Washington Experimental Mathematics Lab

Stability Spectrum for PDEs

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Goal - Find the stability spectrum for mKdV

Motivation To determine the stability of periodic traveling wave solutions to the focusing mKdV equation

Problem To compute the spectrum of the operator obtained by linearizing the focusing mKdV equation about a particular solution

Methods Using the Floquet-Fourier-Hill method to reduce an infinite-dimensional eigenvalue problem to a finite-dimensional one

Stability

We start with a solution $u(x,t) = U(x - Vt) = U(y)$ of the focusing modified Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{1}$$

Looking for nearby solutions, $u(y,t) = U(y) + \epsilon w(y,t)$, $W(y) = e^{-\lambda t} w(y,t)$ satisfies

$$\mathcal{L}W = \lambda W, \qquad \lambda \in \mathbb{C} \tag{2}$$

where

$$\mathcal{L} = -\partial_y^3 + (V + 6U^2)W_y + 12UU_yW. \tag{3}$$

Stability

The perturbation $w(y,t) = e^{\lambda t}W(y)$ remains small if the only allowed λ are purely imaginary.

$\mathcal{L}W = \lambda W$ is an eigenvalue problem and we say $\lambda \in \sigma(\mathcal{L})$ if the associated $W(y)$ is bounded for all $y \in \mathbb{R}$, $\sup_y |W(y)| < \infty$.

Solution to the focusing mKdV equation

The general solution to the focusing mKdV equation is given by:

$$U(y) = \frac{\pm\sqrt{2E}\wp'(\frac{1}{2}(y+y_0), g_2, g_3) + C(2\wp(\frac{1}{2}(y+y_0), g_2, g_3) - \frac{2}{3}V)}{\left(\wp(\frac{1}{2}(y+y_0), g_2, g_3) - \frac{V}{3} - 2\sqrt{-2E}\right)\left(\wp(\frac{1}{2}(y+y_0), g_2, g_3) - \frac{V}{3} + 2\sqrt{-2E}\right)}$$

Coding this solution in Python was a major challenge during the past few weeks.

A simple solution

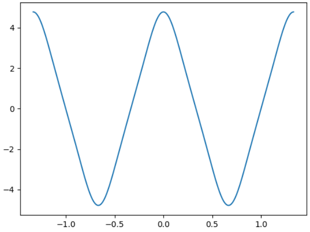


Figure: Plot of the solution $U(y)$ with parameters $V = 10$, $k = 0.8$, and $C = 0$

Spectral density

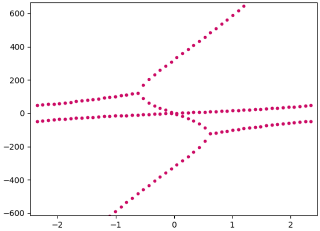


Figure: Plot of $\mathrm{Im}(\lambda)$ against μ, with parameters $V = 10$, $k = 0.8$, and $C = 0$, using 81 Fourier modes and 49 Floquet modes

Numerically computed spectra

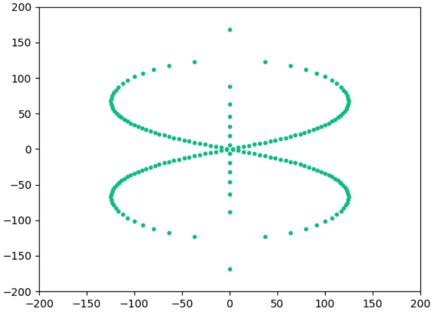


Figure: Focusing mKdV: $V = 10$, $k = 0.8$, $C = 0$

Numerically computed spectra

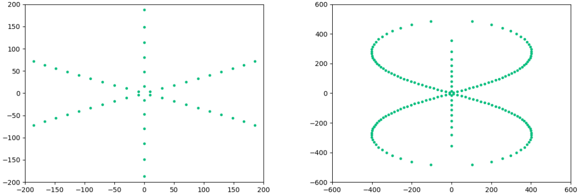


Figure: Focusing mKdV: $V = 10$, $k = 0.75$, $C = 0$

Numerically computed spectra

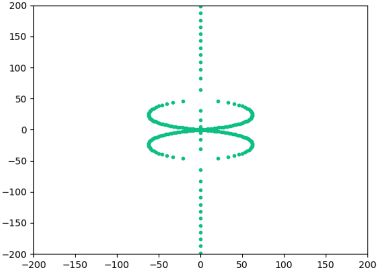


Figure: Focusing mKdV: $V = 10$, $k = 0.85$, $C = 0$

Numerically computed spectra

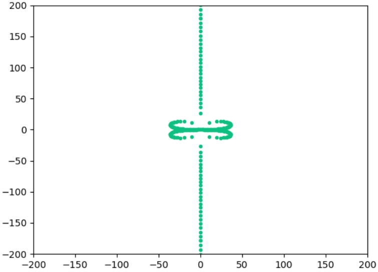


Figure: Focusing mKdV: $V = 10$, $k = 0.9$, $C = 0$

Numerically computed spectra

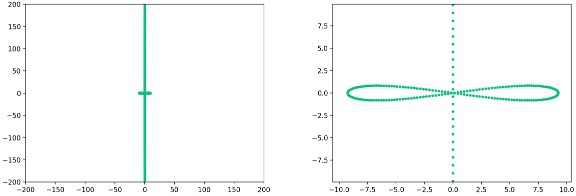


Figure: Focusing mKdV: $V = 10$, $k = 0.99$, $C = 0$

Numerically computed spectra

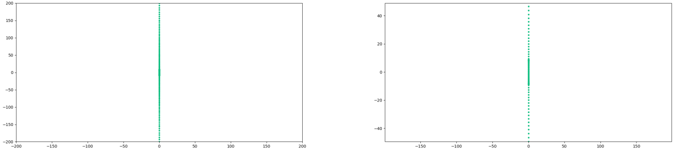


Figure: Focusing mKdV: $V = 10$, $k = 1.01$, $C = 0$

Numerically computed spectra

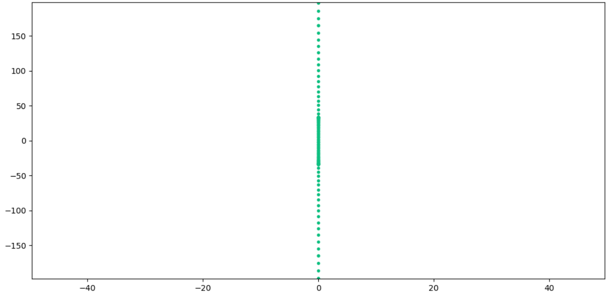
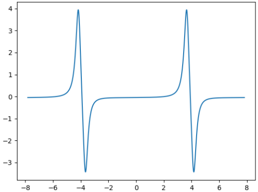


Figure: Focusing mKdV: $V = 10$, $k = 1.8$, $C = 0$

A solution with $C \neq 0$



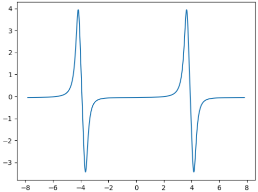


Figure: Solutions when $V = 10$, $k = 0.8$, $C = 10$: (left) using built-in Numpy functions; (right) using Fourier series

Final steps

With the general solution now coded properly, the remaining tasks are:

(1) to compute the spectra of operators centered about such solutions, and

(2) to interpret the stability of these solutions based on their spectra.