic of Thue Equations with our next goal: to investigate Diophantine Equations of higher degrees (> 3). Using what we previously learned about why Thue Equations have finitely many solutions, we can expand on this knowledge to understand why all equations of higher degrees must also have finitely many solutions. While doing so, we will also begin our study of the general case, i.e. plane curves of the form, $f(X, Y) = 0$, where $f \in \mathbb{Z}[X, Y]$.

Also, using our algorithm to compute fundamental solutions to Pell's Equation, we will continue gathering more results to expand on and contribute to currently available data.

References

[1] SageMath. *CoCalc Collaborative Computation Online*, 2016. `https://cocalc.com/`.

[2] U. Zannier. *Lecture notes on Diophantine analysis*, volume 8 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, 2009. With an appendix by Francesco Amoroso.