Washington Experimental Mathematics Lab

Rotation Random Walks

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Rotation Random Walk

Motivation Random walks are a well studied and interesting mathematical topic. We wish to expand the study of random walks to walks on S^1.

Problem On next slide

Methods We began with a simpler case (rational case) and use code to get experimental data for the irrational case

Problem

For a fixed step length $\alpha \in [0, 1)$ we define the following random walk:
Let Y_i be the random variable of our position on $[0, 1)$ after i steps
$Y_0 = 0$
for $i = 1, 2, ...$
Let $X_i \sim Rademacher(1/2)$. (+1 or -1 with equal probability)
Update $Y_i = \{Y_{i-1} + \alpha X_i\}$ (where $\{x\} = x \pmod 1$)
Let

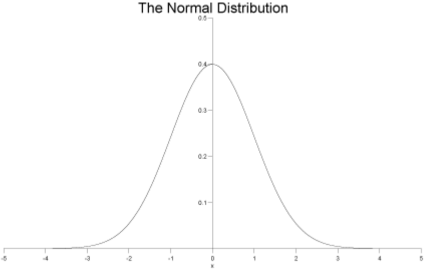
$$Z_i = \begin{cases} +1 & \text{if } Y_i \geq 1/2 \\ -1 & \text{if } Y_i < 1/2 \end{cases}$$

Question: Does the mean $\frac{1}{n}\sum_{i=1}^{n} Z_i$ follow a central limit theorem?

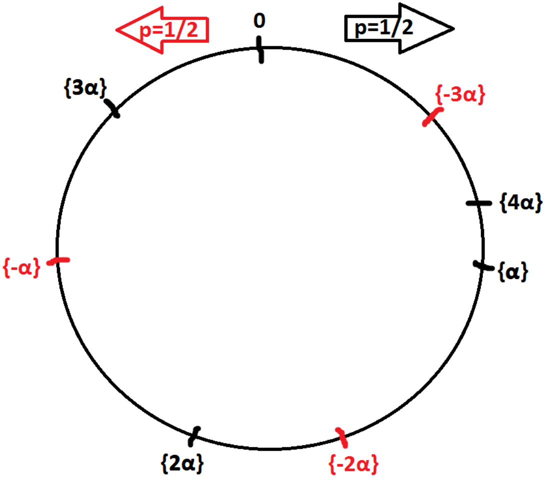
The (Standard) Central Limit Theorem

Let $X_1, ..., X_n$ be independent and identically distributed (iid) random variables with common mean $\mathbb{E}[X_i] = \mu < \infty$ and common variance $Var[X_i] = \sigma^2 < \infty$. Then, as $n \to \infty$,

$$Z_n = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1)$$

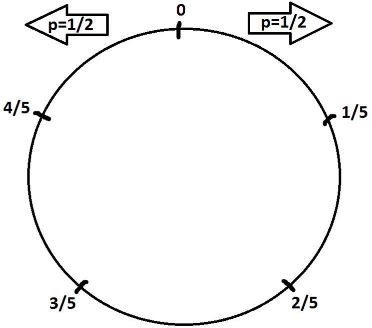


Visual Explanation



Rational Case ($\alpha \in \mathbb{Q}$)

What happens if the steps we take are rational?
The answer is much simpler, and can be reduced to a Markov chain with finite state space, which there already exist CLT's for.



Return Time

We do a first step-analysis using the law of total expectation, conditioning on whether we take a left or right step. Let T_k be the random variable that denotes the first return time to the starting point when there are k positions.

$$\mathbb{E}[T_k] = \mathbb{E}[T_k|R]Pr(R) + \mathbb{E}[T_k|L]Pr(L) = \frac{1}{2}[\mathbb{E}[T_k|L] + \mathbb{E}[T_k|R]] = \mathbb{E}[T_k|R]$$

$$\mathbb{E}[T_k|R] = \frac{1}{2}(\mathbb{E}[T|R^2] + \mathbb{E}[T|RL]) = \frac{1}{2}\mathbb{E}[T_k|R^2] + 1$$

$$\mathbb{E}[T_k|R^2] = \frac{1}{2}(\mathbb{E}[T_k|R^3] + \mathbb{E}[T_k|R^2L]) = \frac{1}{2}(\mathbb{E}[T_k|R^3] + 2 + \mathbb{E}[T_k|R])$$

Plugging this back in, we get:

$$\mathbb{E}[T_k] = \frac{1}{2}\mathbb{E}[T_k|R^2] + 1 = \frac{1}{3}\mathbb{E}[T_k|R^3] + 2$$

Therefore, we get the general pattern:

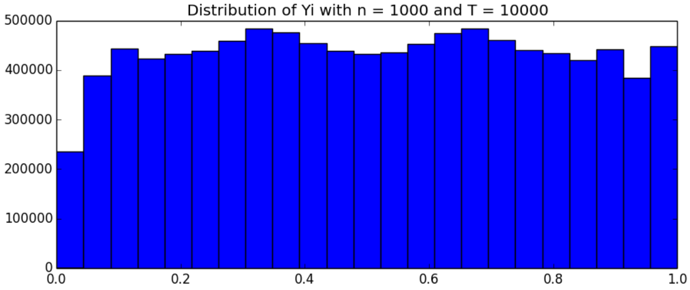
$$\mathbb{E}[T_k] = \frac{1}{j}\mathbb{E}[T_k|R^j] + (j-1)$$

But the base case is $\mathbb{E}[T_k|R^k] = k$ since taking k right steps from the origin gets you back to the origin:

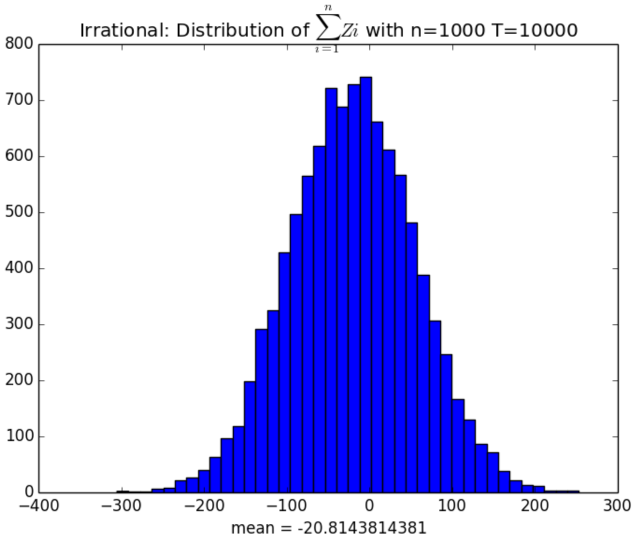
$$\mathbb{E}[T_k] = \frac{1}{k}k + (k-1) = k$$

Irrational Case

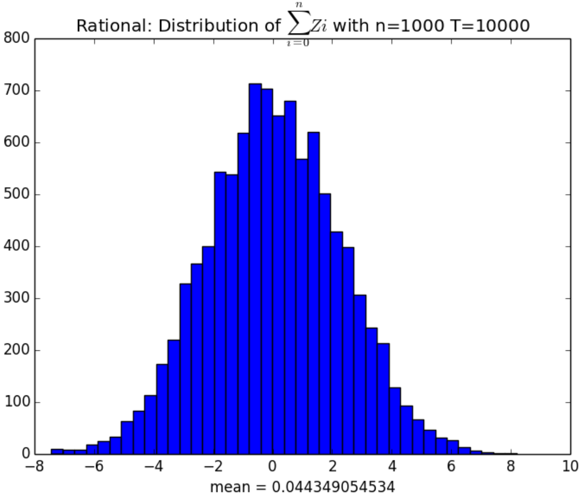
The α trajectory is dense and uniformly distributed.



Experiments: Irrational



Experiments: rational



Future goals

Next steps We will read more literature on showing how a random variable follows a central limit theorem, as well as literature on random walks.

Challenges Experiments for the irrational case are hard because of the difficulty in computing irrational number with computer (floating point problem).