Counting $K$-tuples in discrete sets

Washington Experimental Mathematics Lab
Counting $K$-tuples in Discrete Sets

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Spring 2018
Figure: Integer lattice within the circle $x^2 + y^2 \leq 10$
Primitive Points

Figure: Primitive points in the first quadrant of an integer lattice

"visible points," i.e. \(\gcd(x, y) = 1\)
Primitive Pairs

Definition

Let $\text{Count}(R, k)$ denote the number of matrices

$$A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$a^2 + b^2 + c^2 + d^2 \leq R^2, \quad ad - bc = k, \quad a, b, c, d \in \mathbb{Z}$$

$$\gcd(a, c) = 1, \quad \gcd(b, d) = 1$$
$SL_2(\mathbb{Z})$: set of $2 \times 2$ matrices with determinant 1 and all integer entries

\[
SL_2(\mathbb{Z}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det A = 1 \text{ and } a, b, c, d \in \mathbb{Z} \right\}
\]
SL₂(ℤ) Orbits

Example:

\[
SL₂(ℤ) \cdot \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}
\]
\[
= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} a & a + 3b \\ c & c + 3d \end{pmatrix} : a, b, c, d \in ℤ, \ ad - bc = 1 \right\}
\]

\[
SL₂(ℤ) \cdot \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}
\]
\[
= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} a & 2a + 3b \\ c & 2c + 3d \end{pmatrix} : a, b, c, d \in ℤ, \ ad - bc = 1 \right\}
\]
SL₂(ℤ) Orbits Continued

\[
SL₂(ℤ) \cdot \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} = \left\{ \begin{pmatrix} a & a+3b \\ c & c+3d \end{pmatrix} : ad - bc = 1 \right\}
\]

\[
SL₂(ℤ) \cdot \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \left\{ \begin{pmatrix} a & 2a+3b \\ c & 2c+3d \end{pmatrix} : ad - bc = 1 \right\}
\]

The SL₂(ℤ) orbit of \( \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \) equals the SL₂(ℤ) orbit of \( \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \) if their sets are equal:

\[2a + 3b = a + 3b \Rightarrow a = 0\]
\[2c + 3d = c + 3d \Rightarrow c = 0\]

Since \( a = c = 0 \), \( ad - bc = 0 \) which is a contradiction to our definition that \( ad - bc = 1 \). Therefore, we showed they are not in the same orbit.
For a group of $2 \times 2$ matrices with certain determinant $k$, we proved that there are $\varphi(k) \ SL_2(\mathbb{Z})$ orbits, where $\varphi(k)$ is Euler’s Totient function.

We did this by showing that any $2 \times 2$ matrix can be reduced into the form \[
\begin{pmatrix}
1 & j \\
0 & k
\end{pmatrix},
\]
where $k$ is the determinant of the original matrix and $j$ is an integer such that $0 < j \leq k$.

Note: $gcd(j, k) = 1$ hence, $\varphi(k)$ distinct forms or orbits.
How the Orbits Relate to Count

\[
\frac{\text{Count}(R, k)}{R^2} \to \frac{6\varphi(k)}{k}
\]

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Triples

Definition
Let \( \text{Count}(R, k_1, k_2, k_3) \) denote the number of groups of 3 vectors

\[
V_1, V_2, V_3 \in \mathbb{Z}_\text{prim}^2
\]

where each pair of vectors is inside a ball of radius \( R \) such that

\[
\det(V_1, V_2) = k_1, \\
\det(V_1, V_3) = k_2, \\
\det(V_2, V_3) = k_3.
\]
Example of Orbits of Triples

Consider the matrix \( B = \begin{pmatrix} 1 & 1 & k \\ 0 & 1 & 1 \end{pmatrix} \), where \( k_1 = 1 \), \( k_2 = 1 \), \( k_3 = 1 - k \)

We want to classify our matrix \( B \) by applying any matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) in \( SL_2(\mathbb{Z}) \) to \( B \).

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 & k \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} a & a+b & ak+b \\ c & c+d & ck+d \end{pmatrix}
\]

Thus, we have classified the orbit of \( \begin{pmatrix} 1 & 1 & k \\ 0 & 1 & 1 \end{pmatrix} \) to be all matrices of the form \( \begin{pmatrix} a & a+b & ak+b \\ c & c+d & ck+d \end{pmatrix} \).
Consider the matrix \( \begin{pmatrix} 2 & 1 & 5 \\ 3 & 2 & 8 \end{pmatrix} \), where \( k_1 = 1, \, k_2 = 1, \, k_3 = -2 \)

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 & k \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 5 \\ 3 & 2 & 8 \end{pmatrix}
\]

\( \Rightarrow a = 2, \, b = -1, \, c = 3, \, d = -1 \Rightarrow k = 3. \)

Thus, this matrix is in the orbit of \( \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix} \).
Linear Independence

For a $2 \times 3$ matrix $A = [C_1 \mid C_2 \mid C_3]$ where $C_1, C_2, C_3 \in \mathbb{R}^2$, the $\text{rank}(A) \leq 2$.

This gives us the following cases for independence between the vectors:

1. $C_1, C_2, C_3$ are linearly dependent.
2. $C_1$ and $C_2$ are linearly independent.
3. $C_1$ and $C_3$ are linearly independent.
4. $C_2$ and $C_3$ are linearly independent.
Consider matrix \((V_1 \quad V_2 \quad V_3)\), where vectors \(V_j\) are "columns" in \(\mathbb{Z}_2^{\text{prim}}\). If there exists real numbers \(x\) and \(y\) such that \(V_3 = xV_1 + yV_2\), then the \(SL_2\mathbb{Z}\) orbit of the matrix is

\[
\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z} \cdot (V_1 \quad V_2) \right\}
\]
Future goals

- Continue to understand the decomposition of primitive triples into $SL_2\mathbb{Z}$ orbits
- Write a monster program that will count the density of any $k$-tuple of vectors given any number of determinants as input
- Write our beautiful paper with all the theory behind our findings