Washington Experimental Mathematics Lab

Rotation Random Walks

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Rotation Random Walk

Motivation Random walks are a well studied and interesting mathematical topic. We wish to expand the study of random walks on S^1.

Problem On next slide

Methods We began with a simpler case (rational case) and now are looking to extend the problem to the irrational case

Problem

Algorithm 1 Rotation Random Walk

Fix $\alpha = \frac{1}{k}, k \in \mathbb{N}_{\geq 2}$ to be the step size.
Set $Y_0 = 0$, where $Y_i \in [0, 1)$ is the position of the walk at time i.
for $i = 1, 2, ...n$ **do**

$$X_i = \begin{cases} +1, & \text{w.p } \frac{1}{2} \\ -1, & \text{w.p } \frac{1}{2} \end{cases}$$

$$Y_i = \{Y_{i-1} + \alpha X_i\} \text{ the new position mod 1.}$$

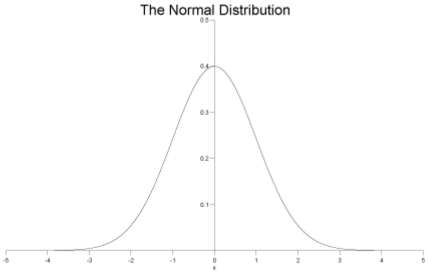
$$Z_i = \begin{cases} +1, & \text{if } Y_i \in [0, \frac{1}{2}) \\ -1, & \text{if } Y_i \in [\frac{1}{2}, 1) \end{cases}$$

Question: The partial sums $S_n = \sum_{i=1}^{n} Z_i$ define a new random walk on $\mathbb{Z}$. Does the mean $\frac{1}{n}\sum_{i=1}^{n} Z_i$ follow a central limit theorem (the Z_i's are not independent!)?

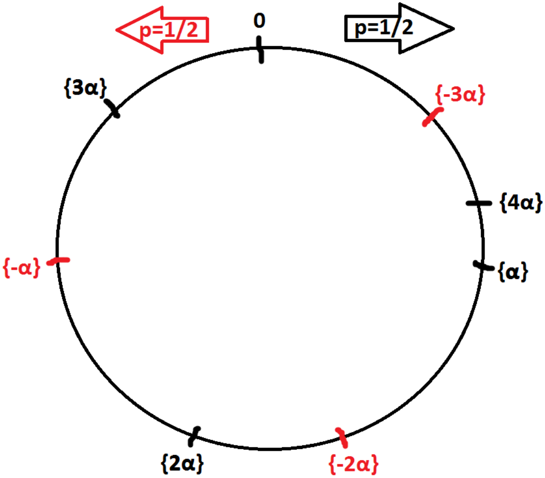
The (Standard) Central Limit Theorem

Let $X_1, ..., X_n$ be independent and identically distributed (iid) random variables with common mean $\mathbb{E}[X_i] = \mu < \infty$ and common variance $Var[X_i] = \sigma^2 < \infty$. Then, as $n \to \infty$,

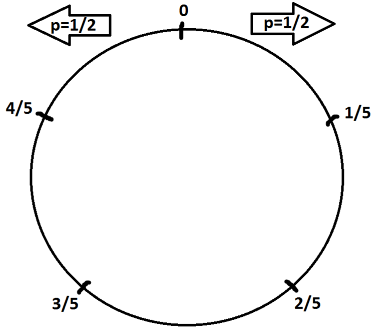
$$Z_n = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, 1)$$



Visual Explanation

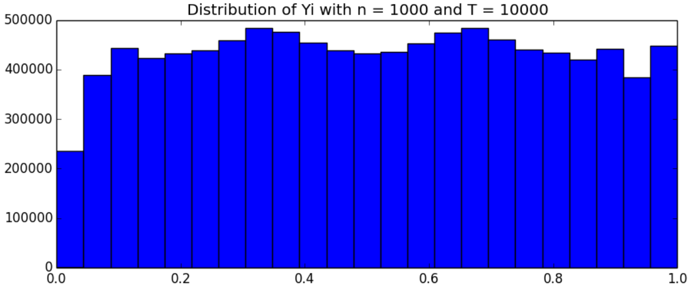


What happens if the steps we take are rational?
The answer is much simpler, and can be reduced to a Markov chain with finite state space, which there already exist CLT's for.



Irrational Case

The α trajectory is dense and uniformly distributed.



Extending Rational Case to Irrationals

We can use the CLT result of the rational case for extending the results to the irrationals.
For the number of steps $= N$ we can choose a rational $\frac{p}{q}$ with $|\frac{p}{q} - \alpha| < \varepsilon_N$. This ε_N is chosen so that for all $n < N$

$$Y_{\alpha,n} \in [0, 1/2) \Leftrightarrow Y_{\frac{p}{q},n} \in [0, 1/2)$$

Thus our Z_is will be the same. Since $Z_{\frac{p}{q},i}$ follows CLT we can bound its distribution close to the normal distribution. Thus we can bound $Z_{\alpha,i}$ close to the normal distribution as $N \to \infty$
(in Analysis this is sometimes called a triple epsilon argument)

Continued Fractions

Continued Fractions give a very good approximation to any irrational number.
Given $0 < \alpha < 1$, we can write

$$\alpha = [a_1, a_2, a_3, ...] = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

Of course, we can terminate after any time $[a_1, ..., a_n]$. Each $[a_1, ..., a_n]$ can be written as $\frac{p_n}{q_n}$, where $\gcd(p_n, q_n) = 1$. We are guaranteed that $|\alpha - \frac{p_n}{q_n}| < \frac{1}{q_n^2}$, a quadratic bound in the denominator q_n.

Continued Fractions (example)

$\sqrt{2}-1 = [2,2,2,2,2,2,...] \approx 0.41421356237$
$[2] = \frac{1}{2} = 0.5$
$[2,2] = \frac{1}{2+\frac{1}{2}} = \frac{2}{5} = 0.4$
$[2,2,2] = \frac{1}{2+\frac{1}{2+\frac{1}{2}}} = \frac{5}{12} = 0.41\bar{6}$
$[2,2,2,2] = \frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}} = \frac{12}{29} \approx 0.41379$
...

Future goals

Next steps We must understand how 'fast' the rational case converge to the normal distribution.
We will conduct more experiments now using continued fractions to approximate the irrational case.

Challenges It is hard to find literature on the rate of convergence for the random walk on a cyclic graph following CLT.