University of Washington

Washington Experimental Mathematics Lab

Counting K-Tuples in Discrete Sets

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December 23, 2017

1 Introduction

We were first interested in counting the density of integer lattice points in a ball of radius R. Using a python program, we found that this number converges to the area of the ball. Intuitively, one can imagine each lattice point as a pixel on a screen. The larger the radius, the more pixels you can fit inside, and thus, the closer the number of pixels will be to the actual area.

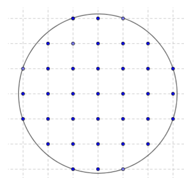

Figure 1: Circle with Lattice

We then started looking as the set of primitive points (m, n), where m and n are both coprime integers. That is, the set of points where $gcd(m, n) = 1$. Our interest was in finding a relationship between the number of pairs of vectors (m, n) in a ball of radius R with determinant k and the area the ball itself.

Geometrically, the determinant is the area of the parallelogram formed by two vectors.

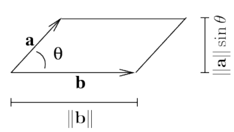

Figure 2: Geometric Depiction of Determinant

Mathematically we know the determinant is:

$$\begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} = m_1 n_2 - n_1 m_2 = k$$

To gather empirical evidence of a convergence for $\dfrac{Count(R, k)}{R^2}$ as R increases, we used python to create a $Count(R, K)$ function which counts the number of integer vector pairs within a ball of radius R that have determinant K.

As stated in the box below, $Count(R, k)$ counts the number of integer vector pairs in a ball of radius R that have a determinant k.

Let $Count(R, k)$ denote the number of matrices

$$A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$a^2 + b^2 + c^2 + d^2 \leq R^2, \quad ad - bc = k, \quad a, b, c, d \in Z$$

$$gcd(a, c) = 1, \quad gcd(b, d) = 1$$

.

Our code implements a Farey Tree to make computing $Count(R, k)$ more efficient. It is based off of the Farey sequence. The Farey sequence of order n is the sequence of completely reduced fractions between 0 and 1 which when in lowest terms have denominators less than or equal to n, arranged in order of increasing size. This helps us generate the set of coprime pairs

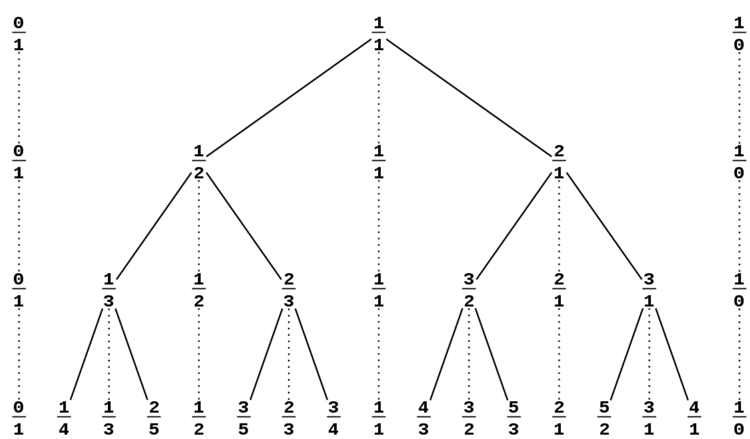

Figure 3: Farey Tree

We calibrated our code against the theorem[1] that states

$$\lim_{R \to \infty} \frac{Count(R, 1)}{R^2} = 6$$

[1] Counting Modular Matrices with Specified Euclidean Norm, Morris Newman

2 Research

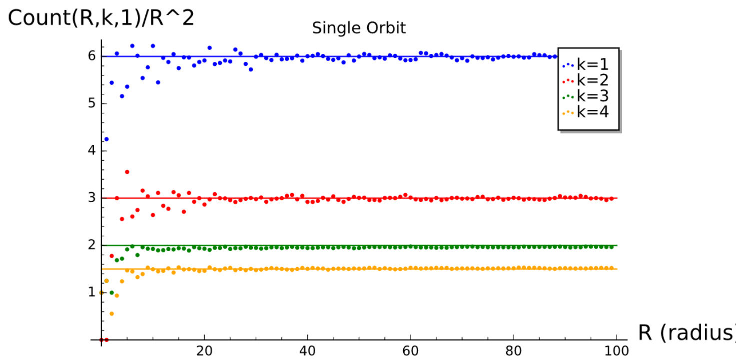

Figure 4: Single orbit $\dfrac{Count(R,k,1)}{R^2} \to \dfrac{6}{k}, \quad \forall k \in \mathbb{Z}$

The first graph shows the result of our code for increasing R and values $k = 1, 2, 3, 4$. Each $Count(R, k)$ can be split up into a certain number of groups or "orbits" based on the determinant k. This graph just looks at a single orbit for each $Count(R, k)$, which we see converges to $\dfrac{6}{k}$ as R increases.

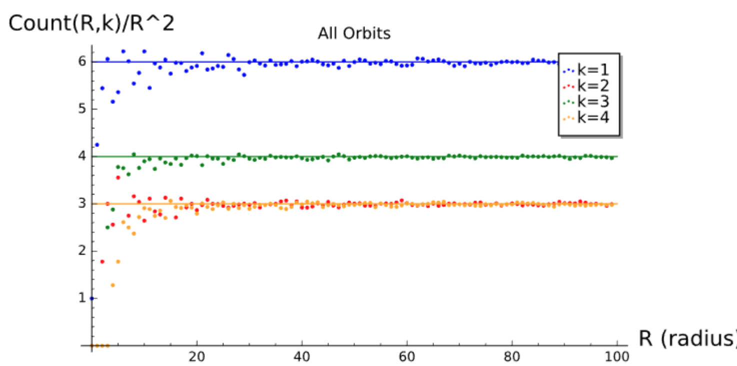

Figure 5: All orbits $\dfrac{Count(R,k)}{R^2} \to \dfrac{6\varphi(k)}{k}, \quad \forall k \in \mathbb{Z}$

The second graph shows the total $Count(R, k)$ for $k = 1, 2, 3, 4$. We see that each of the orbits contributes $6k$ to the total, and that the number of orbits for a given k is equal to $\varphi(k)$. Hence, when you increase R the total count of integer vector pairs with determinant k, proportional to the size of the ball (R^2), converges to $\dfrac{6\varphi(k)}{k}$.

3 Future Goals

In our continuation of this project next quarter, we hope to explore these ideas further by counting vectors in other latices beyond the integer lattice, as well as by counting more than just pairs of vectors (triples, etc.)