Uniformity of Solutions to Diophantine Equations

Rohan Hiatt, Daria Mićović, Blanca Viña Patiño, Bryan Quah

Mentors: Travis Scholl, Amos Turchet

Washington Experimental Mathematics Lab (WXML)
University of Washington

UNIVERSITY of WASHINGTON

What are Diophantine Equations?

- Polynomial equations with integer coefficients, where we look for integer or rational solutions.
- May have infinite or finite number of solutions.
- Named after Diophantus of Alexandria.

Examples:

$$a^2 + b^2 = c^2$$

$$x^2 - Dy^2 = \pm 1$$

$$x^3 + x^2 y + 3xy^2 - y^3 = 17$$

$$y^2 = x^5 - 5x^3 + 4x - 1$$

Motivation

- Investigate methods for solving Diophantine Equations - there is no general algorithm for solving Diophantine Equations (Hilbert’s 10th Problem)
- Understand the set of solutions to specific types of Diophantine Equations
- Search for more rational points for hyperelliptic curves of genus 2

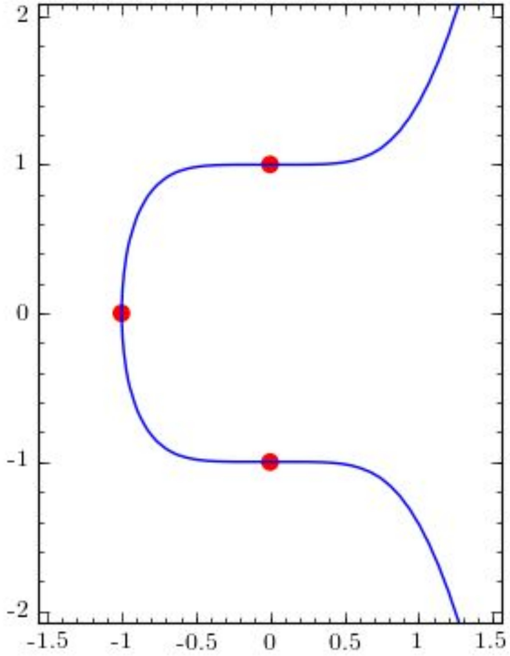
Equations to Plane Curves

Consider: $y^2 = x^5 + 1$

This equation defines a curve in the plane

Solutions to this equation, $x = a$ and $y = b$, correspond to points (a,b) on the curve

When a and b are rational numbers, these are called "rational points"



Faltings' Theorem (1983)

Let K be a number field. If C is an algebraic curve over K of genus $g \geq 2$, then there are only finitely many rational points, i.e., the set $C(K)$ of K-rational points is finite.

If C is a smooth curve of degree *d*, the genus is given by

$$g = \frac{(d-1)(d-2)}{2}$$

Lang Conjecture (1986)

If X is a variety of general type defined over a number field K, then the set $X(K)$ of K-rational points of X is not Zariski dense.

- One of the most important conjectures in Diophantine geometry
- Conjectural analogue of Faltings' theorem in higher dimensions. Caporaso, Harris, and Mazur proved that if the Lang Conjecture is true, then the following conjecture must also be true.

Uniformity Conjecture (1997)

Let $g \geq 2$ be an integer. There exists a number $B(g)$, depending only on g, such that every smooth curve of genus g defined over $\mathbb{Q}$, has at most $B(g)$ rational points.

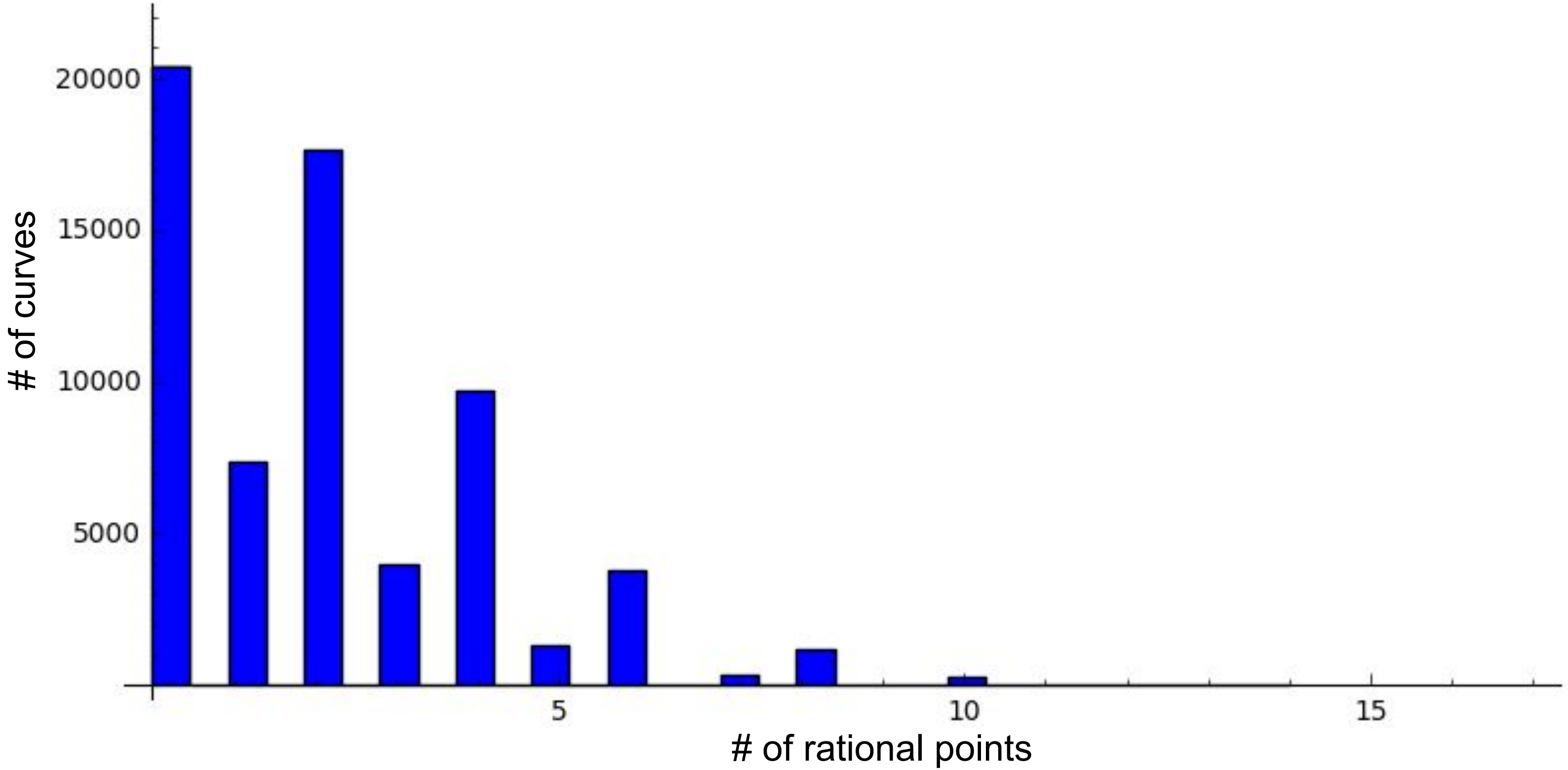
- Theorem (Katz–Rabinoff–Zureick-Brown) (2016):
  - Found a genus-dependent formula for the number of points on a smooth curve of genus at least 3.

$$\#C(\mathbb{Q}) = 84g^2 - 98g + 28$$

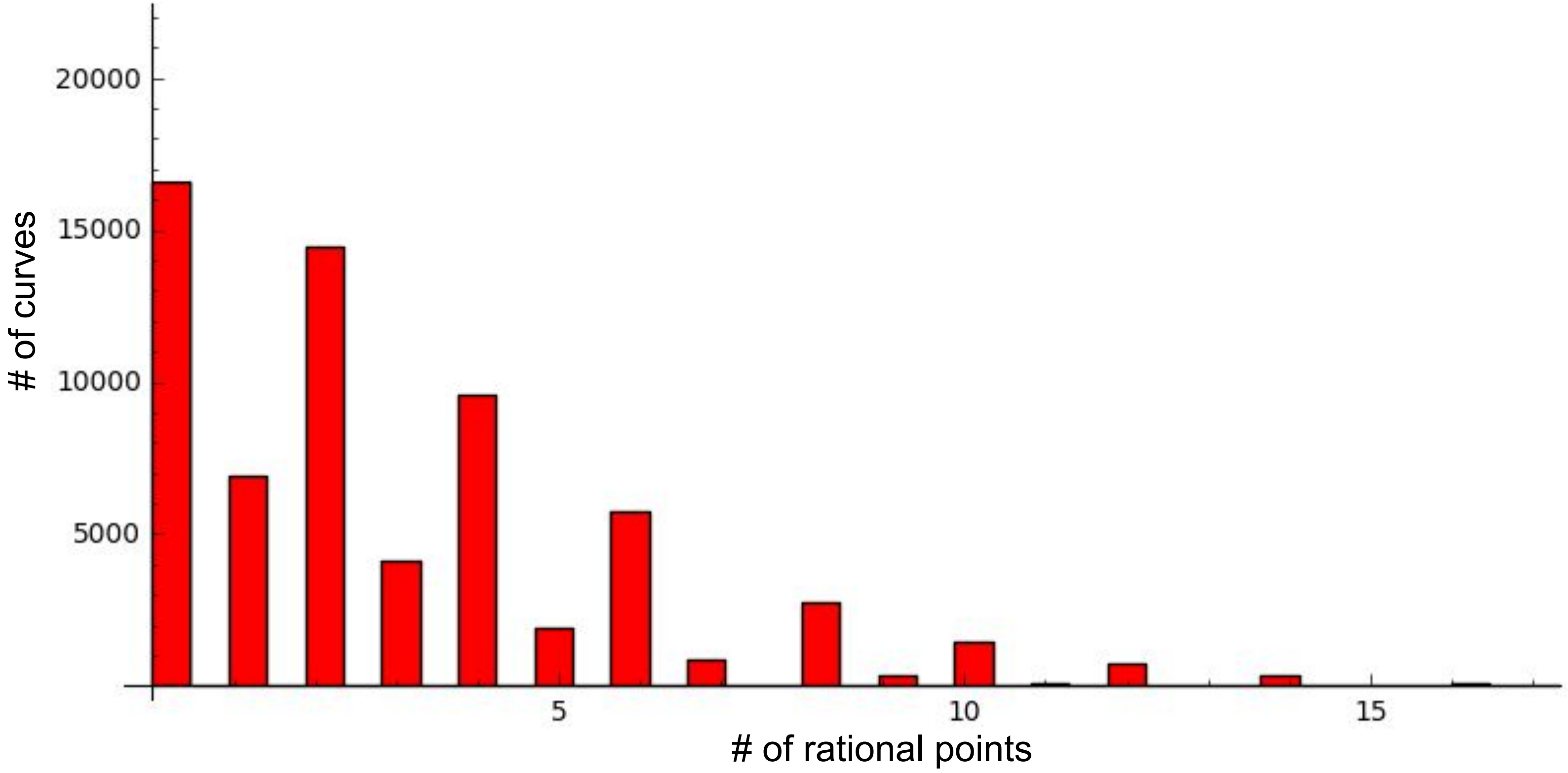
Rational Points on Hyperelliptic Curves

- The **L-Functions and Modular Forms Database (LMFDB)** is an online collection of mathematical objects including a list of genus 2 plane curves
- We imported Michael Stoll's C program to CoCalc and added Sage scripts for increased functionality
- Our code was run using the LMFDB database of genus 2 plane curves
  - We found points not in the database for incomplete curves!

Results - LMFDB Data



Results - Our Data



Future Directions

- Construct a hyperelliptic curve not found in the LMFDB and compute its rational points
- Continue to gather data for higher upper bound values in Stoll's
- Optimize our code to work more efficiently when utilizing parallelization