Washington Experimental Mathematics Lab
Stability Spectrum for PDEs

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Motivation: To determine the stability of periodic solutions to certain PDEs, including the focusing mKdV equation.

Problem: To determine the eigenvalues of a linear operator such that the associated eigenfunctions are bounded.

Methods: Taking advantage of the periodicity of the coefficients using Floquet theory and Fourier series.
Let $\mathcal{L} = -\partial_x^2$ and consider the eigenvalue problem

$$\mathcal{L}y = \lambda y, \quad \lambda \in \mathbb{C},$$

or equivalently, $y'' + \lambda y = 0$.

We say $\lambda \in \sigma(\mathcal{L})$ if the associated $y(x)$ is bounded for all $x \in \mathbb{R}$,

$$\lim_{|x| \to \infty} |y(x)| < \infty.$$
To find $\sigma(\mathcal{L})$, we solve the differential equation $y'' + \lambda y = 0$. The general solution is

$$y = c_1 e^{\sqrt{-\lambda} x} + c_2 e^{-\sqrt{-\lambda} x}.$$ 

In general, $\sqrt{-\lambda} \in \mathbb{C}$, so let $\sqrt{-\lambda} = \alpha + i\beta$ for some $\alpha, \beta \in \mathbb{R}$. 
Letting $\sqrt{-\lambda} = \alpha + i\beta$, rewrite $y = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$ as

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{-(\alpha + i\beta)x}$$

$$= c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{-\alpha x} e^{-i\beta x}$$

$$= c_1 e^{\alpha x} [\cos(\beta x) + i \sin(\beta x)] + c_2 e^{-\alpha x} [\cos(\beta x) - i \sin(\beta x)]$$

We require that the solution $\lim_{|x| \to \infty} |y(x)| < \infty$. Therefore, $\alpha = 0$. 
Example (ct’d.)

With $\alpha = 0$, 

$$y = c_1[\cos(\beta x) + i \sin(\beta x)] + c_2[\cos(\beta x) - i \sin(\beta x)]$$

$$= A \cos(\beta x) + iB \sin(\beta x)$$

is bounded for any $\beta \in \mathbb{R}$. $\sqrt{-\lambda} = i\beta$ only if $\lambda \geq 0$.

So $\sigma(-\partial_{xx}) = [0, \infty)$. 
Figure: Spectrum of the operator $\mathcal{L} = -\partial_x^2$. 
Figure: Spectrum of the operator $\mathcal{L} = -\partial_x^2 + 2q \cos(2x)$ for $q = 2$ from 0 to 10.
Future goals

Next steps  Determining the spectrum of

\[ \mathcal{L} = -\partial_y^3 + (c - 6U^2)\partial_y + 12UU_y, \]

which is used in determining the stability of the solution
\[ u(x, t) = U(y) = U(x - ct) \]

of the focusing mKdV equation:

\[ u_t + 6u^2u_x + u_{xxx} = 0. \]

Challenges  Understanding elliptic functions and generalizing our code.