Washington Experimental Mathematics Lab

Counting K-tuples in Discrete Sets

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Integer Points

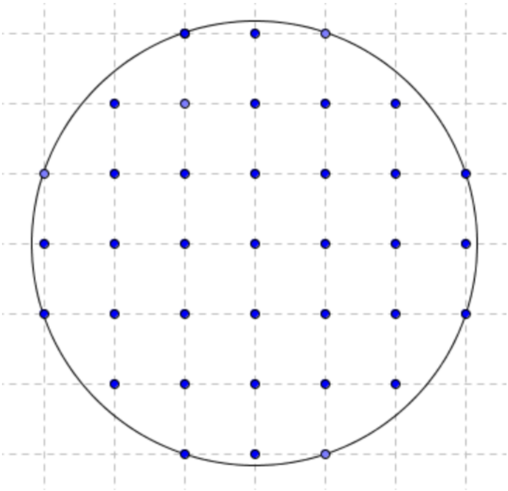


Figure: Integer lattice within the circle $x^2 + y^2 \leq 10$

Primitive Points

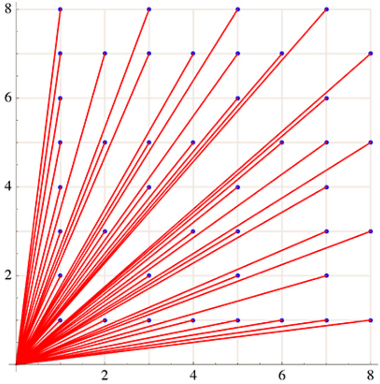


Figure: Primitive points in the first quadrant of an integer lattice

"visible points," i.e. $gcd(x, y) = 1$

Primitive Pairs

Definition

Let $Count(R, k)$ denote the number of matrices

$$A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$a^2 + b^2 + c^2 + d^2 \leq R^2, \quad ad - bc = k, \quad a, b, c, d \in Z$$

$$gcd(a, c) = 1, \quad gcd(b, d) = 1$$

$SL_2(\mathbb{Z})$

$SL_2(\mathbb{Z})$: set of 2×2 matrices with determinant 1 and all integer entries

$$SL_2(\mathbb{Z}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det A = 1 \text{ and } a, b, c, d \in \mathbb{Z} \right\}$$

$SL_2(\mathbb{Z})$ Orbits

Example:

$$SL_2(\mathbb{Z}) \cdot \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} a & a+3b \\ c & c+3d \end{pmatrix} : \text{a}, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

$$SL_2(\mathbb{Z}) \cdot \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} a & 2a+3b \\ c & 2c+3d \end{pmatrix} : \text{a}, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

$SL_2(\mathbb{Z})$ Orbits Continued

$$SL_2(\mathbb{Z}) \cdot \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} = \left\{ \begin{pmatrix} a & a+3b \\ c & c+3d \end{pmatrix} : ad - bc = 1 \right\}$$
$$SL_2(\mathbb{Z}) \cdot \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \left\{ \begin{pmatrix} a & 2a+3b \\ c & 2c+3d \end{pmatrix} : ad - bc = 1 \right\}$$

The $SL_2(\mathbb{Z})$ orbit of $\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$ equals the $SL_2(\mathbb{Z})$ orbit of $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ if their sets are equal:

$$2a + 3b = a + 3b \quad \Rightarrow \quad a = 0$$
$$2c + 3d = c + 3d \quad \Rightarrow \quad c = 0$$

Since $a = c = 0$, $ad - bc = 0$ which is a contradiction to our definition that $ad - bc = 1$. Therefore, we showed they are not in the same orbit.

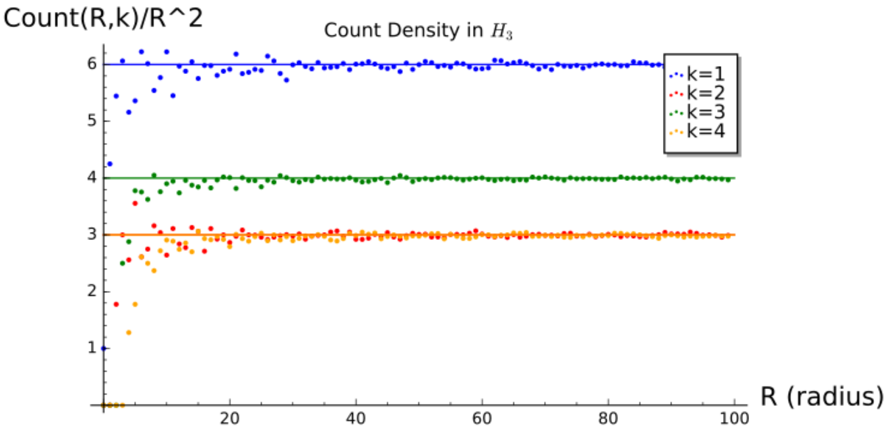
$SL_2(\mathbb{Z})$ Orbits Continued

For a group of 2×2 matrices with certain determinant k, we proved that there are $\varphi(k)$ $SL_2(\mathbb{Z})$ orbits, where $\varphi(k)$ is Euler's Totient function.

We did this by showing that any 2×2 matrix can be reduced into the form $\begin{pmatrix} 1 & j \\ 0 & k \end{pmatrix}$, where k is the determinant of the original matrix and j is an integer such that $0 < j \leq k$.

Note: $gcd(j, k) = 1$ hence, $\varphi(k)$ distinct forms or orbits.

How the Orbits Relate to Count



$$\frac{Count(R,k)}{R^2} \to \frac{6\varphi(k)}{k}$$

Triples

Definition

Let $Count(R, k_1, k_2, k_3)$ denote the number of groups of 3 vectors

$$V_1, V_2, V_3 \in \mathbb{Z}^2_{prim}$$

where each pair of vectors is inside a ball of radius R such that

$$\det(V_1, V_2) = k_1,$$

$$\det(V_1, V_3) = k_2,$$

and

$$\det(V_2, V_3) = k_3.$$

Example of Orbits of Triples

Consider the matrix $B = \begin{pmatrix} 1 & 1 & k \\ 0 & 1 & 1 \end{pmatrix}$, where $k_1 = 1$, $k_2 = 1$, $k_3 = 1 - k$

We want to classify our matrix B by applying any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbb{Z})$ to B.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 & k \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} a & a+b & ak+b \\ c & c+d & ck+d \end{pmatrix}$$

Thus, we have classified the orbit of $\begin{pmatrix} 1 & 1 & k \\ 0 & 1 & 1 \end{pmatrix}$ to be all matrices of the form $\begin{pmatrix} a & a+b & ak+b \\ c & c+d & ck+d \end{pmatrix}$.

Example of Orbits of Triples Continued

Consider the matrix $\begin{pmatrix} 2 & 1 & 5 \\ 3 & 2 & 8 \end{pmatrix}$, where $k_1 = 1$, $k_2 = 1$, $k_3 = -2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 & k \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 5 \\ 3 & 2 & 8 \end{pmatrix}$$

$$\Rightarrow a = 2, b = -1, c = 3, d = -1 \Rightarrow k = 3.$$

Thus, this matrix is in the orbit of $\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix}$

Linear Independence

For a 2×3 matrix $A = [C_1 \mid C_2 \mid C_3]$ where $C_1, C_2, C_3 \in \mathbb{R}^2$, the $rank(A) \leq 2$.

This gives us the following cases for independence between the vectors:

1. C_1, C_2, C_3 are linearly dependent.
2. C_1 and C_2 are linearly independent.
3. C_1 and C_3 are linearly independent.
4. C_2 and C_3 are linearly independent.

$SL_2(\mathbb{Z})$ Orbits of Triples

Consider matrix $\begin{pmatrix} V_1 & V_2 & V_3 \end{pmatrix}$, where vectors V_j are "columns" in $\mathbb{Z}^2_{prim}$. If there exists real numbers x and y such that $V_3 = xV_1 + yV_2$, then the $SL_2\mathbb{Z}$ orbit of the matrix is

$$\left\{ \begin{pmatrix} a & b & ax+by \\ c & d & cx+dy \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z} \cdot \begin{pmatrix} V_1 & V_2 \end{pmatrix} \right\}$$

Future goals

- Continue to understand the decomposition of primitive triples into $SL_2\mathbb{Z}$ orbits
- Write a monster program that will count the density of any k-tuple of vectors given any number of determinants as input
- Write our beautiful paper with all the theory behind our findings