Washington Experimental Mathematics Lab

Counting K-tuples in Discrete Sets

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Background

We are interested in counting the density of integer lattice points in a ball of radius R

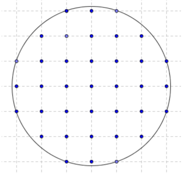

Figure: Circle with Lattice

- Count the set of primitive points (m, n) (where $gcd(m, n) = 1$)
- The density normalized for πR^2 converges to $\frac{6}{\pi^2}$

Counting Vector Pairs

- Counting pairs of integer lattice points (m_1, m_2) and (n_1, n_2) with determinant k
- Determinant is the area of the parallelogram formed by two vectors

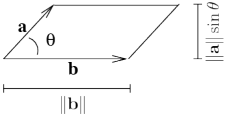

Figure: Geometric Depiction of Determinant

- Determinant:

$$\det \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} = m_1 n_2 - n_1 m_2 = k$$

Progress

- Recall Last Time: We coded a $Count(R, K)$ function which counts the number of integer vectors within a ball (circle) of radius R that have a determinant K

Progress

Definition

Let $Count(R, k)$ denote the number of matrices

$$A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$a^2 + b^2 + c^2 + d^2 \leq R^2, \quad ad - bc = k, \quad a, b, c, d \in Z$$

$$gcd(a, c) = 1, \quad gcd(b, d) = 1$$

Progress

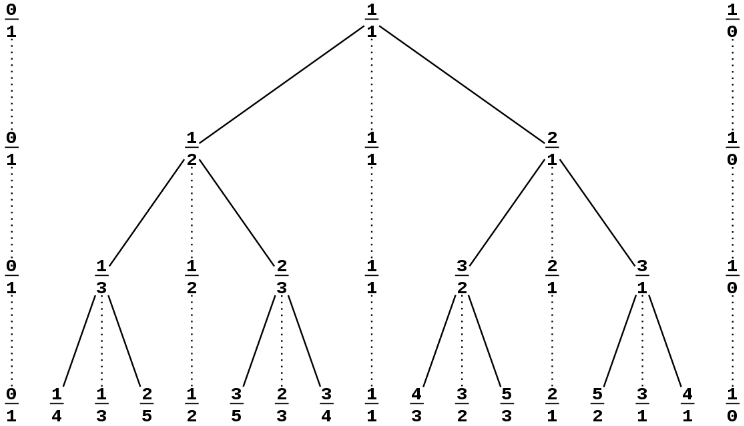

Figure: Farey Tree

Progress

- We wrote code that implements a Farey Tree to make computing *R* more efficient
- We calibrated our code against the theorem[1] that states

$$\lim_{R\to\infty} \frac{Count(R,1)}{R^2} = 6$$

[1] Counting Modular Matrices with Specified Euclidean Norm, Morris Newman

Graph

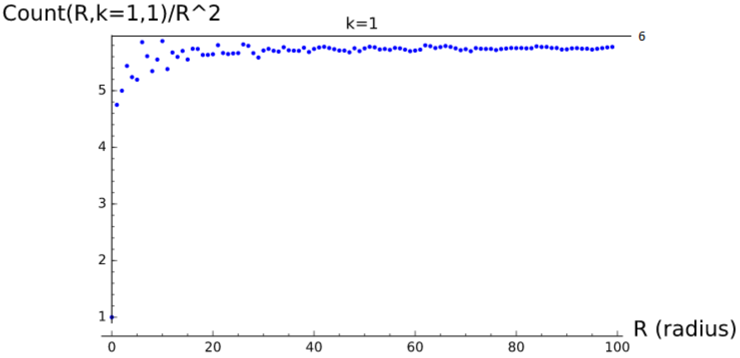

Figure: For $k = 1$, we see $\frac{Count(R,1,1)}{R^2} \to \frac{6}{k} = 6$

Graphs

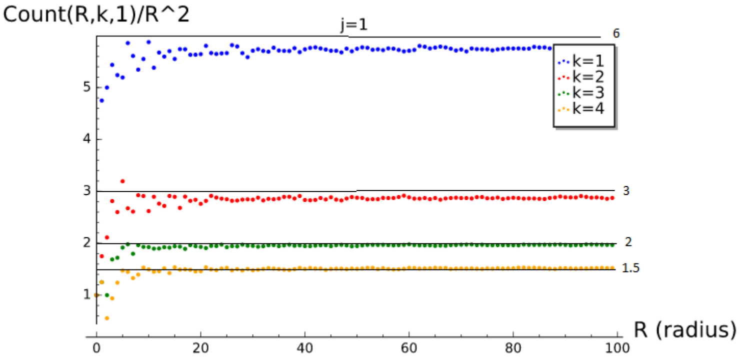

Figure: $$\frac{Count(R,k,1)}{R^2} \to \frac{6\varphi(k)}{k}, \quad \forall k \in \mathbb{Z}$$

Future goals

- Count more than just pairs (triples, etc.)
- Count other visible points not in the integer lattice